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Research Article

Strong global distribution center of graphs

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Abstract: Let $G = (V, E)$ be a graph. A strong global distribution center of G is a dominating set $S \subseteq V$ such that for any $v \in V \setminus S$, there exists a vertex $u \in N[v] \cap S$ with the property $|N[u] \cap S| > |N[v] \cap (V \setminus S)|$. The strong global distribution center number, $\mathrm{gdc}^s(G)$, of a graph G is the minimum cardinality of a strong global distribution center of G. In this paper, we introduce the concept of strong global distribution center. We give some bounds on the $\mathrm{gdc}^s(G)$ for general graphs and classify graphs with extremal values of $\mathrm{gdc}^s(G)$. Also, we compute the strong global distribution center number for some families of graphs and study this parameter for some families of graph products.

Keywords: global distribution center, strong global distribution center, dominating set, graphs products.

AMS Subject classification: 05C69, 05C76

1. Introduction

Throughout this paper, we consider (non trivial) simple graphs that are finite and undirected, without loops or multiple edges. Let $G = (V, E)$ be a graph of order n and size m. For every vertex $v \in V$, the *open neighborhood* of v is defined by $N_G(v) = \{u \in V | uv \in E(G)\}$. Also, the *closed neighborhood* of v is defined by $N_G[v] = N_G(v) \cup \{v\}$. The *minimum* and maximum degree among the vertices of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A subset $S \subseteq V$ is called a *clique* if all elements of S are adjacent to each other and it is called an *independent set* if no elements of S are adjacent.

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For a nonempty subset S of V, the subgraph $G[S]$ of G induced by S has S as its vertex set, and two vertices u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G.

A *cycle* on n vertices is denoted by C_n, while a *path* on n vertices is denoted by P_n. We denote by K_n the *complete graph* on n vertices and by $K_{n,m}$ the *complete bipartite graph* with one partite set of cardinality n and the other of cardinality m. A *star* is a complete bipartite graph of the form $S_n = K_{1,n-1}$. A *double star* with respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. A graph G with $\Delta \leq 1$, is called an *elementary graph*. If we can partition the vertex set of a graph G, into a clique and independent sets, then G is called a *split* graph.

In graph theory, a *dominating set* for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is joined to at least one member of D by some edges. The *domination number*, $\gamma(G)$, is the number of vertices in a smallest dominating set for G. We call a dominating set of cardinality $\gamma(G)$ a γ-set of G. A *total dominating set* of a graph G is a subset S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. We call a dominating set of cardinality $\gamma_t(G)$, a γ_t-set. For the definitions of other graph-theoretical terms see any standard graph theory text such as [1].

For a graph $G = (V, E)$, a dominating set $S \subseteq V$ is called a *global distribution center* if every vertex $v \in V \setminus S$ is adjacent to a vertex in S, say u, where u has at least as many neighbors in S as v has in $V \setminus S$. The *global distribution center number*, $\mathrm{gdc}(G)$, of a graph G is the minimum cardinality of a global distribution center of G. The concept of global distribution center was introduced by Wyatt J. Desormeaux and et. al. in [2] and further developed in [3]. By a global distribution center, S, elements of S, can supplies the demand of the vertices in $V \setminus S$. The elements of S, are distribution centers and when an element of $V \setminus S$ has a demand (for itself and its neighbors in $V \setminus S$), there exists a distribution center, u, that supplies the demand of vertex v (for itself and its neighbors in S). This transfer from u to v occurs over two days. On the first day, each neighbor of $u \in S$ can ship one unit of resource to u, and then, on the second day, vertex u can ship $|N[v] \cap (V \setminus S)|$ units of resource to its neighbor v in $V \setminus S$. In this model, it is possible that all products of vertex u are transferred to v, after the trading between the vertices u and v. This fact leads us to define the strong global distribution center. A non-empty dominating set $S \subseteq V$ is called a *strong global distribution center* if, for every vertex $v \in V \setminus S$, there exists a vertex $u \in N[v] \cap S$, such that

$$|N[u] \cap S| > |N[v] \cap (V \setminus S)|.$$

In this case, we say that u strongly supplies the demand of v, or equivalently, v is strongly supplied by u. The *strong global distribution center number*, $\mathrm{gdc}^s(G)$, of a graph G is the minimum cardinality of a strong global distribution center of G. A strong global distribution center of size $\mathrm{gdc}^s(G)$ is called a gdc^s-set. Clearly, $\mathrm{gdc}(G) \leq \mathrm{gdc}^s(G)$.

