

On k -(total) limited packing in graphs

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Abstract: A set $B \subseteq V(G)$ is called a k -total limited packing set in a graph G if $|B \cap N(v)| \leq k$ for any vertex $v \in V(G)$. The k -total limited packing number $L_{k,t}(G)$ is the maximum cardinality of a k -total limited packing set in G . This concept introduced by Hosseini Moghaddam et al. in 2016. Here, we give some results on the k -total limited packing number of graphs emphasizing trees, especially when $k = 2$. We also study the 2-(total) limited packing number of some product graphs. Ahmadi et al. introduced the concept of k -limited packing partition (k LPP) in 2024. A k LPP of graph G is a partition of $V(G)$ into k -limited packing sets. The minimum cardinality of a k LPP is called the k LPP number of G and is denoted by $\chi_{\times k}(G)$, and we obtain some results for this parameter.

Keywords: limited packing, k -limited packing partition number, graph products.

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1. Introduction and preliminaries

In this work, we consider $G = (V(G), E(G))$ as a finite simple graph. $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$ are used to refer to the *open neighborhood* and *closed neighborhood* of a vertex $v \in V(G)$, respectively. The *minimum* and *maximum degrees* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We refer to [15] as a source for terminology and notation that is not explicitly defined here.

G^- indicates the graph obtained from G by removing its isolated vertices. By $G[S]$, we mean the subgraph induced by the subset S of vertices in G .

A set of vertices $S \subseteq V(G)$ is called a *dominating set* (DS) in G if every vertex not in S is adjacent to at least one vertex in S . The *domination number* of G , denoted

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$\gamma(G)$, is the smallest number of vertices in a dominating set of G . A set $S \subseteq V(G)$ is a *total dominating set* (TDS) in the graph G if every vertex in $V(G)$ is adjacent to a vertex of S . The *total domination number* of G , denoted $\gamma_t(G)$, is the smallest number of vertices in a total dominating set of G .

A set of vertices $S \subseteq V(G)$ with $\delta(G) \geq k - 1$ is said to be a *k -tuple dominating set* (k TD set) in G provided that for every $v \in V(G)$, we have $|N[v] \cap S| \geq k$. The *k -tuple domination number* $\gamma_{\times k}(G)$ of graph G is the number of vertices in a smallest k TD set in G . A *k -tuple domatic partition* (k TD partition) of a graph G is a partition of the vertices of G into k TD sets. The largest number of sets that can be obtained from a vertex partition of G into k TD sets is called the *k -tuple domatic number* and is denoted by $d_{\times k}(G)$. Notice that when $k = 1$, S and $\gamma_{\times 1}(G)$ are the usual dominating set and domination number $\gamma(G)$, respectively. Additionally, $d_{\times 1}(G) = d(G)$ refers to the well-studied domatic number (see [4]).

A vertex subset B of a graph G is called a *packing* (resp. an *open packing*) provided that $|B \cap N[v]| \leq 1$ (resp. $|B \cap N(v)| \leq 1$) for each vertex $v \in V(G)$. The *packing number* $\rho(G)$ and *open packing number* $\rho_o(G)$ are defined as the maximum cardinality of a packing set and an open packing set, respectively. To obtain additional information on these concepts, the reader can refer to [9] and [8].

In 2010, the concept of limited packing (LP) in graphs was introduced by Gallant et al. [7]. A *k -limited packing* (k LP) in a graph G is a set $B \subseteq V(G)$ such that for each vertex v of $V(G)$, the cardinality of the intersection of B and $N[v]$ is at most k . The maximum cardinality of a k -limited packing set in G is called the *k -limited packing number* $L_k(G)$. They also presented some real-world applications of this concept in network security, market situation, NIMBY and codes. This topic was next investigated in numerous papers, such as references [2], [3], [5], [6], [11] and [14]. Similarly, a *k -total limited packing* (k TLP) in G is a set $B \subseteq V(G)$ such that for each vertex v of $V(G)$, the cardinality of the intersection of B and $N(v)$ is at most k . The maximum cardinality of a k -total limited packing set in G is called the *k -total limited packing number* $L_{k,t}(G)$. This topic was initially studied in [10], and some theoretical applications of it were given in [1, 12]. It is worth noting that the latter two concepts are identical to packing and open packing when k equals 1. Notice that a k LP set is a k TLP set, too.

A *k -limited packing partition* (k LPP) of a graph G is a partition of the vertices of G into k LP sets. This topic was first studied in [1]. The smallest number of sets that can be obtained from a vertex partition of G into k LP sets is called the *k -limited packing partition number* (k LPP number) and is denoted by $\chi_{\times k}(G)$. This concept can also be considered as the dual of k TD partition problem. Our main focus for k TLP sets is on $k = 2$. This is because for larger values of k , we lose some significant families of graphs (for instance, $\gamma_{\times k}$ and $d_{\times k}$ cannot be defined for trees when $k \geq 3$) or we encounter trivial problems (for instance, $L_{\times k}(G) = |V(G)|$ and $\chi_{\times k}(G) = 1$ if $k \geq \Delta(G) + 1$). On the other side, many results for $k \in \{1, 2\}$ may be generalized to the general case k . In addition, stronger results may be obtained for small values of k . For the *cartesian product* of graphs G and H , denoted $G \square H$, and the *direct product* of graphs G and H , denoted $G \times H$, the vertex set of the product is $V(G) \times V(H)$.

Their edge sets are defined as follows. In $G \square H$, two vertices are adjacent if they are adjacent in one coordinate and equal in the other. In $G \times H$ two vertices are adjacent if they are adjacent in both coordinates.

Suppose that G is a labeled graph on n vertices, and \mathcal{H} is a sequence of n rooted graphs H_1, H_2, \dots, H_n . If we identify the i^{th} vertex of G with the root of H_i , we obtain a new graph called the *rooted product* graph. This graph is denoted by $G(\mathcal{H})$. We here focus on the special case of rooted product graphs for which \mathcal{H} consists of n isomorphic rooted graph. Assume that v is the root vertex of H , we define the rooted product graph $G \circ_v H = (V, E)$, such that $V = V(G) \times V(H)$ and

$$E = \bigcup_{i=1}^n \{(g_i, h)(g_i, h') : hh' \in E(H)\} \cup \{(g_i, v)(g_j, v) : g_i g_j \in E(G)\}.$$

For $g \in V(G)$, $h \in V(H)$ and $*$ $\in \{\square, \times, \circ_v\}$, we call $G^h = \{(g, h) \in V(G * H) \mid g \in V(G)\}$ a G -layer through h , and ${}^g H = \{(g, h) \in V(G * H) \mid h \in V(H)\}$ an H -layer through g in $G * H$.

Notice that the subgraphs induced by the H -layers (resp. the G -layers) of $G \circ_v H$ (or $G \square H$) are isomorphic to H (resp. to G). However, there are no edges between the vertices of G^h and the vertices of ${}^g H$ in direct product $G \times H$.

The *corona product* of two graphs G with $V(G) = v_1, \dots, v_n$ and H is defined as the graph created by taking one copy of G , $|V(G)|$ copies of H and joining $v_i \in V(G)$ to every vertex in the i^{th} copy of H . The corona product of the graphs G and H is denoted by $G \odot H$.

