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Research Article

Roman domination value in graphs

P. Roushini Leely Pushpam[1], Sampath Padmapriea[2,*]

[1]Department of Mathematics, D.B. Jain College, Chennai 600 097, Tamil Nadu, India
roushinip@yahoo.com

[2]Department of Mathematics, Sri Sairam Engineering College, Chennai 600 044, Tamil Nadu, India
*prieas@gmail.com

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Abstract: For a graph $G = (V, E)$, a set $S \subseteq V$ is a *dominating set* if every vertex in $V \setminus S$ has a neighbour in S. The *domination number*, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G and a dominating set of minimum cardinality is called a $\gamma(G)$*-set*. Cockayne et al. defined a *Roman dominating function* (RDF) on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *Roman domination number*, denoted by $\gamma_R(G)$, is the minimum weight of an RDF in G. An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$*-function*. Eunjeong Yi introduced the *domination value of* v, denoted by $DV_G(v)$, to be the number of $\gamma(G)$-sets to which v belongs. In this paper, we extend the idea of domination value to Roman domination. For a vertex $v \in V$, we define the *Roman domination value*, denoted by $R_G(v)$, as $R_G(v) = \sum_{f\in\mathcal{F}} f(v)$, where $\mathcal{F}$ denote the set of all $\gamma_R(G)$-functions. We also study some basic properties of Roman domination value of vertices for a given graph and determine the Roman domination value for the vertices of a complete k-partite graph.

Keywords: domination, Roman domination, Roman domination value.

AMS Subject classification: 05C69

1. Introduction

By a graph $G = (V, E)$, we mean a simple, finite, undirected graph with $|V| = n$. For any vertex $v \in V$, the *open neighbourhood* of v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighbourhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighbourhood* of S is $N(S) = \cup_{v\in S} N(v)$ and the *closed neighbourhood* of S is $N[S] = N(S) \cup S$. We denote by $\triangle(G)$ and $\delta(G)$, respectively, the *maximum degree* and minimum degree of G. For graph theoretic terminology, we refer to Chartrand and Lesniak. [1].

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* *Corresponding Author*

A set of vertices S is a *dominating set* if $N[S] = V$ or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G and a dominating set S of minimum cardinality is called a $\gamma(G)$-set. The literature on domination and its variations in graphs has been surveyed and detailed in the two books by Haynes et al. [5, 6].

In 1999, Slater [12] introduced the notion of the number of dominating sets of G, which he denoted by $HED(G)$ in honour of Steve Hedetniemi; further, he used $\#\gamma(G)$ to denote the number of $\gamma(G)$-sets. Mynhardt [8] characterized vertices that belong to all $\gamma(T)$-sets for a tree T. In 2012, Yi [14] introduced the *domination value of* v, denoted by $DV_G(v)$, to be the number of $\gamma(G)$-sets to which v belongs and he used the notation $\tau(G)$ to denote the number of $\gamma(G)$-sets. Kang [7] initiated the study of total domination value in graphs. For additional article on the topic of domination value in graphs see [15].

Cockayne et al. [4] defined a *Roman dominating function* (RDF) on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of f is $f(V) = \sum_{v \in V} f(v)$. The *Roman domination number*, denoted by $\gamma_R(G)$, is the minimum weight of an RDF in G. An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$-function. This definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart entitled "Defend the Roman Empire!" [13]. Also Roman domination has been studied in [2, 3, 9–11]. Let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1, 2$. Note that there exists a 1-1 correspondence between the function $f : V \to \{0, 1, 2\}$ and the ordered partition (V_0, V_1, V_2) of V. Thus, we write $f = (V_0, V_1, V_2)$. Following [7, 14], we denote by $\tau_R(G)$, the number of $\gamma_R(G)$-functions. For a graph G, let $\mathcal{F} = \{f_i = (V_0^i, V_1^i, V_2^i), 1 \le i \le \tau_R(G)\}$ be the set of all $\gamma_R(G)$-functions and $\Gamma_i = \{V_1^i \cup V_2^i\}, 1 \le i \le \tau_R(G)$.

In this paper, we extend this idea of domination value to Roman domination. Analogous to [7, 14], we define the *Roman domination value* of a vertex $v \in V$, denoted by $R_G(v)$, to be the value $\sum_{f \in \mathcal{F}} f(v)$.

For practical purposes, a vertex with high Roman domination value gains importance. For instance, facility locations such as hospitals, fire stations, mobile towers can be located in such vertices which will minimize the cost involved and further, will increase the coverage area. In certain practical scenarios, one may also opt for minimizing $|V_1|$ (see Figure 1).

In this paper, analogues to the results obtained in [7, 14], for (total) domination value in graphs, we observe some basic properties of Roman domination value in graphs and determine Roman domination value of vertices in complete k-partite graphs.

2. Basic Properties of Roman Domination value of graphs

Clearly for any vertex $v \in V$, $0 \le R_G(v) \le 2\tau_R(G)$. In this section, we consider the lower and upper bounds of the Roman domination value for a fixed vertex v_0 of a

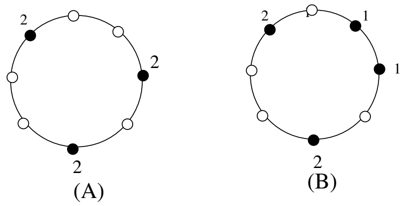

Figure 1. Graphs illustrating the concept of minimizing $|V_1|$ and not minimizing $|V_1|$ respectively.

graph G and for $v \in N[v_0]$. We state the following Observations 1, 2 and 3 that are analogous to those of the domination value [14] and the total domination value [7].

Observation 1. $\sum_{v \in V(G)} R_G(v) = \tau_R(G).\gamma_R(G)$

Observation 2. If there is an isomorphism of graphs carrying a vertex v in G to a vertex v' in G', then $R_G(v) = R_{G'}(v')$.