Global defensive alliance and global 1-defensive alliance are two concepts related to the global distribution center and strong global distribution center. A non-empty dominating set $S \subseteq V$ is a global defensive alliance if $|N[v] \cap S| \geq |N[v] \cap (V \setminus S)|$ and is a global 1-defensive alliance if $|N[v] \cap S| > |N[v] \cap (V \setminus S)|$ for any $v \in S$ (see [5–7]). Even though these concepts are similar, the corresponding parameters can easily be shown to be incomparable. Also in modeling of a supply–demand distribution center, we note that an global defensive alliance, we need only one day for transferring the products in vertices of S to vertices in $V \setminus S$.

In this paper, we give bounds for $\mathrm{gdc}^s(G)$ and classify the graphs, in which satisfy in extremal bounds. Also we compute the $\mathrm{gdc}^s(G)$ for some families of graphs and finally, we study this parameter for graphs products.

2. Basic results

In this section, we give some bounds for $\mathrm{gdc}^s(G)$. At first, we need the following simple facts.

Observation 1. If G is a graph with connected components $G_1, G_2, \ldots, G_k$, then $gdc^s(G) = \sum_{i=1}^{k} gdc^s(G_i)$.

Observation 2. i) $gdc^s(K_n) = \lceil \frac{n+1}{2} \rceil$,
ii) $gdc^s(K_{1,n}) = 2$,
iii) $gdc^s(S_{p,q}) = 2$.

By the definition of a strong global distribution center, we have $\gamma(G) \leq \mathrm{gdc}^s(G)$. In the following theorem, we give a stronger result.

Theorem 3. *Let G be a graph without isolated vertices. Then $\gamma_t(G) \leq \mathrm{gdc}^s(G)$.*

Proof. Let S be a gdc^s-set of G and H be the induced subgraph $G[S]$. Let S_1 be the set of isolated vertices of H. If $S_1 = \varnothing$, then S is a total domination of G and hence $\gamma_t(G) \leq \mathrm{gdc}^s(G)$. Suppose that $S_1 = \{v_1, v_2, \ldots, v_k\}$. Since G has no isolated vertices, then each vertex v_i is adjacent to a vertex $w_i \in V(G) \setminus S$. Let $S' = (S \setminus S_1) \cup \{w_1, w_2, \ldots, w_k\}$. We will show that S' is a total dominating set. Let $x \in V \setminus S'$. If $x = v_i$ for some $1 \leq i \leq k$, then v_i is adjacent to $w_i \in S'$. If $x \neq v_i$ for any $1 \leq i \leq k$, then $x \in V \setminus S$. Since S is a gdc^s-set, x is adjacent to a non-isolated vertex of $G[S]$, and hence x is adjacent to a vertex in $S \setminus S_1$. This fact implies that S' is a dominating set. Note that every w_i is adjacent to a vertex of $S \setminus S_1$ and hence $G[S']$ has no isolated vertices. Thus, S' is a total dominating set, and $|S'| \leq |S|$. Hence, $\gamma_t(G) \leq \mathrm{gdc}^s(G)$. □

Note that there exists a graph G with $\gamma_t(G) = \mathrm{gdc}^s(G)$, but no gdc^s-set is a total dominating set. We have $\gamma_t(C_5) = \mathrm{gdc}^s(C_5) = 3$, but there is no gdc^s-set that is a total dominating set. Even though, there are some families of graphs with $\gamma_t(G) = \mathrm{gdc}^s(G)$ (such as star and double star graphs), the gap between $\mathrm{gdc}^s(G)$ and $\gamma_t(G)$ can be arbitrary large. We have, $\mathrm{gdc}^s(K_n) - \gamma_t(K_n) = \lceil \frac{n+1}{2} \rceil - 2 = \lceil \frac{n-3}{2} \rceil$.