Here, we first discuss k TLP, especially when $k = 2$, and give several sharp bounds for it. Then, we improve some of these inequalities for trees. In Section 3, we bound L_2 and $L_{2,t}$ for the cartesian product, direct product and rooted product graphs. In Section 4, we give a lower bound for $\chi_{\times k}$, and determine the values of $\chi_{\times 2}$ for the corona product. For the sake of convenience, for any graph G by an $\eta(G)$ -set with $\eta \in \{L_k, \gamma_t, \rho, \rho_o, L_{k,t}\}$ we mean a k LP set, TD set, packing set, open packing set and k TLP set in G of cardinality $\eta(G)$, respectively.

2. Results on k -total limited packing

If G is a graph of order n and $k \geq n - 1$, then $L_{k,t}(G) = n$. Note that $k \geq \Delta(G)$ is a weaker condition than the previous one. Therefore, we only need to compute the k TLP number for those graphs G such that $k < \Delta(G)$.

It was proved in [1] that the problem of computing the 2TLP number is NP-complete, even for bipartite graphs and for chordal graphs. Consequently, it would be desirable to bound this parameter in terms of several invariants of graphs. Several bounds on the k TLP number (emphasizing trees, especially when $k = 2$) were given in the following. If $B \subseteq V(G)$ and $|B| = k$, then $|B \cap N(v)| \leq k$ for each vertex v of $V(G)$. So, $k \leq L_{k,t}(G) \leq n$.

Theorem 1. *Let G be a graph of order $n \geq 2$ with degree sequence d_1, d_2, \dots, d_n such that $d_1 \leq d_2 \leq \dots \leq d_n$. Then*

$$L_{k,t}(G) \leq \max \{x \mid d_1 + d_2 + \dots + d_x \leq kn\},$$

and this bound is sharp.

Proof. Let $\mathcal{B} = \{v_1, v_2, \dots, v_{|\mathcal{B}|}\}$ be an $L_{k,t}(G)$ -set. Then

$$d_1 + d_2 + \dots + d_{|\mathcal{B}|} \leq \deg(v_1) + \deg(v_2) + \dots + \deg(v_{|\mathcal{B}|}) \leq k|\mathcal{B}| + k(n - |\mathcal{B}|).$$

So $d_1 + d_2 + \dots + d_{|\mathcal{B}|} \leq kn$. Therefore, $L_{k,t}(G) \in \{x \mid d_1 + d_2 + \dots + d_x \leq kn\}$. The sharpness of this bound can be seen as follows. Suppose that G is a complete graph of order at least $k + 2$. Then, it is easy to see that $L_{k,t}(G) = k$. On the other hand, $k = L_{k,t}(G) \leq \max \{x \mid x(n - 1) \leq kn\} = k$. □

Lemma 1. *If G is a graph of order n , then $L_{k,t}(G) \leq n + k - \Delta(G)$.*

Proof. Assume that w is a vertex of maximum degree in G . As we mentioned at the beginning of this section, we assume that $k < \Delta(G)$. Otherwise, $L_{k,t}(G) = n \leq n + k - \Delta(G)$. Let S be an $L_{k,t}(G)$ -set. Since $|N(w) \cap S| \leq k$, there is at least $\Delta(G) - k$ vertices in $N(w) \setminus S$. Hence, $|\overline{S}| \geq \Delta(G) - k$. Therefore, we have $L_{k,t}(G) = |S| = n - |\overline{S}| \leq n + k - \Delta(G)$. □

We now define the family Ω consisting of all graphs G constructed as follows, and in the next theorem we show that Ω is the set of all graphs G of order n satisfying $L_{2,t}(G) = n + 2 - \Delta(G)$. Suppose that G is a graph of order n such that $V(G) = A \cup B$ has the following conditions:

- (i) $|A \cap B| = 3$,
- (ii) $G[A]$ has a spanning star, and each component of $G[B]$ is a path or a cycle, and
- (iii) for every vertex $v \in \overline{B}$, we have $|N(v) \cap B| \leq 2$.

Figure 1 depicts a representative member of Ω .

Theorem 2. *If G is a graph of order n , then $L_{2,t}(G) \leq n + 2 - \Delta(G)$. Furthermore, $L_{2,t}(G) = n + 2 - \Delta(G)$ if and only if $G \in \Omega$.*

Proof. Suppose that S is an $L_{2,t}(G)$ -set, and w is a vertex of maximum degree in G . Notice that each component of $G[S]$ is a path or a cycle, and we have $L_{2,t}(G) = |S| = n - |\overline{S}| \leq n + 2 - \Delta(G)$ by Lemma 1. If $L_{2,t}(G) = n + 2 - \Delta(G)$, then $|\overline{S}| = \Delta(G) - 2$, $(V(G) \setminus N[w]) \subseteq S$ and $|N[w] \cap S| = 3$. Based on the above argument, we have $G \in \Omega$ with $N[w] = A$ and $S = B$.

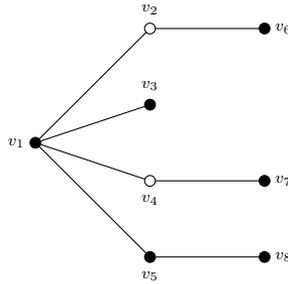


Figure 1. A graph $H \in \Omega$ with $A = \{v_1, v_2, v_3, v_4, v_5\}$ and $B = \{v_1, v_3, v_5, v_6, v_7, v_8\}$.

Let now $G \in \Omega$, then it suffices to prove that $L_{2,t}(G) \geq n + 2 - \Delta(G)$. Assume now that $A \cap B = \{u_1, u_2, u_3\}$ and $|A| = a + 1$, where w is a vertex of degree a in $G[A]$. We claim that $\Delta(G) = a$. Each vertex $v \in B$ in $G[B]$ is at most of degree two. So, each of the vertices u_1, u_2 and u_3 is adjacent to at most two vertices in B . On the other hand, each of u_1, u_2, u_3 is adjacent to at most $a - 2$ vertices in $A \setminus \{u_1, u_2, u_3\}$. Thus, $\deg(u_1) \leq a$, $\deg(u_2) \leq a$ and $\deg(u_3) \leq a$. For each vertex $v \in A \setminus \{u_1, u_2, u_3\}$, v is adjacent to at most $a - 3$ vertices in $A \setminus \{u_1, u_2, u_3, v\}$ and to at most two vertices in B . So, $\deg(v) \leq a - 1$ for every $v \in A \setminus \{u_1, u_2, u_3\}$. For each vertex $v' \in B \setminus \{u_1, u_2, u_3\}$, v' is adjacent to at most $a - 2$ vertices in $A \setminus \{u_1, u_2, u_3\}$ and to at most two vertices in B . Thus, $\deg(v') \leq a$ for every $v' \in B \setminus \{u_1, u_2, u_3\}$. Hence, $\Delta(G) \leq a$. But $\deg(w) \geq a$, which implies that $\Delta(G) = a$. Note that B is a 2TLP set of G with $|B| = n - |A| + 3 = n + 2 - \Delta(G)$. Therefore, we have $L_{2,t}(G) \geq n + 2 - \Delta(G)$. \square

Theorem 3. Let G be an r -regular graph of order n such that $L_{k,t}(G) = n + k - r$ for $k \leq r - 1$. Then, we have $r \geq \frac{n}{2}$.