Cycles, paths and the Petersen graph are some of the graphs that admit automorphism. Let $\mathcal{P}$ denote the Petersen graph with $V(\mathcal{P}) = \{v_j, 1 \leq j \leq 10\}$ (Refer Figure 2). It is clear that $\gamma_R(\mathcal{P}) = 6$. Since $\mathcal{P}$ is vertex transitive, it suffices to compute $R_G(v_1)$. It was obtained in [14] that $\{v_1, v_3, v_7\}$, $\{v_1, v_4, v_{10}\}$ and $\{v_1, v_8, v_9\}$ are the only $\gamma(\mathcal{P})$-sets containing the vertex v_1. Now $f_1 : V \rightarrow \{0, 1, 2\}$ defined by $f_1(v_1) = f_1(v_3) = f_1(v_7) = 2, ; f_1(x) = 0$ for all $x \in V(\mathcal{P}) \setminus \{v_1, v_3, v_7\}$, $f_2 : V \rightarrow \{0, 1, 2\}$ defined by $f_2(v_1) = f_2(v_4) = f_2(v_{10}) = 2, f_2(x) = 0$ for all $x \in V(\mathcal{P}) \setminus \{v_1, v_4, v_{10}\}$ and $f_3 : V \rightarrow \{0, 1, 2\}$ defined by $f_3(v_1) = f_3(v_8) = f_3(v_9) = 2, f_3(x) = 0$ for all $x \in V(\mathcal{P}) \setminus \{v_1, v_8, v_9\}$ are the only three $\gamma_R(\mathcal{P})$-functions that assign 2 to v_1 and all other $\gamma_R(\mathcal{P})$-functions assign 0 to v_1 and hence $R_G(v_1) = 6$. Hence for all $v \in V,\ R_G(v) = 6$.

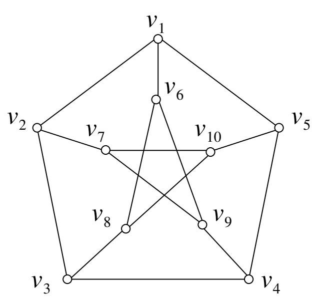

Figure 2. The Petersen graph

Observation 3. Let G be the disjoint union of two graphs G_1 and G_2. Then $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2)$ and $\tau_R(G) = \tau_R(G_1).\tau_R(G_2)$. For $v \in V(G_1)$, $R_G(v) = R_{G_1}(v).\tau_R(G_2)$.

The following statement and its proof for the case of domination value and total domination value can be found in [14] and [7], respectively.

Theorem 4. *For a fixed $v_0 \in V$, we have $\tau_R(G) \leq \sum_{v \in N[v_0]} R_G(v) \leq \gamma_R(G)\tau_R(G)$ and both the bounds are sharp.*

Proof. The upper bound follows from Observation 1. Also every Γ_i must contains a vertex in $N[v_0]$. Otherwise Γ_i fails to dominate v_0 and hence the lower bound follows. Let v_0 be the pendant vertex of any spider. Then $\tau_R(G) = \sum_{v \in N[v_0]} R_G(v)$ and hence the lower bound is sharp (Refer Figure 3(B)). Let v_0 be the root of complete binary tree. Then $\sum_{v \in N[v_0]} R_G(v) = \gamma_R(G)\tau_R(G)$ and hence the upper bound is sharp (Refer Figure 3(A)). □

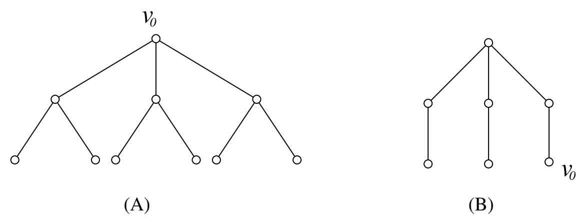

Figure 3. **Graphs that attain the upper and lower bound, respectively, of Theorem 4**

The rest of the results in this section for $\tau_R(G)$ and $R_G(v)$ on a general graph G and vertices $v \in V(G)$ are analogous to those obtained in [7, 14] for the number of minimum (total) dominating sets and the (total) domination value of a general graph, while there are similarities and differences.

Theorem 5. *Let H be a spanning subgraph of G. If $\gamma_R(H) = \gamma_R(G)$, then $\tau_R(H) \leq \tau_R(G)$.*

Proof. Since $\gamma_R(H) = \gamma_R(G)$, every $\gamma_R(H)$-function is also a $\gamma_R(G)$-function. Hence $\tau_R(H) \leq \tau_R(G)$. □

Theorem 6. *Let G be any graph on $n = 2m \geq 4$ vertices. If G or $\overline{G}$ is mK_2, then for any $v \in V$, $R_G(v) + R_{\overline{G}}(v) = 3^{\frac{n}{2}} + 3$.*

Proof. Without loss of generality, assume $G = mK_2$ and label the vertices of G by $v_1, v_2, \cdots, v_{2m}$. Further, assume that the vertex v_{2k-1} is adjacent to the vertex $v_{2k}, 1 \leq k \leq m$. Clearly $\gamma_R(K_2) = 2$.
Consider the vertex v_1. Now the number of $\gamma_R(G)$-functions that assign 2 to the vertex v_1 is $(m-1)C_{(m-1)}2^0 + (m-1)C_{(m-2)}2^1 + \cdots + (m-1)C_2 2^{m-3} + (m-1)C_1 2^{m-2} + (m-1)C_0 2^{m-1} = (1+2)^{m-1} = 3^{m-1}$ [nC_r denotes the number of ways of selecting r objects out of n objects]. Similarly, the number of $\gamma_R(G)$-functions that assign 1 to the vertex v_1 is 3^{m-1}. Now, by Observation 3, $R_G(v_1) = 2(3^{m-1}) + 3^{m-1} = 3 \times 3^{m-1} = 3^m = 3^{\frac{n}{2}}$. Hence by Observation 2, for any $v \in V$, $R_G(v) = 3^{\frac{n}{2}}$.
Consider the graph $\overline{G}$ and the vertex v_1. Clearly $\gamma_R(\overline{G}) = 3$. Now $f_1 : V(\overline{G}) \to \{0, 1, 2\}$ defined by $f_1(v_1) = 2,\ f_1(v_2) = 1,\ f_1(x) = 0$ for all $x \in V \setminus \{v_1, v_2\}$ and $f_2 : V(\overline{G}) \to \{0, 1, 2\}$ defined by $f_2(v_1) = 1,\ f_2(v_2) = 2,\ f_2(x) = 0$ for all $x \in V \setminus \{v_1, v_2\}$ are the only $\gamma_R(G)$-functions that assign positive weight to v_1 and all other $\gamma_R(G)$-functions assign 0 to that vertex. Hence $R_{\overline{G}}(v_1) = 3$. This is true for every vertex in $\overline{G}$. Hence for $v \in V,\ R_G(v) + R_{\overline{G}}(v) = 3^{\frac{n}{2}} + 3$. □