Theorem 4. *Let G be a graph of order n and size m, with $\delta(G) \geq 1$. Then $\mathrm{gdc}^s(G) \leq m - n + \lfloor \frac{3\gamma_t(G)}{2} \rfloor$.*

Proof. Let S be a γ_t-set of G. If S is a $\mathrm{gdc}^s(G)$-set, we are done. Thus, we can assume that there is a non-empty subset X of vertices of $V \setminus S$, that are not supplied by their neighbors in S. This fact, means that each vertex v in X has a demand of at least two and this implies that v is adjacent to at least one vertex in $V \setminus S$. So v is incident to at least one edge not covered by S. Since S is a total domination, then there are at least $n - \gamma_t(G)$ edges between S and $V \setminus S$, and at least $\lceil \frac{\gamma_t(G)}{2} \rceil$ edges between the vertices of S. Thus there are at most

$$m - (n - \gamma_t(G)) - \lceil \frac{\gamma_t(G)}{2} \rceil = m - n + \lfloor \frac{\gamma_t(G)}{2} \rfloor$$

uncovered edges. From an uncovered edge $e = xy$, add x to S, and repeat this process until we get a total dominating set S', such that S' covered set. Clearly $V \setminus S'$ is an independent set and hence S' is an strong global distribution center set. Thus $\mathrm{gdc}^s(G) \leq |S'| \leq |S| + m - n + \lfloor \frac{\gamma_t(G)}{2} \rfloor = m - n + \lfloor \frac{3\gamma_t(G)}{2} \rfloor$. □

The bound is sharp for $G = C_4$, since $\gamma_t(C_4) = 2$ and $\mathrm{gdc}^s(C_4) = 3$.

Corollary 1. *If T is a tree of order n, then $\mathrm{gdc}^s(T) \leq \lfloor \frac{3\gamma_t(G)-2}{2} \rfloor$.*

Proof. The result follows by Theorem 4 and the fact that $m = n - 1$. □

The bound is sharp for stars and double stars.

Theorem 5. *Let G be a connected graph of order $n \geq 2$.*
i) $\lceil \frac{\delta}{2} \rceil + 1 \leq \mathrm{gdc}^s(G) \leq n - \lfloor \frac{\Delta}{2} \rfloor$.
ii) If G is a triangle free graph, then $\delta + 1 \leq \mathrm{gdc}^s(G) \leq n - \Delta + 1$.

Proof. i) For the upper bound, consider a vertex v with degree Δ. Let S be the set of vertices formed by removing the $\lfloor \frac{\Delta}{2} \rfloor$ of neighbors of v from V. Clearly S is a dominating set and every vertices of $V \setminus S$ has demand at most $\lfloor \frac{\Delta}{2} \rfloor$ on S. But every vertex of S is adjacent to v and v supplies each vertex of $V \setminus S$ with $\lceil \frac{\Delta}{2} \rceil + 1$ vertices. For the lower bound, let S be a strong global distribution set. If $|S| \leq \lceil \frac{\delta}{2} \rceil$, then $|N[v] \cap (V \setminus S)| \geq \lfloor \frac{\delta}{2} \rfloor + 1$, for any $v \in V \setminus S$. But every vertex in S, can supplies the demand of $v \in V \setminus S$ by at most $|S|$ vertices, which is a contradiction.

ii) Let v be a vertex of G of degree Δ. Consider $u \in N(v), S = N(v) \setminus \{u\}$ and $S' = V \setminus S$. Since G ia a triangle free graph, hence S is an independent set and then

v strong supplies every vertices of S. Hence, S' is a strong distribution center of size $n - \Delta + 1$.
For the lower bound, consider a gdc^s-set, S, and $v \in V \setminus S$. Thus, there exists $u \in N(v) \cap S$, such that $|N[u] \cap S| > |N[v] \cap (V \setminus S)|$. Note that $(N(v) \cap S) \cap (N(u) \cap S) = \varnothing$, since G is a triangle free graph. Hence

$$|S| \geq |N(u) \cap S| + |N(v) \cap S| > |N(v) \cap (V \setminus S)| + |N(v) \cap S| = |N(v)| \geq \delta,$$

and the result is follows. □

The lower and upper bounds of (i) and (ii) are tight. The upper and lower bounds of (i) are sharp for complete and star graphs, respectively. In addition lower and upper bounds in (ii) are sharp for star graphs.