Proof. If $r = n - 1$, then G is a complete graph with $L_{k,t}(G) = k$ for $1 \leq k \leq n - 2$. So, let $r \leq n - 2$. Now assume that $w \in V(G)$ and that S is an $L_{k,t}(G)$ -set with $|S| = n + k - r$. Since $|N(w) \cap S| \leq k$, it follows that $|N(w) \cap \bar{S}| \geq r - k$. Obviously, $|\bar{S}| = n - |S| = r - k$. Thus, there are exactly $r - k$ vertices, namely v_1, v_2, \dots, v_{r-k} , in $N(w) \cap \bar{S}$. In particular, $\bar{S} = \{v_1, v_2, \dots, v_{r-k}\}$ and $w \in S$. Let $U = V(G) \setminus N[w]$, clearly $U \subseteq S$, and $U \neq \emptyset$ since $r \leq n - 2$. If $u \in U$, then $|N(u) \cap S| \leq k$. So, any vertex $u \in U$ is adjacent to all vertices in \bar{S} , i.e., every vertex $v_i \in \bar{S}$ is adjacent to all $n - r - 1$ vertices in U . Note that v_i has at least one neighbor in $N[w]$, and $\deg(v_i) = r$. Therefore, $n - r - 1 + 1 \leq r$ and we get $r \geq \frac{n}{2}$. \square

Remark 1. If G is an r -regular graph of order n with $r < \frac{n}{2}$, then $L_{k,t}(G) < n + k - r$ for $k \leq r - 1$. In fact, the bound of Lemma 1 is not sharp under these conditions.

Before we state the Theorem 4, we introduce a new concept needed in that theorem. For any tree T with order $n \geq 3$, $\delta'(T)$ denotes the minimum degree of T among all

non-leaf vertices.

Theorem 4. *Let T be a tree of order $n \geq 3$ for which $\delta'(T) \geq c$ for some positive integer $c \geq 4$. Then, we have $L_{2,t}(T) \leq \frac{c-2}{c-1}n - c + 4$.*

Proof. We prove this theorem by induction on the order of tree T . Since $\delta'(T) \geq c$, we have $n \geq c+1$. If $n \in \{c+1, c+2, \dots, 2c-1\}$, then $T \in \{K_{1,c}, K_{1,c+1}, \dots, K_{1,2c-2}\}$, respectively. Hence, $L_{2,t}(T) = 3 \leq \frac{c-2}{c-1}n - c + 4$. Assume that for all tree T' of order $n' < n$ with $\delta'(T') \geq c$, we have $L_{2,t}(T') \leq \frac{c-2}{c-1}n' - c + 4$. Now let T be a tree of order $n \geq 2c$ such that $\delta'(T) \geq c$ and let S be an $L_{2,t}(T)$ -set. We root T at r , and suppose v' is a leaf of T at the furthest distance from r , and v'' is the parent of v' . Assume that L is the set of all leaves in $N(v'')$. Since v'' is adjacent to at least $c - 1$ leaves, it follows that $|L| \geq c - 1$. Suppose that T'' be obtained from T by deleting all the vertices of L . By the induction hypothesis, we have $L_{2,t}(T'') \leq \frac{c-2}{c-1}|V(T'')| - c + 4 \leq \frac{c-2}{c-1}(n - (c - 1)) - c + 4 = \frac{c-2}{c-1}n - 2c + 6$. On the other hand, $|L \cap S| \leq |N(v'') \cap S| \leq 2$. Therefore, we get $L_{2,t}(T) \leq L_{2,t}(T'') + 2 \leq \frac{c-2}{c-1}n - 2c + 8 \leq \frac{c-2}{c-1}n - c + 4$. \square

Proposition 1. *Let G be a graph without isolated vertices such that $\Delta(G) \geq 2$, then*

$$L_{2,t}(G) \leq \frac{\Delta(G)^2 + 1}{\delta(G)} \rho_o(G).$$

Proof. Let $v \in V(G)$ be an arbitrary vertex, then the set of all vertices at distance at most two from v has at most $\Delta(G)^2 + 1$ vertices. Thus, $\rho_o(G) \geq \frac{2n}{\Delta(G)^2 + 1}$, by the greedy algorithm. Moreover, $L_{k,t}(G) \leq \frac{kn}{\delta(G)}$ [10], and we get

$$\rho_o(G) \geq \frac{2n}{\Delta(G)^2 + 1} = \frac{2n\delta(G)}{(\Delta(G)^2 + 1)\delta(G)} \geq L_{2,t}(G) \frac{\delta(G)}{\Delta(G)^2 + 1}.$$

Therefore, we infer that $L_{2,t}(G) \leq \frac{\Delta(G)^2 + 1}{\delta(G)} \rho_o(G)$. \square

We can improve the above bounds for trees as follows.

Proposition 2. *If T is a given tree with $\Delta(T) \geq 2$, then*

$$L_{2,t}(T) \leq 2\rho_o(T).$$

Proof. We know that $L_{2,t}(T) \leq 2\gamma_t(T)$ ([10]). On the other hand, we have $\rho_o(T) = \gamma_t(T)$ for every tree T with at least two vertices ([13]). As a consequence, we have $L_{2,t}(T) \leq 2\gamma_t(T) = 2\rho_o(T)$. \square

We next state a relevant result on 2- total limited packings of graphs, which will be needed later.

Lemma 2. (Hosseini Moghaddam et al. [10]) Let G be a graph of order n such that $\Delta(G) \geq 2$, then $\rho_o(G) + 1 \leq L_{2,t}(G)$.

According to the previous lemma and the Proposition 2, we can say $\rho_o(T) + 1 \leq L_{2,t}(T) \leq 2\rho_o(T)$ for every tree T with $\Delta(T) \geq 2$. The next theorem shows how equality occurs in these bounds.

Theorem 5. Let T be a tree such that $\Delta(T) \geq 2$. Then,

- (i) $\rho_o(T) + 1 = L_{2,t}(T)$ if and only if T is a star with at least three vertices, and
- (ii) $L_{2,t}(T) = 2\rho_o(T)$ if and only if for every $L_{2,t}(T)$ -set S and every $\gamma_t(T)$ -set D , we have $|N(s) \cap D| = 1$ and $|N(d) \cap S| = 2$ for any $s \in S$ and any $d \in D$.

Proof. Let T be the star $K_{1,x}$ with $x \geq 2$. Then, $\rho_o(T) = 2$ and $L_{2,t}(T) = 3$. Therefore, $\rho_o(T) + 1 = L_{2,t}(T)$.

It remains for us to prove the converse. Assume now that T is a tree with $\rho_o(T) + 1 = L_{2,t}(T)$. We claim that $diam(T) \leq 2$. Suppose to the contrary that there exist two vertices $v_1, v_4 \in V(T)$ such that $d(v_1, v_4) = 3$ and let $P = v_1v_2v_3v_4$ be the path between them. Assume that S_1 is a $\rho_o(T)$ -set, then $|V(P) \cap S_1| \leq 2$. We now consider three cases as follows.

Case 1. Let $V(P) \cap S_1 = \emptyset$. Set $S_2 = S_1 \cup \{v_1, v_2\}$, we show that S_2 is a 2TLP set of T . Since $|N(v_i) \cap S_1| \leq 1$ and $|N(v_i) \cap \{v_1, v_2\}| \leq 1$ for $1 \leq i \leq 4$, it follows that $|N(v_i) \cap S_2| \leq 2$ for every $v_i \in V(P)$. Let now w be a vertex outside of P , so $|N(w) \cap \{v_1, v_2\}| \leq 1$ because T has no cycle. Thus, $|N(w) \cap S_2| \leq 2$ for every vertex w outside P . Therefore, we conclude that S_2 is a 2TLP set of T .