Theorem 7. *Let G be any graph on $n = 2m \geq 4$ vertices. If G or $\overline{G}$ is mK_2 and $|V_1^i|, 1 \leq i \leq \tau_R(G)$ is minimized, then for any $v \in V(G)$, $R_G(v) + R_{\overline{G}}(v) = 2^{\frac{n}{2}} + 3$.*

Proof. Let G be the graph as described in Theorem 6. In this case, $|V_1| = 0$. The number of $\gamma_R(G)$-functions that assign 2 to v_1 is 2^{m-1}. Hence $R_G(v_1) = 2 \times 2^{m-1} = 2^m = 2^{\frac{n}{2}}$.
For the graph $\overline{G}$, as in the proof of Theorem 6, we see that $R_{\overline{G}(v_1)} = 3$. Thus for any $v \in V, R_G(v) + R_{\overline{G}}(v) = 2^{\frac{n}{2}} + 3$. □

The following Observation is immediate.

Observation 8. Let G be a graph of order $n \geq 2$ such that $\triangle(G) = n - 1$. Then $\gamma_R(G) = 2$ and $R_G(v) = 0$ or 2 or 3 for any $v \in V$.

It is clear from Observation 8, that for $G = K_n,\ n \geq 3$, $R_G(v) = 2$ for each $v \in V(K_n)$. Further $\tau_R(G) = n$.

Theorem 9. *Let G be a graph of order $n \geq 3$ such that $\triangle(G) = n - 2$. Then for any $v \in V$, $\gamma_R(G) = 3$ and $R_G(v) \leq 3$. Further $R_G(v) = 3$ if and only if deg$(v) = n - 2$ and there exists a vertex $w \in V$ such that deg$(w) = n - 2$ where $vw \notin E(G)$.*

Proof. Let deg$(v) = n - 2$. Then $\gamma_R(G) > 2$ and there is only one vertex say w such that $vw \notin E(G)$. Clearly, $f : V(G) \to \{0, 1, 2\}$ defined by $f(v) = 2,\ f(w) = 1,\ f(x) = 0$ for all $x \in V(G) \setminus \{v, w\}$ is a $\gamma_R(G)$-function. Hence $\gamma_R(G) = 3$. Since v dominates $N[v]$, we see that the number of $\gamma_R(G)$-functions assigning a positive weight to v is at most two. If exactly one function assigns positive weight to v, then

$\deg(w) < n-2$ and in this case, $R_G(v) = 2$. If two $\gamma_R(G)$-functions assign positive weights to v, then clearly $\deg(w) = n-2$. In this case, $R_G(v) = 3$. □

Theorem 10. *Let G be a graph of order $n \geq 4$ and $\triangle(G) = n-3$ with $deg_G(u) = \triangle(G)$.*

(i) If G is disconnected, then $\gamma_R(G) = 4$ and $R_G(u) = 2, 3$ or 6.
(ii) If G is connected, then $\gamma_R(G) = 4$ and $R_G(u) \leq 2n$.

Proof. We have the following four cases.

Case (*i*). Neither v nor w is adjacent to any vertex in $N(u)$.
Clearly G is a disconnected graph. Let $G' = G[V(G) \setminus \{v, w\}]$. Then $|V(G')| = n-2$ and $\deg_{G'}(u) = n-3 = (n-2)-1$. By Observation 8, $\gamma_R(G') = 2$ and $R'_G(u) = 2$. Let $G'' = G[\{v, w\}]$. Clearly, $\gamma_R(G) = 4$ and $\gamma_R(G'') = 2$. Suppose that $\deg(v) = \deg(w) = 0$, then clearly $\tau_R(G'') = 1$. Now by Observation 3, $R_G(u) = 2$ (Refer Figure 4(A)).

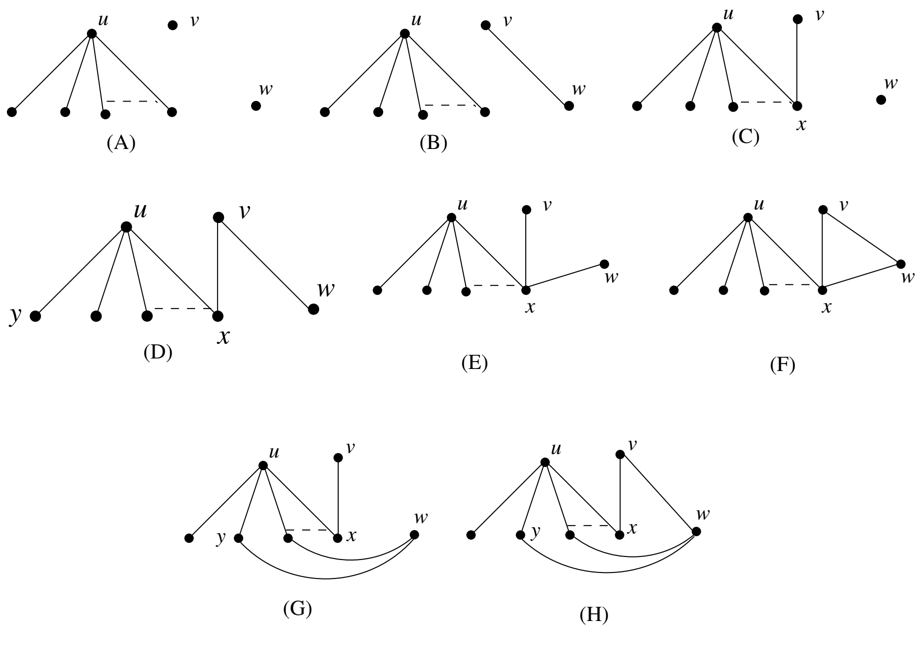

Figure 4. **Graphs illustrating the various cases of Theorem 10 and Theorem 11.**