Proposition 1. *Let G be a graph of order n. Then $\mathrm{gdc}^s(G) = n$ if and only if G is an elementary graph*

Proof. If G has a vertex, v, of degree at least two, then $V \setminus \{x\}$, where $x \in N(v)$ is a strong global distribution center for G and this is a contradiction. Hence $\Delta \leq 1$ and this fact implies that G is an elementary graph. □

Proposition 2. *Let G be a connected graph of order n. Then $\mathrm{gdc}^s(G) = n - 1$ if and only if $G \in \{P_3, C_3, C_4, K_4\}$.*

Proof. Let G be a connected graph of order n and $\mathrm{gdc}^s(G) = n - 1$. If there exists a vertex v of degree at least four, then $V \setminus \{x, y\}$, where $x, y \in N(v)$, is a strong global distribution center of G, which is a contradiction. Hence $\Delta \leq 3$. At first suppose that $\Delta = 3$ and v be a vertex of degree 3. Let $N(v) = \{x, y, z\}$. If two neighbors of v, say x and y, are not adjacent, then $V \setminus \{x, y\}$ is a strong global distribution set of G, which is a contradiction. Thus all neighbors of v are adjacent and so $G = K_4$. Now suppose that $\Delta = 2$. If all vertices of G have degree two, then G is a cycle and we conclude that $G = C_3$ or $G = C_4$. If G has a vertex of degree one, then G is a path and hence $G = P_3$. □

Proposition 3. *Let G be a connected graph. Then $\mathrm{gdc}^s(G) = 2$ if and only if G is a split graph with complete part of size 2.*

Proof. Suppose that $\mathrm{gdc}^s(G) = 2$ and S be a gdc^s-set of size 2. Then $V \setminus S$ is an independent set, since $|N[v] \cap V \setminus S| \leq 1$ for any $v \in V \setminus S$. Thus G is a split graph. □

Consider the family ,$\mathfrak{F}$, includes all connected graphs G, with this property that G has a dominating set S of size three, such that $G[V \setminus S]$ is an elementary graph of order $n-3$. Suppose that N_1 be the non-isolated vertices of $G[V \setminus S]$. We construct the following sub-families of this family as follows:
The family $\mathfrak{F}_1$ contains all graphs in $\mathfrak{F}$ with $G[S] \cong K_3$ and if $|N_1| = 0$, then $N(u) \cap (V \setminus S) \neq \emptyset$ for any $u \in S$.

The family $\mathfrak{F}_2$ contains all graphs in $\mathfrak{F}$, with $G[S] \cong P_3 : x - z - y$ and all vertices of N_1 are adjacent to z and in addition
a) If $|N_1| = 0$, then both two sets $N(x) \cap (V \setminus S)$ and $N(y) \cap (V \setminus S)$ are non-empty sets,
b) If $|N_1| = 2$, then at least one of two sets $N(x) \cap (V \setminus S)$ and $N(y) \cap (V \setminus S)$ is non-empty set.

The family $\mathfrak{F}_3$ contains all graphs in $\mathfrak{F}$, with $G[S] \cong K_1 \cup K_2$, where $V(K_1) = \{z\}$ and $V(K_2) = \{x, y\}$, such that
a) The sets $N(x) \cap (V \setminus S), N(y) \cap (V \setminus S)$ and $N(z) \cap (V \setminus S)$ are non-empty sets,
b) If $|N(y) \cap (V \setminus S)| = 1$ (or $|N(x) \cap (V \setminus S)| = 1$), then $N(y) \cap (V \setminus S) \subsetneqq N(x) \cap (V \setminus S)$ (or $N(x) \cap (V \setminus S) \subsetneqq N(y) \cap (V \setminus S)$).

Theorem 6. *Let G be a connected graph of order n. Then $\mathrm{gdc}^s(G) = 3$ if and only if $G \in \mathfrak{F}_1, G \in \mathfrak{F}_2$ or $G \in \mathfrak{F}_3$.*