Case 2. Assume $V(P) \cap S_1 = \{v_i\}$ for $1 \leq i \leq 4$. First, let $i = 1$ or 4 , by using similar techniques as in the previous case, $S_1 \cup \{v_2, v_3\}$ is a 2TLP set of T . If $i = 2$ or 3 , then $S_1 \cup \{v_1, v_4\}$ is a 2TLP set of T .

Case 3. Suppose $V(P) \cap S_1 = \{v_i, v_j\}$ for some $1 \leq i \neq j \leq 4$. If $(i, j) \in \{(1, 2), (1, 4), (2, 3), (3, 4)\}$, then $S_1 \cup \{v_3, v_4\}$, $S_1 \cup \{v_2, v_3\}$, $S_1 \cup \{v_1, v_4\}$ and $S_1 \cup \{v_1, v_2\}$ are 2TLP sets of T , respectively.

In each case, we observe that $L_{2,t}(T) \geq \rho_o(T) + 2$, which contradicts the assumption $\rho_o(T) + 1 = L_{2,t}(T)$. Therefore, we deduce that $diam(T) \leq 2$, and T is a star with at least three vertices.

Let now T be a tree with $\Delta(T) \geq 2$. As mentioned earlier, we know that $L_{2,t}(T) = 2\rho_o(T)$ if and only if $L_{2,t}(T) = 2\gamma_t(T)$. Let S be an $L_{2,t}(T)$ -set, and D be a $\gamma_t(T)$ -set. We now restate the proof of Theorem 7 in [10]. We set $U = \{(s, d) \in V(G) \times V(G) | s \in S, d \in D \text{ and } s \in N(d)\}$, and count the members of U in two ways. Since $|N(s) \cap D| \geq 1$ for any $s \in S$, it follows that there is at least one vertex $d \in D$ such that $s \in N(d)$. Thus, $|S| \leq |U|$. On the other hand, for any $d \in D$ we have $|N(d) \cap S| \leq 2$. Hence, there exists at most two vertices $s_1, s_2 \in S$ such that $s_1, s_2 \in N(d)$. So, we get $|U| \leq 2|D|$, and $|S| \leq 2|D|$. Therefore, $L_{2,t}(T) = 2\gamma_t(T)$ if and only if the following statements hold:

- (1) for any $s \in S$, we have $|N(s) \cap D| = 1$,
- (ii) for any $d \in D$, we have $|N(d) \cap S| = 2$.

□

Theorem 6. *If G is a graph, then for any edge $e \in E(G)$,*

$$L_{k,t}(G) \leq L_{k,t}(G - e) \leq L_{k,t}(G) + 2.$$

Furthermore, these bounds are sharp.

Proof. Any k TLP set of G is also a k TLP set of $G - e$, so $L_{k,t}(G) \leq L_{k,t}(G - e)$. Moreover, if C is a cycle on n vertices, then $L_{2,t}(C) = L_{2,t}(C - e) = n$ for every edge $e \in E(C)$.

Suppose now that B is an $L_{k,t}(G - e)$ -set and $e = uv$. If $u, v \in B$, then $B - \{u, v\}$ is a k TLP set of G , and hence $L_{k,t}(G) \geq |B| - 2$. If $u \in B$ and $v \notin B$, then $B - \{u\}$ is a k TLP set of G , and $L_{k,t}(G) \geq |B| - 1$. If $u, v \notin B$, then B is a k TLP set of G , and we have $L_{k,t}(G) \geq |B|$. Therefore, $L_{k,t}(G - e) \leq L_{k,t}(G) + 2$.

Let G be a double star $ST(x, y)$, which is the graph obtained by joining the centers of two stars $K_{1,x}$ and $K_{1,y}$ with an edge, such that $x, y \geq k + 1$. Assume that the center of stars are u and v , respectively. Then, $L_{k,t}(G - e) = L_{k,t}(G) + 2$ for $e = uv$. □

Theorem 7. *Let G have a unique $L_{2,t}(G)$ -set B . Then every leaf of G belongs to B .*

Proof. Let B be a unique $L_{2,t}(G)$ -set, and let there exist a leaf $l \notin B$ with the support vertex v . If $v \in B$ and $|N(v) \cap B| \leq 1$, then $B' = B \cup \{l\}$ is a 2TLP set which is greater than B , a contradiction. So if $v \in B$, then $|N(v) \cap B| = 2$. Let $u \in N(v) \cap B$. We can easily see that $B'' = (B \setminus \{u\}) \cup \{l\}$ is an $L_{2,t}(G)$ -set, which is impossible because B is unique. Hence $v \notin B$.

If some neighbor of v , say u' , belongs to B , then $B'' = (B \setminus \{u'\}) \cup \{l\}$ is an $L_{2,t}(G)$ -set. This contradicts the assumption. Therefore, we deduce that $N[v] \cap B = \emptyset$. So $B \cup \{l\}$ is a 2TLP set, which is a contradiction with the maximality of B . Hence $l \in B$. □

If $diam(G) = 1$, then G is a complete graph, and we know that $L_{2,t}(K_n) = 2$. What can be said about the 2TLP number of graphs with diameter 2? The following theorem is an answer to this question.

Theorem 8. *If $c \geq 3$ is a positive integer, then there exists a graph G with $diam(G) = 2$ such that $L_{2,t}(G) = c$.*

Proof. In what follows, we construct a graph G with diameter 2 for which $L_{2,t}(G) = c$. Suppose that $V_1 = \{v_1, v_2, \dots, v_c\}$ and $V_2 = \{u_1, u_2, \dots, u_{\frac{c(c-1)}{2}}\}$ with $V_1 \cap V_2 = \emptyset$. Let G be a graph with vertex set $V(G) = V_1 \cup V_2$ such that $G[V_1] = cK_1$, $G[V_2] =$

$K_{\frac{c(c-1)}{2}}$ and each pair of distinct vertices in V_1 has a unique common neighbor in V_2 . Obviously, $diam(G) = 2$. It remains to show that $L_{2,t}(G) = c$. We know $|V(G)| = c + \frac{c(c-1)}{2}$ and $\Delta(G) = \frac{c(c-1)}{2} + 1$. Hence, by Theorem 2, $L_{2,t}(G) \leq |V(G)| + 2 - \Delta(G) = c + 1$. But $G \notin \Omega$, so $L_{2,t}(G) \leq c$. On the other hand, V_1 is a 2TLP set of G . Thus, $L_{2,t}(G) = c$. \square

Theorem 9. *Assume that $a \geq 3$ and b are two integers with $a + 1 \leq b \leq 2a$. Then, there exists a tree T for which $\rho_o(T) = a$ and $L_{2,t}(T) = b$.*

Proof. Let $a \geq 3$ and b be two integers such that $a + 1 \leq b \leq 2a$, and $b = a + x$ with $1 \leq x \leq a$. In what follows, we construct a tree T with $\rho_o(T) = a$ and $L_{2,t}(T) = a + x$ for $a \geq 3$ and $1 \leq x \leq a$. We distinguish two cases based on the value of x .