Suppose that $\deg(v) = \deg(w) = 1$, then $vw \in E(G)$. Clearly there exist exactly three $\gamma_R(G)$-functions $f_i : V(G'') \to \{0, 1, 2\},\ 1 \leq i \leq 3$ defined as $f_1(v) = 2,\ f_1(w) = 0$, $f_2(w) = 2,\ f_2(v) = 0$ and $f_3(v) = f_3(w) = 1$. Hence $\tau_R(G'') = 3$. Now by Observation 3, $R_G(u) = 6$ (Refer Figure 4(B)).

Case (*ii*). Exactly one of v or w is adjacent to a vertex in $N(u)$.
Without loss of generality, assume that v is adjacent to a vertex say x in $N(u)$.

Suppose G is disconnected, then $vw \notin E(G)$. Let $G' = G[V(G) \setminus \{w\}]$. Then $\gamma_R(G') = 3$. Now $|V(G')| = n-1$ and $\deg_{G'}(u) = n-3 = (n-1)-2$. By Theorem 9, $R_{G'}(u) \leq 3$.
If $1 \leq \deg(v) \leq n-4$, then function $f: V(G') \to \{0,1,2\}$ defined by $f(u) = 2,\ f(v) = 1,\ f(r) = 0$ for all $r \in V \setminus \{u,v\}$ is the only $\gamma_R(G')$-function. Hence $R_{G'}(u) = 2$.
Suppose $\deg(v) = n-3$, then there exists two $\gamma_R(G)$-functions $f_i : V(G') \to \{0,1,2\},\ i = 1,2$ that are defined as
$f_1(u) = 2,\ f_1(v) = 1,\ f_1(r) = 0$ for all $r \in V \setminus \{u,v\}$ and
$f_2(u) = 1,\ f_2(v) = 2,\ f_2(r) = 0$ for all $r \in V \setminus \{u,v\}$.
Hence $R_{G'}(u) = 3$.
Let $G'' = G[\{w\}]$. Then $\gamma_R(G'') = 1$ and $\tau_R(G'') = 1$. Now by Observation 3, $R_G(u) = 2$ or 3 (Refer Figure 4(C)).
Suppose that G is connected, then $vw \in E(G)$. Clearly $\gamma_R(G) = 4$ and $1 \leq \deg(v) \leq n-3$.(Refer Figure 4(D)).
Suppose $1 \leq \deg(v) \leq n-4$, then there exist exactly three $\gamma_R(G)$-functions $f_i : V(G) \to \{0,1,2\}, 1 \leq i \leq 3$ that are defined as
$f_1(u) = f_1(v) = 2,\ f_1(r) = 0$ for all $r \in V \setminus \{u,v\}$,
$f_2(u) = f_2(w) = 2,\ f_2(r) = 0$ for all $r \in V \setminus \{u,w\}$ and
$f_3(u) = 2,\ f_3(v) = f_3(w) = 1,\ f_3(r) = 0$ for all $r \in V \setminus \{u,v,w\}$.
Hence $R_G(u) = 6$.
Suppose $\deg(v) = n-3$, then there exist exactly four $\gamma_R(G)$-functions $f_i : V(G) \to \{0,1,2\},\ 1 \leq i \leq 4$ that are defined as
$f_1(u) = f_1(v) = 2,\ f_1(r) = 0$ for all $r \in V \setminus \{u,v\}$,
$f_2(u) = f_2(w) = 2,\ f_2(r) = 0$ for all $r \in V \setminus \{u,w\}$,
$f_3(u) = 2,\ f_3(v) = f_3(w) = 1,\ f_3(r) = 0$ for all $r \in V \setminus \{u,v,w\}$ and
$f_4(u) = f_4(y) = 1,\ f_4(v) = 2,\ f_4(r) = 0$ for all $r \in V \setminus \{u,v,y\}$ where y is the vertex such that $y \in N(u)$ but $y \notin N(v)$.
Hence $R_G(u) = 7$. (Refer Figure 4(D)).

Case *(iii)*. Both v and w are adjacent to a vertex in $N(u)$.
Let x be a vertex in $N(u)$ which is adjacent to both v and w. In this case, x is not adjacent to at least two members of $N(u)$ (otherwise $\deg_G(x) > \deg_G(u) = \triangle(G)$). Since $ux, vx, wx \in E(G)$, clearly, $f : V(G) \to \{0,1,2\}$ defined by $f(u) = f(x) = 2,\ f(r) = 0$ for all $r \in V(G) \setminus \{u,x\}$ is a $\gamma_R(G)$-function with weight 4.
If $vw \notin E(G)$, then $|N(v) \cap N(w)| \leq n-3$.
Suppose $\deg_G(v) = \deg_G(w) = n-3$. Then there are $|N(v) \cap N(w)|$ $\gamma_R(G)$-functions that assign 2 to u. Also the functions $f_i : V(G) \to \{0,1,2\},\ 1 \leq i \leq 3$ defined by $f_1(u) = 2,\ f_1(v) = f_1(w) = 1$ and $f_1(r) = 0$ for all $r \in V \setminus \{u,v,w\},\ f_2(v) = 2,\ f_2(u) = f_2(w) = 1$ and $f_2(r) = 0$ for all $r \in V \setminus \{u,v,w\}, f_3(w) = 2, f_3(u) = f_3(v) = 1$ and $f_3(r) = 0$ for all $r \in V \setminus \{u,v,w\}$ assign 2 or 1 to u. Hence $R_G(u) = 2+1+1+2|N(v) \cap N(w)| \leq 4 + 2(n-3) \leq 2n-2$.
Suppose $\deg_G(v) = n-3$ and $\deg_G(w) < n-3$. Then there are $|N(v) \cap N(w)|$ $\gamma_R(G)$-functions that assign 2 to u. Also the functions $f_i : V(G) \to \{0,1,2\},\ i = 1,2$ defined by $f_1(u) = 2,\ f_1(v) = f_1(w) = 1$ and $f_1(r) = 0$ for all $r \in V \setminus \{u,v,w\}$,