Proof. Let $S = \{x, y, z\}$ be a gdc^s-set of size three. Consider the induced subgraph H of G induced by $V \setminus S$. Since for any $u \in S$, we have $|N[u] \cap S| \leq 3$, we conclude that $|N[v] \cap V \setminus S| \leq 2$ for any $v \in V \setminus S$. Hence H is an elementary graph. Thus $G \in \mathfrak{F}$. Suppose that N_1 is the set of non-isolated vertices of $G[V \setminus S]$. Since S is a dominating set, we conclude that every vertex of $V \setminus S$ is adjacent to a vertex of S, specially the vertices of N_1 adjacent to a vertex of degree two of $G[S]$. At first Suppose that $G[S] \cong K_3$. If N_1 is non-empty, then $G \in \mathfrak{F}_1$. If N_1 is empty, and for a vertex $x \in S, N(x) \cap (V \setminus S) = \varnothing$, then $(V \setminus S) \cup \{x\}$ is an independent set of size $n-2$ and hence G is a split graph with clique part of size two. Then $\mathrm{gdc}^s(G) = 2$ by Theorem 3 , which is a contradiction. Hence $G \in \mathfrak{F}_1$. If $G[S] \cong P_3 : x - z - y$, then all elements of N_1 are adjacent to z. If N_1 is empty and one of two vertices x or y do not have a neighbor in $V \setminus S$, then G is a split graph with clique part of size 2, which is a contradiction. Hence both sets $N(x) \cap (V \setminus S)$ and $N(y) \cap (V \setminus S)$ are non-empty sets and this implies that $G \in \mathfrak{F}_2$. The same argument works, when $|N_1| = 2$. If $G[S] \cong K_1 \cup K_2$, then $N(u) \cap (V \setminus S) \neq \varnothing$ for any $u \in S$, since otherwise G is an split graph, which is a contradiction by Theorem 3. Also If $|N(x) \cap V \setminus S| = 1$, and $\{w\} = N(x) \cap (V \setminus S) \subseteq N(y) \cap (V \setminus S)$, then $(V \setminus S) \setminus \{w\} \cup \{x, z\}$ is an independent set of G of order $n-2$. Hence G is a split graph, which is a contradiction by Theorem 3. Hence $G \in \mathfrak{F}_3$.

The converse of theorem is obvious. □

In the following theorem, we study the effect of edge deletion on strong global distribution number.

Theorem 7. *For any graphs G, if $e \in E(G)$, then*

$$\mathrm{gdc}^s(G) - 1 \leq \mathrm{gdc}^s(G - e) \leq \mathrm{gdc}^s(G) + 2.$$

Proof. Let S be a gdc^s-set of G and $e = uv$ be an arbitrary edge of G. Clearly, if $u, v \in V \setminus S$, then S is a strong global distribution center for $G - e$. Also if $u \in S$ and $v \in V \setminus S$, then $S \cup \{v\}$ is a strong global distribution center for $G - e$. Finally, assume that $u, v \in S$. Now if $u', v' \in V \setminus S$ are adjacent to u and v, respectively, then $S \cup \{u', v'\}$ is a strong global distribution center for $G - e$. This means that $\mathrm{gdc}^s(G - e) \leq \mathrm{gdc}^s(G) + 2$. For the lower bound, suppose that S be a gdc^s-set of $G - e$ and $e = uv$. If at least one of vertices, u or v, is a member of the set S, then S is also a strong global distribution center for G. Therefore suppose that $u, v \notin S$. In this case, $S \cup \{u\}$ is a strong global distribution center for G. Thus $\mathrm{gdc}^s(G) \leq \mathrm{gdc}^s(G - e) + 1$. □

If $G = P_4$, $e = uv$ and $e' = uv'$, where u and v are vertices of degree 2 and v' is a vertex of degree one, then $\mathrm{gdc}^s(G) = 2$, $\mathrm{gdc}^s(G - e) = 4$ and $\mathrm{gdc}^s(G - e') = 3$. If $G = C_4$, then $\mathrm{gdc}^s(G) = 3$ and $\mathrm{gdc}^s(G - e) = 2$ for any edge e of C_4. If $G = C_5$, then $\mathrm{gdc}^s(G) = \mathrm{gdc}^s(G - e)$ for any edge e of C_5. These examples, show that all numbers in interval $[\mathrm{gdc}^s(G) - 1, \mathrm{gdc}^s(G) + 2]$ are feasible numbers.

3. gdc^s-set of some familes of graphs

In this section, we study the strong global distribution center of some families of graphs.