Case 1. First, let $x = a$. Assume $P = v_1v_2 \dots v_a$ is a path. We add two leaves u_{i_1} and u_{i_2} to each v_i , and obtain tree T . Let S_1 be a $\rho_o(T)$ -set. If $|S_1| \geq a + 1$, by the Pigeonhole principle, there is at least one vertex v_i such that $|N(v_i) \cap S_1| \geq 2$, which is impossible. Hence, $\rho_o(T) \leq a$. On the other hand, $\{u_{1_1}, u_{2_1}, \dots, u_{a_1}\}$ is a 1TLP set of T , so $\rho_o(T) = a$.

Let S_2 be an $L_{2,t}(T)$ -set. Similarly, if $L_{2,t}(T) \geq 2a + 1$, there exists at least one vertex v_i such that $|N(v_i) \cap S_2| \geq 3$, a contradiction. Thus, $L_{2,t}(T) \leq 2a$. Moreover, $\{u_{1_1}, u_{1_2}, u_{2_1}, u_{2_2} \dots, u_{a_1}, u_{a_2}\}$ is a 2TLP set of T , hence $L_{2,t}(T) = 2a = b$.

Case 2. Suppose now that $1 \leq x \leq a - 1$. Consider the star $T' = K_{1,a}$ with $V(T') = \{r, v_1, v_2, \dots, v_a\}$ and $\deg(r) = a$. Let T be the tree obtained from T' by adding two leaves u_i and u'_i to each v_i for $1 \leq i \leq x - 1$ and one leaf u_i to each v_i for $x \leq i \leq a - 1$. We show that $\rho_o(T) = a$ and $L_{2,t}(T) = b$. Since $T \notin \Omega$, it follows that $L_{2,t}(T) < |V(T)| + 2 - \Delta(T)$ by Theorem 2. Notice that $|V(T)| = 2a + x - 1$ and $\Delta(T) = a$, thus $L_{2,t}(T) \leq a + x$. On the other hand, $\{u_1, u_2, \dots, u_{a-1}, u'_1, u'_2, \dots, u'_{x-1}, v_1, v_a\}$ is a 2TLP set of T , so $L_{2,t}(T) = a + x = b$. Since T is a tree with at least two vertices, $\rho_o(T) = \gamma_t(T)$ [10]. Moreover, $\{r, v_1, v_2, \dots, v_{a-1}\}$ is a TD set of T , and hence $\gamma_t(T) \leq a$. Thus, $\rho_o(T) \leq a$. It is readily verified that $\{u_1, u_2, \dots, u_{a-1}, v_a\}$ is a 1TLP set of T . Therefore, $\rho_o(T) = a$. \square

3. On 2-(total) limited packing number of some graph products

Theorem 10. *For any graphs G and H , $L_{2,t}(G \square H) \geq \max\{L_{2,t}(G)\rho(H), \rho(G)L_{2,t}(H)\}$. Moreover, this bound is sharp.*

Proof. Let P_G and P_H be an $L_{2,t}(G)$ -set and a $\rho(H)$ -set, respectively. Set $P = P_G \times P_H$, and suppose to the contrary that P is not a 2TLP set of $G \square H$. Therefore, there exists a vertex $(x, y) \in V(G) \times V(H)$ adjacent to three distinct vertices $(g_1, h_1), (g_2, h_2), (g_3, h_3) \in P$. We distinguish the following cases.

Case 1. $h_1 = h_2 = h_3$. If y is adjacent to h_1 , then $x = g_1 = g_2 = g_3$, which is impossible. So, $y = h_1 = h_2 = h_3$. In such a situation, x is adjacent to $g_1, g_2, g_3 \in P_G$, which contradicts the fact that P_G is a 2TLP set in G .

Case 2. At least two vertices from $\{h_1, h_2, h_3\}$, say h_1 and h_2 , are distinct. By the adjacency rule of the Cartesian product graphs, we deduce that $\{h_1, h_2\} \subseteq N_H[y] \cap P_H$. This contradicts the fact that P_H is a packing in H .

Therefore, P is a 2TLP set in $G \square H$. Hence, $L_{2,t}(G \square H) \geq |P| = L_{2,t}(G)\rho(H)$. Similarly, we have $L_{2,t}(G \square H) \geq \rho(G)L_{2,t}(H)$.

We can show that this bound is sharp in the following way. Let G' be any connected graph on the set of vertices $\{v'_1, \dots, v'_n\}$. Let $G = G' \odot K_1$, in which $N_G(v'_i) \setminus N_{G'}(v'_i) = \{v_i\}$ for each $1 \leq i \leq n$. We now consider the graph $G \square K_r$ for $r \geq 3$, and let Q be an $L_{2,t}(G \square K_r)$ -set. Clearly, $\rho(G) = n$ and $L_{2,t}(K_r) = 2$. It is not difficult to see that $|Q \cap (\{v_i, v'_i\} \times V(K_r))| \leq 2$ for each $1 \leq i \leq n$. This implies that

$$\begin{aligned} L_{2,t}(G) &= |Q| = |Q \cap V(G \square K_r)| = |Q \cap (\cup_{i=1}^n (\{v_i, v'_i\} \times V(K_r)))| \\ &= \sum_{i=1}^n |Q \cap (\{v_i, v'_i\} \times V(K_r))| \leq 2n = L_{2,t}(K_r)\rho(G). \end{aligned} \tag{3.1}$$

On the other hand, $L_{2,t}(G) \leq 2n$ since G has $2n$ vertices. We also know $\rho(K_r) = 1$, so $\max\{L_{2,t}(G)\rho(K_r), \rho(G)L_{2,t}(K_r)\} = L_{2,t}(K_r)\rho(G)$.

According to this theorem, we have $L_{2,t}(G \square K_r) \geq \rho(G)L_{2,t}(K_r)$. Therefore, this bound is sharp.

To show the sharpness of this bound, we present a simpler example by considering $G = K_{m,n}$ and $H = K_2$ for $m, n \geq 2$. We know that $L_{2,t}(K_{m,n}) = 4$, $\rho(K_2) = 1$ and $L_{2,t}(K_2) = \rho(K_{m,n}) = 2$. It is easy to see that $L_{2,t}(K_{m,n} \square K_2) = 4$. □

Theorem 11. *Let G and H be graphs with i_G and i_H isolated vertices, respectively. Then,*

$$L_{2,t}(G \times H) \geq \max \{ \rho_o(G^-)L_{2,t}(H^-), L_{2,t}(G^-)\rho_o(H^-) \} + i_G|V(H)| + i_H|V(G)| - i_Gi_H$$

and this bound is sharp.