$f_2(v) = 2,\ f_2(u) = f_2(w) = 1$ and $f_2(r) = 0$ for all $r \in V \setminus \{u, v, w\}$ assign 2 or 1 to u. Hence $R_G(u) = 2 + 1 + 2|N(v) \cap N(w)| \leq 3 + 2(n-3) \leq 2n-3$.
Suppose $\deg_G(v) < n-3$ and $\deg_G(w) < n-3$. Then there are $|N(v) \cap N(w)|$ $\gamma_R(G)$-functions that assign 2 to u. Also the function $f : V(G) \to \{0, 1, 2\}$ defined by $f(u) = 2,\ f(v) = f(w) = 1$ and $f_1(r) = 0$ for all $r \in V \setminus \{u, v, w\}$ assign 2 to u. Hence $R_G(u) = 2 + 2|N(v) \cap N(w)| \leq 2 + 2(n-3) \leq 2n-4$ (Refer Figure 4(E)).
If $vw \in E(G)$, then $|N(v) \cap N(w)| \leq n-4$.
Suppose $\deg_G(v) = \deg_G(w) = n-3$. Then there are $|N(v) \cap N(w)|$ $\gamma_R(G)$-functions that assign 2 to u. Also the functions $f_i : V(G) \to \{0, 1, 2\},\ 1 \leq i \leq 5$ defined by $f_1(u) = 2,\ f_1(v) = f_1(w) = 1$ and $f_1(r) = 0$ for all $r \in V \setminus \{u, v, w\}$, $f_2(u) = f_2(v) = 2$ and $f_2(r) = 0$ for all $r \in V \setminus \{u, v\}$, $f_3(u) = f_3(w) = 2$ and $f_3(r) = 0$ for all $r \in V \setminus \{u, w\}$ assign 2 to u, $f_4(u) = f_4(y) = 1,\ f_4(v) = 2$ and $f_4(r) = 0$ for all $r \in V \setminus \{u, v, y\}$ and $f_5(u) = f_5(y) = 1,\ f_5(w) = 2$ and $f_5(r) = 0$ for all $r \in V \setminus \{u, w, y\}$ where y is a vertex in $N(u)$ such that $y \notin N(v) \cap N(w)$ assign 1 to u. Hence $R_G(u) = 3(2) + 1 + 1 + 2|N(v) \cap N(w)| \leq 8 + 2(n-4) \leq 2n$. Suppose $\deg_G(v) = \deg_G(w) = n-3$ such that $y \notin N(v)$ and $z \notin N(w)$, then as above $R_G(u) \leq 2n$.
Suppose $\deg_G(v) = n-3$ and $\deg_G(w) < n-3$ or vice-versa, then there are $|N(v) \cap N(w)|$ $\gamma_R(G)$-functions that assign 2 to u. Also the functions $f_i : V(G) \to \{0, 1, 2\}, i = 1, 2, 3, 4$ defined by $f_1(u) = 2,\ f_1(v) = f_1(w) = 1$ and $f_1(r) = 0$ for all $r \in V \setminus \{u, v, w\}$, $f_2(u) = f_2(v) = 2$ and $f_2(r) = 0$ for all $r \in V \setminus \{u, v\}$, $f_3(u) = f_3(w) = 2$ and $f_3(r) = 0$ for all $r \in V \setminus \{u, w\}$ assign 2 to u and $f_4(u) = f_4(y) = 1,\ f_4(v) = 2$ and $f_4(r) = 0$ for all $r \in V \setminus \{u, v, y\}$ where y is a vertex in $N(u)$ such that $y \notin N(v) \cap N(w)$. Hence $R_G(u) = 3(2) + 1 + 2|N(v) \cap N(w)| \leq 7 + 2(n-4) \leq 2n-1$.
Suppose $\deg_G(v) < n-3$ and $\deg_G(w) < n-3$, then there are $|N(v) \cap N(w)|$ $\gamma_R(G)$-functions that assign 2 to u. Also the functions $f_i : V(G) \to \{0, 1, 2\},\ i = 1, 2, 3$ defined by $f_1(u) = 2,\ f_1(v) = f_1(w) = 1$ and $f_1(r) = 0$ for all $r \in V \setminus \{u, v, w\}$, $f_2(u) = f_2(v) = 2$ and $f_2(r) = 0$ for all $r \in V \setminus \{u, v\}$, $f_3(u) = f_3(w) = 2$ and $f_3(r) = 0$ for all $r \in V \setminus \{u, w\}$ assign 2 to u. Hence $R_G(u) = 3(2) + 2|N(v) \cap N(w)| \leq 6 + 2(n-4) \leq 2n-2$ (Refer Figure 4(F)).
Case(*iv*): No vertex in $N(u)$ is adjacent to both v and w.
Let $x \in N(u) \cap N(v)$ and $y \in N(u) \cap N(w)$. Suppose that $vw \notin E(G)$.
If $N(x) \cup N(y) = V(G)$, then a function $f : V(G) \to \{0, 1, 2\}$ defined by $f(x) = f(y) = 2,\ f(r) = 0$ for all $r \in V(G) \setminus \{x, y\}$ is a $\gamma_R(G)$-function and in this case $R_G(u) = 2$.
If $N(x) \cup N(y) \subset V(G)$, then a function $f : V(G) \to \{0, 1, 2\}$ defined by $f(u) = 2,\ f(v) = f(w) = 1\ f(r) = 0$ for all $r \in V(G) \setminus \{u, x\}$ is a $\gamma_R(G)$-function and in this case $R_G(u) = 2$.
Suppose that $vw \in E(G)$. Since there is no vertex in $N(u)$ that is adjacent to both v and w, there exists only three $\gamma_R(G)$-functions $f_i : V(G) \to \{0, 1, 2\},\ i = 1, 2, 3$ that are defined as $f_1(u) = 2,\ f_1(v) = 2,\ f_1(r) = 0$ for all $r \in V(G) \setminus \{u, v\}$ or $f_2(u) = 2,\ f_2(w) = 2,\ f_2(r) = 0$ for all $r \in V(G) \setminus \{u, w\}$ or $f_3(u) = 2,\ f_3(v) = f_3(w) = 1,\ f_3(r) = 0$ for all $r \in V(G) \setminus \{u, v, w\}$ and $R_G(u) = 6$. □