Theorem 8. *Let m and n be two integers with $1 \leq m \leq n$. Then $\mathrm{gdc}^s(K_{m,n}) = m + 1$.*

Proof. Suppose that V_1 and V_2 are partite of $K_{m,n}$ of sizes m and n, respectively. For an arbitrary vertex $u \in V_2$, it is not difficult to see that $V_1 \cup \{u\}$ is a strong global distribution center for $K_{m,n}$. Hence $\mathrm{gdc}^s(K_{m,n}) \leq m + 1$. Now suppose that S be a gdc^s-set of $K_{m,n}$ and set $|V_1 \cap S| = k$ and $|V_2 \cap S| = l$. If $k < m$, then there are $v \in V_1 \cap (V \setminus S)$ and $u \in V_2 \cap S$, such that

$$k + 1 = |N[u] \cap S| > |N[v] \cap (V \setminus S)| = n - l + 1,$$

and hence $|S| = k + l > n$. Thus $|S| \geq m + 1$. If $k = m$, since $l \geq 1$, we conclude that $|S| \geq m + 1$. Therefore the result follows. □

Theorem 9. *For any non-trivial path P_n of order n,*

$$\mathrm{gdc}^s(P_n) = \begin{cases} \lceil \frac{3n}{5} \rceil, & \text{if } n = 5k+1 \text{ or } n = 5k+3 \\ \lfloor \frac{3n}{5} \rfloor, & \text{otherwise.} \end{cases}$$

Proof. For $n \leq 4$, the result is clear. Assume that $n \geq 5$. Let S be a gdc^s-set of P_n. Label the vertices of P_n as $v_1, v_2, \ldots, v_n$ where $e_i = v_i v_{i+1}$ is an edge of P_n for $1 \leq i \leq n-1$. Consider five consecutive vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$ of the path P_n. It is not difficult to see that at least three of this consecutive vertices must be belong to S. Hence $|S| \geq \frac{3n}{5}$. Without loss of generality, we can assume that the set $T = \{v_i | i = 5j, 5j-2, 5j-3, j = 1, 2, \ldots \lfloor \frac{n}{5} \rfloor\}$ is a subset of S. Let $n = 5k + r$. For $r = 0$, set $S = T$, for $r = 1, 2$, set $S = T \cup \{v_{5k+1}\}$ and for $r = 3, 4$ set, $S = T \cup \{v_{5k+2}, v_{5k+3}\}$. Hence the result is follows. □

Theorem 10. *For a cycle C_n of order n,* $\mathrm{gdc}^s(C_n) = \lceil \frac{3n}{5} \rceil$.

Proof. Let S be a gdc^s-set of P_n. For $n = 4$ the result is obvious. Suppose that $n \geq 5$. If $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$, then for any five consecutive vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$, at least three of them must be belong to S. Hence $|S| \geq \frac{3n}{5}$. Without loss of generality, we can assume that $v_1 \notin S$. Since at least three vertices of five vertices $v_{n-3}, v_{n-2}, v_{n-1}, v_n, v_1$ are belong to S, by a similar argument as proof of Theorem 9, we can conclude that $gdc^s(C_n) = \lceil \frac{3n}{5} \rceil$. □

4. Graphs products

In this section, we study the strong global distribution center for some families of graphs products. Reader can see [4] for further information of graphs products. The *direct product* $G \times H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, and two vertices (a, b) and (a', b') being adjacent in $G \times H$ if and only if $aa' \in E(G)$ and $bb' \in E(H)$. The *cartesian product* of two graphs G and H, denoted by $G \Box H$, is a graph with vertex set $V(G) \times V(H)$, where two vertices (a, b) and (a', b') are adjacent if $aa' \in E(G)$ and $b = b'$ or $a = a'$ and $bb' \in E(H)$. The *strong product*, $G \boxtimes H$, of graphs G and H is $(G \times H) \cup (G \Box H)$.

Theorem 11. *Let G_1 and G_2 be graphs of orders n_1 and n_2, respectively, Then*

$$\mathrm{gdc}^s(G_1 \Box G_2) \leq \min\{n_1 \mathrm{gdc}^s(G_2), n_2 \mathrm{gdc}^s(G_1)\}.$$

Proof. Let $G = G_1 \Box G_2$, $V = V(G_1) \times V(G_2)$, and $S_i \subseteq V(G_i)$, $i \in \{1, 2\}$ be the gdc^s-sets of G_i. Suppose that $S = S_1 \times V(G_2)$. If $(v, w) \in V \setminus S$, then $v \in V(G_1) \setminus S_1$.