Proof. Suppose first that G and H are graphs without isolated vertices. Let P_G and P_H be a $\rho_o(G)$ -set and an $L_{2,t}(H)$ -set, respectively. Set $P = P_G \times P_H$, and assume for the sake of contradiction that P is not a 2TLP set of $G \times H$. Hence, there exists a vertex $(x, y) \in V(G \times H)$ adjacent to three distinct vertices $(g, h), (g', h'), (g'', h'') \in P$. Then $g = g' = g''$ because P_G is a $\rho_o(G)$ -set. So $h \neq h' \neq h''$ and $|N(y) \cap P_H| \geq 3$, a contradiction. Therefore, P is a 2TLP set in $G \times H$, and $L_{2,t}(G \times H) \geq |P| = \rho_o(G)L_{2,t}(H)$. We have $L_{2,t}(G \times H) \geq L_{2,t}(G)\rho_o(H)$ by a similar fashion. We have

$$L_{2,t}(G \times H) = L_{2,t}(G^- \times H^-) + i_G|V(H)| + i_H|V(G)| - i_Gi_H.$$

Therefore,

$$L_{2,t}(G \times H) \geq \max \{ \rho_o(G^-)L_{2,t}(H^-), L_{2,t}(G^-)\rho_o(H^-) \} + i_G|V(H)| + i_H|V(G)| - i_G i_H.$$

In what follows, we show that this bound is sharp. Let G be a bipartite graph without isolated vertices. Then, $L_{2,t}(G \times K_2) = L_{2,t}(2G) = 2L_{2,t}(G)$. On the other hand, $L_{2,t}(G \times K_2) \geq \max \{ \rho_o(G)L_{2,t}(K_2), L_{2,t}(G)\rho_o(K_2) \} = \max \{ 2\rho_o(G), 2L_{2,t}(G) \} = 2L_{2,t}(G)$. \square

Theorem 12. *Let G be a graph of order n with i_G isolated vertices. If H is a rooted graph at v , then*

$$n(L_{2,t}(H) - 1) + i_G \leq L_{2,t}(G \circ_v H) \leq nL_{2,t}(H).$$

Furthermore, these bounds are sharp.

Proof. Note that any H -layer in $G \circ_v H$ is isomorphic to H . So, each $L_{2,t}(G \circ_v H)$ -set intersects every H -layer is at most $L_{2,t}(H)$ vertices. Hence, $L_{2,t}(G \circ_v H) \leq nL_{2,t}(H)$. On the other side, let P_H be an $L_{2,t}(H)$ -set. We can readily observe that $P = \bigcup_{g \in V(G)} (\{g\} \times (P_H \setminus \{v\}))$ is a 2TLP set in $G \circ_v H$, so we have $n(L_{2,t}(H) - 1) \leq |P| \leq L_{2,t}(G \circ_v H)$.

Suppose that there exists an $L_{2,t}(H)$ -set P_H not consisting of v . Notice that $P = \bigcup_{g \in V(G)} (\{g\} \times P_H)$ is a 2TLP set in $G \circ_v H$, and we have $nL_{2,t}(H) = |P| \leq L_{2,t}(G \circ_v H)$. Thus, we conclude that $L_{2,t}(G \circ_v H) = nL_{2,t}(H)$ in this case.

Assume now that v has degree two in all subgraphs induced by every $L_{2,t}(H)$ -set P_H , that is $\deg_{H[P_H]}(v) = 2$. Suppose that for every 2TLP set P'_H in H , $|P'_H \setminus N[v]| \leq L_{2,t}(H) - 3$. Assume that P is an $L_{2,t}(G \circ_v H)$ -set. Note that exactly i_G components of $G \circ_v H$ are isomorphic to H , which implies that each of these components has exactly $L_{2,t}(H)$ vertices in P . Moreover, we have one component isomorphic to $G^- \circ_v H$. Let $P^- = P \cap (G^- \circ_v H)$ and $P_g^- = P^- \cap gH$ for every $g \in V(G^-)$. We now show that $L_{2,t}(G^- \circ_v H) = (n - i_G)(L_{2,t}(H) - 1)$. Assume that $L_{2,t}(G^- \circ_v H) > (n - i_G)(L_{2,t}(H) - 1)$. Hence, there exists at least one vertex $g_1 \in V(G^-)$ for which $|P_{g_1}^-| = L_{2,t}(H)$. Otherwise, $|P^-| = \sum_{g_i \in V(G^-)} |P_{g_i}^-| \leq \sum_{g_i \in V(G^-)} (L_{2,t}(H) - 1) = (n - i_G)(L_{2,t}(H) - 1)$, which is a contradiction.

Since G^- has no isolated vertex, there exists a vertex $g_2 \in V(G^-)$ for which $g_1 g_2 \in E(G)$. If $|P_{g_2}^-| = L_{2,t}(H)$, then both (g_1, v) and (g_2, v) have three neighbors in P^- , which is impossible. Therefore $|P_{g_2}^-| \leq L_{2,t}(H) - 1$ for all $g_2 \in N_G(g_1)$. Now let g_2 be an arbitrary neighbor of g_1 in G^- . If $|P_{g_2}^-| = L_{2,t}(H) - 1$, then $|N_{(G^- \circ_v H)[g_2 H]}[(g_2, v)] \cap P_{g_2}^-| = 2$. This implies that $|N_{(G^- \circ_v H)}(g_1, v) \cap P^-| \geq 3$ or $|N_{(G^- \circ_v H)}(g_2, v) \cap P^-| \geq 3$, a contradiction. Thus, $|P_{g_2}^-| \leq L_{2,t}(H) - 2$ for all $g_2 \in N_G(g_1)$.

The above argument provides a guarantee that for every vertex $g_1 \in V(G)$ such that $|P_{g_1}^-| = L_{2,t}(H)$, we have $|P_{g_2}^-| \leq L_{2,t}(H) - 2$ for all $g_2 \in N_G(g_1)$. This implies that

$$L_{2,t}(G^- \circ_v H) \leq \frac{n - i_G}{2} L_{2,t}(H) + \frac{n - i_G}{2} (L_{2,t}(H) - 2) = (n - i_G)(L_{2,t}(H) - 1),$$

which contradicts the assumption $L_{2,t}(G^- \circ_v H) > (n - i_G)(L_{2,t}(H) - 1)$. So $L_{2,t}(G^- \circ_v H) \leq (n - i_G)(L_{2,t}(H) - 1)$. It follows that $L_{2,t}(G^- \circ_v H) = (n - i_G)(L_{2,t}(H) - 1)$ by using the corresponding inequality obtained from the first steps of the proof. Therefore, $L_{2,t}(G \circ_v H) = L_{2,t}(G^- \circ_v H) + i_G L_{2,t}(H) = (n - i_G)(L_{2,t}(H) - 1) + i_G L_{2,t}(H) = n(L_{2,t}(H) - 1) + i_G$. \square

Theorem 13. For any graphs G and H ,

$$L_2(G \square H) \leq \min\{L_2(G)|V(H)|, L_2(H)|V(G)|\},$$

and this bound is sharp.

Proof. Let $V(H) = \{v_1, v_2, \dots, v_{|V(H)|}\}$. It is obvious that $G \square H$ contains $|V(H)|$ disjoint G -layers. Suppose now that P is an $L_2(G \square H)$ -set, thus $P_i = P \cap G^{v_i}$ is a 2LP set in $(G \square H)[G^{v_i}]$ for each $1 \leq i \leq |V(H)|$. Therefore, $|P_i| \leq L_2(G)$, which leads to

$$L_2(G \square H) = |P| = \sum_{i=1}^{|V(H)|} |P_i| \leq L_2(G)|V(H)|.$$

Similarly, we have $L_2(G \square H) \leq L_2(H)|V(G)|$.

For sharpness consider $G = P_2$ and $H = K_{m,n}$ for $m, n \geq 2$. We observe that $L_2(K_{m,n}) = 2$ [7], and $L_2(P_2) = 2$. It is easy to see that $L_2(G \square H) = 4$. \square

Theorem 14. Let G and H be graphs with i_G and i_H isolated vertices, respectively. Then,

$$L_2(G \times H) \geq \max \{ \rho_o(G^-)L_2(H^-), L_2(G^-)\rho_o(H^-), \rho(G^-)L_{2,t}(H^-), L_{2,t}(G^-)\rho(H^-) \} + i_G|V(H)| + i_H|V(G)| - i_G i_H.$$

Moreover, this bound is sharp.