Theorem 11. *Let G be a graph of order $n \geq 4$ and $\triangle(G) = n-3$. Let $deg_G(u) = \triangle(G)$ and $|V_1^i|$, $1 \leq i \leq \tau_R(G)$ be minimized.*

(i) If G is disconnected, then $\gamma_R(G) = 4$ and $R_G(u) = 2, 3$ or 4.

(ii) If G is connected, then $\gamma_R(G) = 4$ and $R_G(u) \leq 2n-4$.

Proof. Since $\deg_G(u) = \triangle(G) = n-3$, there exist two vertices say v and w such that $uv, uw \notin E$. We consider the following four cases.

Case (*i*). Neither v nor w is adjacent to any vertex in $N(u)$.
As in the proof of Theorem 10, $\gamma_R(G) = 4$ and $R_G(u) = 2$.
Suppose $\deg(v) = \deg(w) = 1$, then $vw \in E(G'')$. Clearly there are only two $\gamma_R(G'')$-functions $f_i : V(G'') \to \{0,1,2\}$, $i = 1,2$ defined as $f_1(v) = 2$, $f_1(w) = 0$ and $f_2(w) = 2$, $f_2(v) = 0$. Hence $\tau_R(G'') = 2$. Now by Observation 3, $\gamma_R(G) = 4$ and $R_G(u) = 0$ or 4 (Refer Figure 4(B)).

Case (*ii*). Exactly one of v and w is adjacent to a vertex in $N(u)$.
As in the proof of Theorem 10, $\gamma_R(G) = 4$ and $R_G(u) = 2$ or 3.
Suppose G is connected, then $vw \in E$. Clearly $\gamma_R(G) = 4$ and there exists at least one vertex $y \in N(u)$ that is not adjacent to v. Now there exist two $\gamma_R(G)$-functions $f_i : V \to \{0,1,2\}$, $i = 1,2$ such that $\gamma_R(G) = 4$ that are defined as $f_1(u) = 2$, $f_1(v) = 2$, $f_1(r) = 0$ for all $r \in V \setminus \{u,v\}$ and $f_2(u) = 2$, $f_2(w) = 2$, $f_2(r) = 0$ for all $r \in V \setminus \{u,w\}$. Hence $R_G(u) = 4$ (Refer Figure 4(D)).

Case (*iii*). Both v and w are adjacent to a vertex in $N(u)$.
As in the proof of Theorem 10, $\gamma_R(G) = 4$.
If $vw \notin E(G)$, then $|N(v) \cap N(w)| \leq n-3$. There are $|N(v) \cap N(w)|$ $\gamma_R(G)$-functions that assign 2 to u. Hence $R_G(u) = 2|N(v) \cap N(w)| \leq 2(n-3) \leq 2n-6$ (Refer Figure 4(E)).
If $vw \in E(G)$, then $|N[v] \cap N[w]| \leq n-4$. There are $|N(v) \cap N(w)| - \gamma_R(G)$-functions that assign 2 to u. Also two other functions $f_i : V(G) \to \{0,1,2\}$, $i = 1,2$ defined by $f_1(u) = f_1(v) = 2$ and $f_1(r) = 0$ for all $r \in V \setminus \{u,v\}$, $f_2(u) = f_2(w) = 2$ and $f_2(r) = 0$ for all $r \in V \setminus \{u,w\}$ assign 2 to u. Hence $R_G(u) = 4 + 2\,|N(v) \cap N(w)| \leq 4 + 2(n-4) \leq 2n-4$ (Refer Figure 4(F)).

Case(*iv*). No vertex in $N(u)$ is adjacent to both v and w.
Let $x \in N(u) \cap N(v)$ and $y \in N(u) \cap N(w)$.
Suppose that $vw \notin E(G)$. Then, as in the proof of Theorem 10, $\gamma_R(G) = 4$ and $R_G(u) = 2$.
Suppose that $vw \in E(G)$. Since there is no vertex in $N(u)$ that is adjacent to both v and w there exist only two $\gamma_R(G)$-functions say f_1 and f_2 such that $f_1 : V(G) \to \{0,1,2\}$ defined by $f_1(u) = 2$, $f_1(v) = 2$, $f_1(r) = 0$ for all $r \in V \setminus \{u,v\}$ or $f_2 : V(G) \to \{0,1,2\}$ defined by $f_2(u) = 2$, $f_2(w) = 2$, $f_2(r) = 0$ for all $r \in V(G) \setminus \{u,w\}$ with weight 4 and hence $\gamma_R(G) = 4$ and $R_G(u) = 4$. □

3. Roman Domination Value on Complete k-partite graphs

Yi [14] determined the number of minimum dominating sets and the domination value of vertices on paths, cycles and complete multipartite graphs. Kang [7] determined the number of minimum total dominating sets and the total domiantion value of vertices on paths, cycles and complete multipartite graphs. As shown in [7, 14], determining the (total) domination value of a vertex in a general graph can be very challenging. In this section, we determine the number of minimum Roman dominating functions and Roman domination value of vertices on complete multipartite graphs. For a complete k-partite graph G with $k \geq 2$, let $V(G)$ be partitioned into k-partite sets $X_1, X_2, \ldots, X_k$ with $|X_j| = m_j$, where $1 \leq j \leq k$. We recall that, for a $\gamma_R(G)$-function f on $V(G)$, $V_i = \{u \in V(G) : f(u) = i\}$ where $i \in \{0, 1, 2\}$.
First, we determine $\gamma_R(G)$ where G is a complete k-partite graph. It was observed in [4] that, if $min\{m, n\} \neq 2$, then $\gamma_R(K_{m,n}) \in \{2, 4\}$.