Hence there exists a vertex $u \in S_1$ such that $|N_{G_1}[u] \cap S_1| > |N_{G_1}[v] \cap (V \setminus S_1)|$. Now $(u, w) \in S$ is adjacent to (v, w) and

$$\begin{aligned} |N_G[(v,w)] \cap (V \setminus S)| &= |N_{G_1}[v] \cap (V(G_1) \setminus S_1)| + deg_{G_2}(w) \\ &< |N_{G_1}[u] \cap S_1| + deg_{G_2}(w) \\ &= |N_G[(u,w)] \cap S| \end{aligned}$$

Therefore S is a strong global distribution center for G. Similarly, the set $S = V(G_1) \times S_2$ is also a strong global distribution center for G. □

The above bound is sharp. We have $\mathrm{gdc}^s(C_3 \Box K_2) = 4 = 2\mathrm{gdc}^s(C_3)$.

By the same argument as Theorem 11, we can prove the following theorem.

Theorem 12. *Let G_1 and G_2 be graphs of orders n_1 and n_2, respectively, Then*
a)$\mathrm{gdc}^s(G_1 \times G_2) \leq \min\{n_1\mathrm{gdc}^s(G_2), n_2\mathrm{gdc}^s(G_1)\}$,
b)$\mathrm{gdc}^s(G_1 \boxtimes G_2) \leq \min\{n_1\mathrm{gdc}^s(G_2), n_2\mathrm{gdc}^s(G_1)\}$.

The *join* $G \vee H$ of G and H is obtained from $G \cup H$ by joining each vertex of G to every vertex of H

Theorem 13. *For any graph G with maximum degree Δ,*

$$\mathrm{gdc}^s(G \vee K_1) \leq min\{\mathrm{gdc}(G) + 1, \Delta + 2\}$$

.

Proof. Assume that S be an gdc-set of G and the vertex v be the a vertex of G of maximum degree and also $u \in V(K_1)$. It can be easily shown that the sets $S \cup \{u\}$ and $N_G[v] \cup \{u\}$ are the strong global distribution centers for the graph $G \vee K_1$. Thus the proof is complete. □

The graph $W_n = K_1 \vee C_{n-1}$ is called a *wheel graph.*

Proposition 4. *For the wheel graph W_n, for $n \geq 4$,*

$$\mathrm{gdc}^s(W_n) = \begin{cases} 3, & \text{if n=4 or n=5} \\ \\ 4, & \text{otherwise.} \end{cases}$$

Proof. Note that $3 \leq \mathrm{gdc}\ ^s(W_n) \leq 4$ by Theorems 13, and 5. Now the results is followed by a simple calculation. □

Let G be a graph of order n and let $H_1, H_2, \cdots H_n$, be n graphs. The *generalized corona product*, is the graph obtained by taking one copy of graphs $G, H_1, H_2, \ldots, H_n$ and joining the ith vertex of G to every vertex of H_i. This product is denoted by $G \circ \wedge_{i=1}^{n} H_i$. If each H_i is isomorphic to a graph H, then generalized corona product is called the *corona product* of G and H and is denoted by $G \circ H$.

Theorem 14. *Let G be a graph of order n and $H_1, H_2, \cdots, H_n$ be n graphs. If $\delta(G) > \Delta(H_i)$ for any graph H_i, then $\mathrm{gdc}^s(G \circ \wedge_{i=1}^{n} H_i) = n$.*

Proof. Let $S = V$. If $v \in V \setminus S$, then v is a vertex of H_i for an $1 \leq i \leq n$ and has demand $\Delta(H_i) + 1$. If u is the vertex of G, corresponding to H_i, then $u \in S$ and $|N[u] \cap S| \geq \delta(G) + 1 > \Delta(H_i) + 1 \geq |N[v] \cap (V \setminus S)|$ and the result is followed. □

Corollary 2. *i) Let G be a graph of order n, without isolated vertex, then $\mathrm{gdc}^s(G \circ K_1) = n$,*
ii) Let T be a tree of order n. Then $\mathrm{gdc}^s(T \circ K_1) = n$.

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