Proof. Assume first that G and H are graphs without isolated vertices. Let P_G, P'_G, P_H and P'_H be an $L_{2,t}(G)$ -set, a $\rho_o(G)$ -set, a $\rho(H)$ -set and an $L_2(H)$ -set, respectively. Set $P = P_G \times P_H$ and $P' = P'_G \times P'_H$, and suppose to the contrary that P and P' are not 2LP sets of $G \times H$. So there exist vertices $(x, y), (x', y') \in V(G \times H)$ such that $|N[(x, y)] \cap P| \geq 3$ and $|N[(x', y')] \cap P'| \geq 3$, respectively.

If $(x, y) \in P$, then (x, y) is adjacent to two distinct vertices $(g, h), (g', h') \in P$. So $|N[y] \cap P_H| \geq 2$, which is impossible. If $(x, y) \in V(G \times H) \setminus P$, then (x, y) is adjacent to three distinct vertices $(g, h), (g', h'), (g'', h'') \in P$. We observe that $h = h' = h''$ because P_H is a $\rho(H)$ -set. Hence $g \neq g' \neq g''$ and $|N(x) \cap P_G| \geq 3$, a contradiction. Therefore, P is a 2LP set in $G \times H$ and $L_2(G \times H) \geq |P| \geq L_{2,t}(G)\rho(H)$. We have $L_2(G \times H) \geq \rho(G)L_{2,t}(H)$ by a similar method.

If $(x', y') \in P'$, then (x', y') is adjacent to two distinct vertices $(g_1, h_1), (g_2, h_2) \in P'$. We have $g_1 = g_2$ because P'_G is a $\rho_o(G)$ -set. Thus, $h_1 \neq h_2$ and $|N[y'] \cap P'_H| \geq 3$, which is impossible. If $(x', y') \in V(G \times H) \setminus P'$, then (x', y') is adjacent to three

distinct vertices $(g_1, h_1), (g_2, h_2), (g_3, h_3) \in P'$. $g_1 = g_2 = g_3$ since P'_G is a $\rho_o(G)$ -set, so $h_1 \neq h_2 \neq h_3$ and $|N[y'] \cap P'_H| \geq 3$, a contradiction. Therefore P' is a 2LP set in $G \times H$, and $L_2(G \times H) \geq \rho_o(G)L_2(H)$. We get $L_2(G \times H) \geq L_2(G)\rho_o(H)$ by a similar fashion. Therefore,

$$L_2(G \times H) \geq \max \{ \rho_o(G)L_2(H), L_2(G)\rho_o(H), \rho(G)L_{2,t}(H), L_{2,t}(G)\rho(H) \}.$$

We now suppose that G and H are arbitrary graphs. Then,

$$\begin{aligned} L_2(G \times H) &= L_2(G^- \times H^-) + i_G|V(H)| + i_H|V(G)| - i_Gi_H \geq \\ &\max \{ \rho_o(G^-)L_2(H^-), L_2(G^-)\rho_o(H^-), \rho(G^-)L_{2,t}(H^-), L_{2,t}(G^-)\rho(H^-) \} + \\ &\quad i_G|V(H)| + i_H|V(G)| - i_Gi_H. \end{aligned}$$

In what follows, we show the sharpness of this bound. Let G be a bipartite graph without isolated vertices. Then, $L_2(G \times K_2) = L_2(2G) = 2L_2(G)$. On the other hand,

$$\begin{aligned} L_2(G \times K_2) &\geq \max \{ \rho_o(G)L_2(K_2), L_2(G)\rho_o(K_2), \rho(G)L_{2,t}(K_2), L_{2,t}(G)\rho(K_2) \} = \\ &\max \{ 2\rho_o(G), 2L_2(G), 2\rho(G), L_{2,t}(G) \} = 2L_2(G). \end{aligned}$$

□

We end this section by studying the 2LP number of rooted product graphs.

Theorem 15. *Let G be a graph of order n . If H is a graph with root v , then*

$$L_2(G \circ_v H) = \begin{cases} L_2(G) + n(L_2(H) - 1) & \text{if } v \in P_H \text{ for every } L_2(H)\text{-set } P_H, \\ nL_2(H) & \text{if } v \notin P_H \text{ for some } L_2(H)\text{-set } P_H. \end{cases}$$

Proof. We consider two cases based on the membership of v to $L_2(H)$ -sets.

Case 1. Assume that v belongs to any $L_2(H)$ -set P_H , and P' be an $L_2(G)$ -set. Set $P = (P' \times \{v\}) \cup (V(G) \times (P_H \setminus \{v\}))$. It can be readily seen that P is a 2LP set in $G \circ_v H$. Therefore, $L_2(G \circ_v H) \geq |P' \times \{v\}| + |V(G) \times (P_H \setminus \{v\})| = L_2(G) + n(L_2(H) - 1)$. On the other hand, let B be an $L_2(G \circ_v H)$ -set. Then $B_g = B \cap {}^gH$ is a 2LP set in $(G \circ_v H)[{}^gH]$ for every $g \in V(G)$. Note that B_g is not an $L_2((G \circ_v H)[{}^gH])$ -set for some $g \in V(G)$ since v belongs to every $L_2(H)$ -set. Hence, $|B \cap {}^gH| = |B_g| \leq L_2(H) - 1$ if $(g, v) \notin B$, which means $|B \cap ({}^gH \setminus \{(g, v)\})| \leq L_2(H) - 1$. Also if $(g, v) \in B$, then $|B \cap ({}^gH \setminus \{(g, v)\})| \leq L_2(H) - 1$ as well. In addition, $B \cap G^v$ is a 2LP set in $(G \circ_v H)[G^v]$. Thus, $|B \cap G^v| \leq L_2(G)$, and we have

$$L_2(G \circ_v H) = |B| = |B \cap G^v| + \sum_{g \in V(G)} |B \cap ({}^gH \setminus \{(g, v)\})| \leq L_2(G) + n(L_2(H) - 1).$$

Therefore, $L_2(G \circ_v H) = L_2(G) + n(L_2(H) - 1)$.

Case 2. Assume that there exists an $L_2(H)$ -set P_H for which $v \notin P_H$. Let ${}^gP_H = \{g\} \times P_H$ for every $g \in V(G)$, and let $P'' = \cup_{g \in V(G)} {}^gP_H$. We can easily see that P'' is a 2LP set in $G \circ_v H$, so $L_2(G \circ_v H) \geq |P''| = nL_2(H)$. On the other hand, let P be an $L_2(G \circ_v H)$ -set. We can easily observe that the set $P_g = P \cap {}^gH$ is a 2LP set in $(G \circ_v H)[{}^gH]$ for every $g \in V(G)$. So $L_2(H) = L_2(G \circ_v H)[{}^gH] \geq |P_g|$. Therefore, $L_2(G \circ_v H) = |P| = \sum_{g \in V(G)} |P_g| \leq \sum_{g \in V(G)} L_2(H) = nL_2(H)$. This leads to $L_2(G \circ_v H) = nL_2(H)$. \square

4. Results on vertex partitioning into k -limited packings

In the previous sections, we studied about k -limited packings in graphs. In the following, we state two theorems for the vertex partitioning into k -limited packing sets. As we said before, $\chi_{\times k}(G)$ is the minimum cardinality of k LPP in G . In the next theorem, we discuss the relationship between $L_k(G)$ and $\chi_{\times k}(G)$.