Observation 12. [4] For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

Lemma 1. *Let $G = K_{m_1,\, m_2, \ldots,\, m_k}$ be a complete k-partite graph such that $k \geq 2$ and $m_k \geq m_{k-1} \geq \cdots \geq m_2 \geq m_1$. Then*

$$\gamma_R(G) = \begin{cases} 2 & \text{if } m_1 = 1 \\ 3 & \text{if } m_1 = 2 \\ 4 & \text{if } m_1 \geq 3 \end{cases}$$

Proof. Let G be a complete k-partite graph as described. Note that $\gamma(G) \in \{1, 2\}$, where $\gamma(G) = 1$ if and only if $m_1 = 1$. If $m_1 = 1$, then $\gamma_R(G) = 2$ by Observation 12 and the definition of Roman domination number.
Now, suppose $m_1 \geq 2$. If f is a function on $V(G)$ such that either $|V_1| = 2$ and $|V_2| = 0$, or $|V_1| = 0$ and $|V_2| = 1$, then f fails to be a Roman dominating function of G; thus, $\gamma_R(G) \geq 3$. By Observation 12, $\gamma_R(G) \in \{3, 4\}$.
First, suppose $m_1 = 2$. Let $X_1 = \{w_1, w_2\}$ and let $g : V(G) \to \{0, 1, 2\}$ be a function defined by $g(w_1) = 2, g(w_2) = 1$ and $g(v) = 0$ for each $v \in V(G) - X_1$. Since g is a Roman dominating function of G with $g(V(G)) = 3$, $\gamma_R(G) \leq 3$. Thus, $\gamma_R(G) = 3$.
Second, suppose $m_1 \geq 3$. We show that $\gamma_R(G) = 4$ by claiming that $\gamma_R(G) \geq 4$. Assume, to the contrary, that there exists a Roman dominating function f on G with $f(V(G)) = 3$. Then f must satisfy one of the following: (i) $|V_1| = 3$ and $|V_2| = 0$ or (ii) $|V_1| = 1$ and $|V_2| = 1$. Suppose f satisfies $|V_1| = 3$ and $|V_2| = 0$. Since $m_1 \geq 3$ and $k \geq 2$, $|V(G)| \geq 6$ and there exists a vertex v in G with $f(v) = 0$ and v is not adjacent to any vertex with the value assigned 2 under f. So, f is not a Roman dominating function of G. Now, suppose f satisfies $|V_1| = 1$ and $|V_2| = 1$. Assume that $V_2 = \{u\} \subset X_a$ for some $a \in \{1, 2, \ldots, k\}$, where $m_a \geq 3$. Since each vertex in $X_a - \{u\}$ is not adjacent to u and there exists a vertex in $X_a - \{u\}$ with the value 0 assigned under f, f fails to form a Roman dominating function of G. So, $\gamma_R(G) \geq 4$. □

Next we determine the Roman domination value of vertices in a complete k-partite graph.

Theorem 13. *Let $G = K_{m_1,\, m_2, \ldots,\, m_k}$ be a complete k-partite graph of order $n \geq 3$ with $k \geq 2$ and $m_k \geq m_{k-1} \geq \cdots \geq m_2 \geq m_1$. Then*

(a) If $m_1 = 1$, then

$$R_G(v) = \begin{cases} 2, & \text{if } v \in X_i, \text{ with } |X_i| = 1, \\ 0, & \text{if } v \in X_i, \text{ with } |X_i| \geq 2. \end{cases}$$

(b) If $m_1 = 2$, then

$$R_G(v) = \begin{cases} 3, & \text{if } v \in X_i, \text{ with } |X_i| = 2, \\ 0, & \text{if } v \in X_i, \text{ with } |X_i| \geq 3. \end{cases}$$

(c) If $m_1 = 3$, then

$$R_G(v) = \begin{cases} 2(n - m_i + 2), & \text{if } v \in X_i, \text{ with } |X_i| = 3, \\ 2(n - m_i) & \text{if } v \in X_i, \text{ with } |X_i| \geq 4. \end{cases}$$

(d) If $m_1 \geq 4$, then $R_G(v) = 2(n - m_i)$ for each $v \in X_i$, where $i \in \{1, 2, \cdots, k\}$.

Proof. Let G be a complete k-partite graph of order $n \geq 3$ as described.
(a) Let $m_1 = 1$ and s be the number of partite sets of $V(G)$ consisting of exactly one element, that is, $|X_i| = 1$ for $i \in \{1, 2, \ldots, s\}$. By Lemma 1, $\gamma_R(G) = 2$. Since $n \geq 3$, $|V_2| = 1$ and $|V_1| = 0$. So, for each minimum Roman dominating function f on $V(G)$ and for $v \in V(G)$, we have $f(v) \in \{0, 2\}$ and $f(w) = 2$ for exactly one vertex w satisfying $\deg_G(w) = n - 1$. Thus, $R_G(v) = 2$ if $v \in \cup_{i=1}^{s} X_i$ and $R_G(v) = 0$ if $v \in V(G) \setminus \cup_{i=1}^{s} X_i$.
(b) Let $m_1 = 2$ and d be the number of partite sets of $V(G)$ consisting of exactly two elements, that is, $|X_i| = 2$ for $i \in \{1, 2, \ldots, d\}$. By Lemma 1, $\gamma_R(G) = 3$. Since $n \geq 4$, $|V_1| = |V_2| = 1$. If $V_1 \subset X_i$ and $V_2 \subset X_j$ with $i \neq j$, then each vertex in $X_j - V_2$ that is assigned the value 0 is not adjacent to any vertex assigned the value 2 under the function being considered. So, $V_1 \cup V_2 \subseteq X_i$ for some $i \in \{1, 2, \ldots, d\}$. For each minimum Roman dominating function f on $V(G)$, we have $f(X_i) = 3$ for exactly one $i \in \{1, 2, \ldots, d\}$. If $f(X_i) = 3$ and $X_i = \{w_i, w_i'\}$, then f satisfies either $f(w_i) = 1$ and $f(w_i') = 2$, or $f(w_i) = 2$ and $f(w_i') = 1$. So, $R_G(v) = 3$ if $v \in \cup_{i=1}^{d} X_i$ and $R_G(v) = 0$ if $v \in V(G) \setminus \cup_{i=1}^{d} X_i$.
(c) Let $m_1 = 3$ and t be the number of partite sets of $V(G)$ consisting of exactly three elements, that is, $|X_i| = 3$ for $i \in \{1, 2, \ldots, t\}$. By Lemma 1, $\gamma_R(G) = 4$. Since $n \geq 6$, $|V_2| \geq 1$. Let $f : V(G) \to \{0, 1, 2\}$ be a minimum Roman dominating function of G with $f(V(G)) = 4$. Then f satisfies one of the following two cases:

(i) $|V_1| = 2$ and $|V_2| = 1$;

(ii) $|V_1| = 0$ and $|V_2| = 2$.

We note that $|V_1| = 2$ and $|V_2| = 1$ imply $V_1 \cap V_2 \subset X_i$, with $|X_i| = 3$ for some $i \in \{1, 2, \ldots, t\}$, and that $|V_1| = 0$ and $|V_2| = 2$ imply that $V_2 \cap X_i \neq \emptyset$ and $V_2 \cap X_j \neq \emptyset$ for distinct $i, j \in \{1, 2, \ldots, k\}$. If $|X_i| = 3$, then both (i) and (ii) can occur for the vertices belonging to X_i. So, for $v \in X_i$ with $|X_i| = 3$, we have $R_G(v) = 4 + 2(n - m_i) = 2(n - m_i + 2)$. If $|X_i| \geq 4$,then only (ii) can occur, that is, for distinct $w \in X_i$ and $w' \in X_j$ with $i \neq j$, $f(w) = f(w') = 2$ and $f(v) = 0$ for each $v \in V(G) \setminus \{w, w'\}$. So, for $v \in X_i$ with $|X_i| \geq 4, R_G(v) = 2(n - m_i)$.
(d) Let $m_1 \geq 4$. By Lemma 1, $\gamma_R(G) = 4$. Since $k \geq 2$ and $m_1 \geq 4$, $|V_1| = 0$ and $|V_2| = 2$. For any minimum Roman dominating function f on $V(G)$, there are exactly two vertices $w \in X_i$ and $w' \in X_j$, where $i \neq j$, such that $f(w) = f(w') = 2$ and $f(v) = 0$ for each $v \in V(G) \setminus \{w, w'\}$. For $w \in X_i$, then number of minimum Roman dominating functions f satisfying $f(w) = 2$ equals the number of elements in $V(G) - X_i$. So $R_G(v) = 2(n - m_i)$ for $v \in X_i$. □

Next, based on Observation 12, Lemma 1 and Theorem 13, we determine $\tau_R(G)$ when G is a complete k-partite graph.

Corollary 1. *Let $G = K_{m_1, m_2, \ldots, m_k}$ be a complete k-partite graph of order $n \geq 3$ with $k \geq 2$ and $m_k \geq m_{k-1} \geq \cdots \geq m_2 \geq m_1$. If $m_1 = 1$($m_1 = 2$ and $m_1 = 3$ respectively), let s (d and t respectively) denote the number of partite sets of $V(G)$ with cardinality 1(2 and 3 respectively). Then*

$$\tau_R(G) = \begin{cases} s & \text{if } m_1 = 1, \\ 2d & \text{if } m_1 = 2, \\ 3t + \dfrac{1}{2}\left\{n^2 - \displaystyle\sum_{i=1}^{k} m_i^2\right\} & \text{if } m_1 = 3, \\ \dfrac{1}{2}\left\{n^2 - \displaystyle\sum_{i=1}^{k} m_i^2\right\} & \text{if } m_1 \geq 4. \end{cases}$$

Proof. Let G be a complete k-partite graph of order $n \geq 3$ as described. By Observation 1, Lemma 1 and Theorem 13, we have the following.
If $m_1 = 1$, then $\displaystyle\sum_{i=1}^{s} 2(1) = 2\tau_R(G)$, and thus $\tau_R(G) = s$.
If $m_1 = 2$, then $\displaystyle\sum_{i=1}^{d} 3(2) = 3\tau_R(G)$, and thus $\tau_R(G) = 2d$.
If $m_1 = 3$, then

$$\sum_{i=1}^{t} 2(n - m_i + 2)(3) + \sum_{i=t+1}^{k} 2(n - m_i)m_i = 4\tau_R(G)$$

$$\sum_{i=1}^{t} 4(3) + \sum_{i=t+1}^{k} 2(n - m_i)m_i = 4\tau_R(G);$$

Thus,

$$\begin{aligned}\tau_R(G) &= 3t + \frac{1}{2}\sum_{i=1}^{k}(nm_i - m_i^2)\\ &= 3t + \frac{1}{2}\left(n\sum_{i=1}^{k} m_i - \sum_{i=1}^{k} m_i^2\right)\\ &= 3t + \frac{1}{2}\left\{n^2 - \sum_{i=1}^{k} m_i^2\right\}.\end{aligned}$$

If $m_1 \geq 4$, then $\sum_{i=1}^{k} 2(n-m_i)m_i = 4\tau_R(G)$ and thus $\tau_R(G) = \frac{1}{2}\sum_{i=1}^{k}(nm_i - m_i^2) = \frac{1}{2}\left(n^2 - \sum_{i=1}^{k} m_i^2\right)$. □

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