Theorem 16. *If G is a graph of order $n \geq 2$, then $\chi_{\times k}(G) \geq 2\sqrt{n} - L_k(G)$.*

Proof. We first prove that $\chi_{\times k}(G) \times L_k(G) \geq n$. Let $\{B_1, B_2, \dots, B_{\chi_{\times k}(G)}\}$ be a k LPP of G . Then,

$$\chi_{\times k}(G) \times L_k(G) = \sum_{i=1}^{\chi_{\times k}(G)} L_k(G) \geq \sum_{i=1}^{\chi_{\times k}(G)} |B_i| = n$$

and equality holds when each set B_i is an $L_k(G)$ -set. So $\chi_{\times k}(G) + L_k(G) \geq \chi_{\times k}(G) + \frac{n}{\chi_{\times k}(G)}$.

On the other hand, $\chi_{\times k}(G) \leq \frac{n}{k}$ because every subset of $V(G)$ of cardinality at most k is a k LP set. We observe that the function $g(x) = x + \frac{n}{x}$ is decreasing for $1 \leq x \leq \sqrt{n}$, and it is increasing for $\sqrt{n} \leq x \leq \frac{n}{k}$. Therefore $\chi_{\times k}(G) + L_k(G) \geq 2\sqrt{n}$.

This bound is sharp for the complete graph K_4 , the cycle C_4 and the star S_4 . \square

We end with a study of the 2LPP number of corona product graphs. If v is a vertex of maximum degree in G , then $|N_{G \odot H}[v]| = \Delta(G) + 1 + |V(H)|$. So we need at least $\lceil \frac{\Delta(G)+1+|V(H)|}{2} \rceil$ 2-limited packing sets in every 2LPP of $G \odot H$. Therefore $\chi_{\times 2}(G \odot H) \geq \lceil \frac{\Delta(G)+1+|V(H)|}{2} \rceil$.

Theorem 17. *If G and H are two graphs, then*

$$\chi_{\times 2}(G \odot H) \in \{\chi_{\times 2}(G), \chi_{\times 2}(G) + 1, \chi_{\times 2}(G) + 2, \dots, \chi_{\times 2}(G) + \lceil \frac{|V(H)|}{2} \rceil\}.$$

Proof. Let $\mathbb{P} = \{P_1, P_2, \dots, P_{\chi_{\times 2}(G)}\}$ be a 2LPP of G , and let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(H) = \{u_1, u_2, \dots, u_{n'}\}$.

Since G is a subgraph of $G \odot H$, $\chi_{\times 2}(G) \leq \chi_{\times 2}(G \odot H)$. That $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G)$ can be seen as follows. If $|N_G[v_i] \cap (\cup_{j=1}^{\chi_{\times 2}(G)} P_j)| \leq 2\chi_{\times 2}(G) - n'$ for each $v_i \in V(G)$, then we place the vertices of each copy of H in the members of \mathbb{P} such that $|N_{G \odot H}[v_i] \cap P_j| \leq 2$ for each $1 \leq i \leq n$ and $1 \leq j \leq \chi_{\times 2}(G)$. So the equality holds.

We now have two cases based on the behavior of $|V(H)|$ and prove the upper bound.

- Let $|V(H)|$ be even. In the worst case, if there exists a vertex $v_i \in V(G)$ such that $2\chi_{\times 2}(G) - 1 \leq |N_G[v_i] \cap (\cup_{j=1}^{\chi_{\times 2}(G)} P_j)| \leq 2\chi_{\times 2}(G)$, then we add new sets $P_{\chi_{\times 2}(G)+1}, P_{\chi_{\times 2}(G)+2}, \dots, P_{\chi_{\times 2}(G)+\lceil \frac{|V(H)|}{2} \rceil}$ to \mathbb{P} and put the vertices of each copy of H in these sets two by two until there are no vertices left. Therefore $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G) + \lceil \frac{|V(H)|}{2} \rceil$.

- Let $|V(H)|$ be odd. If there exists a vertex $v_i \in V(G)$ such that $|N_G[v_i] \cap (\cup_{j=1}^{\chi_{\times 2}(G)} P_j)| = 2\chi_{\times 2}(G)$, then $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G) + \lceil \frac{|V(H)|}{2} \rceil$ as before. Hence $\chi_{\times 2}(G) \leq \chi_{\times 2}(G \odot H) \leq \chi_{\times 2}(G) + \lceil \frac{|V(H)|}{2} \rceil$.

In what follows, we show that $\chi_{\times 2}(G \odot H)$ can take all values between $\chi_{\times 2}(G)$ and $\chi_{\times 2}(G) + \lceil \frac{|V(H)|}{2} \rceil$. It is enough to consider the graph G as $K_{a,a}$ and graph H as $\overline{K_{a+b-1}}$ for $a \geq 1$ and $b \geq 0$. Let $\mathbb{P} = \{P_1, P_2, \dots, P_{\chi_{\times 2}(G)}\}$ be a 2LPP of $G = K_{a,a}$. Now we define a labeling for the vertices in each 2-limited packing set P_j in such a way that if $v_i \in V(G)$ belongs to P_j , then v_i has the label j for every $1 \leq i \leq n$ and $1 \leq j \leq \chi_{\times 2}(G)$. For example, we assign the labels from $\{1, 2, \dots, a\}$ to the vertices of G as shown in Figure 2, which is equivalent to a 2LPP for G with the smallest cardinality.

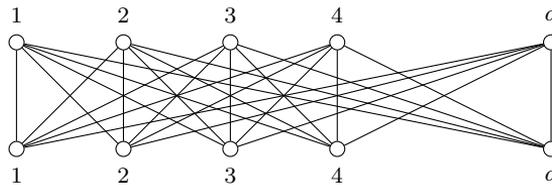


Figure 2. An example for the labeling vertices of $V(K_{a,a})$.

We first show that $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G) + \lceil \frac{b}{2} \rceil$. We have $|N_{G \odot H}[v_i]| = |N_G(v_i)| + 1 + |V(H)| = 2a + b$ for each $v_i \in V(G)$. $(a + 1)$ closed neighbors of each $v_i \in V(G)$ are labeled as above. If $v_i \in P_k$, i.e., v_i has label k , then $|N_G[v_i] \cap P_k| = 2$ and $|N_G[v_i] \cap P_j| = 1$ for each $j \neq k$, as we see in figure 2. It is clear that j has $a - 1$ values. We put $(a - 1)$ unlabeled neighbors of v_i one by one in the sets P_j with the previous condition. Thus, b vertices are still unlabeled. We consider the new labels $a + 1, a + 2, \dots, a + \lceil \frac{b}{2} \rceil$, and then label the remaining b vertices two by two with them. Therefore, $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G) + \lceil \frac{b}{2} \rceil$.

Note that if $a = 1$ and $b \geq 1$, then $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G) + \lceil \frac{b+1-1}{2} \rceil = \chi_{\times 2}(G) + \lceil \frac{|V(H)|}{2} \rceil$. If $b = 0$, then $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G)$. By putting the appropriate values of $a \neq 1$ and $b \neq 0$, the rest of the possible values for $\chi_{\times 2}(G \odot H)$ can be obtained. \square

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