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Research Article

The Sombor index and multiplicative Sombor index of some products of graphs

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Abstract: The Sombor index is a vertex degree-based topological index which it was defined by Ivan Gutman in 2021. We study the Sombor index and the multiplicative Sombor index on some products of graphs, crown graphs, shelf graphs, Ice-cream graphs, helm graphs, flower graphs, generalized Sierpiński graphs, t-Mycielskian graphs, t-ciclo graphs, and t-estella graphs. Then we provide some upper and lower bounds for them.

Keywords: Sombor index, Mycielskian graphs, Estella graphs.

AMS Subject classification: 05C09, 05C92

1. Introduction

Let Θ denote a connected, finite, and simple graph characterized by an order of $|V(\Theta)| = n$ and a size of $|E(\Theta)| = m$. Here, $V(\Theta) = \{w_1, w_2, ..., w_n\}$ represents the set of vertices, while $E(\Theta)$ signifies the set of edges associated with the graph Θ. The number of first neighbors of the vertex w_i denoted by $d_\Theta(w_i)$ and it is called the degree of w_i. Isolated and pendent vertices are vertices of degree zero and degree one, respectively. Let $\delta(\Theta) = \min\{d_\Theta(w_1), \ldots, d_\Theta(w_n)\}$ and $\Delta(\Theta) = \max\{d_\Theta(w_1), \ldots, d_\Theta(w_n)\}$. A sequence $w = w_1, w_2, \ldots, w_{n-1}, w_n = v$ in Θ where $w_iw_j \in E(\Theta)$, $1 \le i \le n-1$ is called a $w-v$ walk in Θ. If a walk with n vertices has not repeated vertex, then it is said to be a path and denoted by P_n. A cycle C_n is a closed path of order n. The complete graph K_n is defined as a graph consisting of n vertices, in which every pair of vertices is adjacent.

In the general case, the topological index based on vertex degree [12] of graph Θ is defined by

$$TI(\Theta) = \sum_{w_iw_j \in E(\Theta)} B(d_\Theta(w_i), d_\Theta(w_j))$$

for some function B with condition $B(r,s) = B(s,r)$ for all non-negative real number r and s. In [12], Gutman listed 26 types of topological indices based on vertex degree, for instance, the different type of Zagreb indices [2], Randić indices [7], and Sombor index. A molecular graph [17] is defined as a connected graph in which the vertices represent atoms and the edges signify the covalent bonds that link these atoms together. Topological indices contain information on the atom-connectivity molecular refractivity, the nature of atoms, molecular volume, the bond multiplicity, etc.

The Sombor index was proposed by Gutman [12] in the following manner

$$SO(\Theta) = \sum_{w_i w_j \in E(\Theta)} \sqrt{(d^2_\Theta(w_i) + d^2_\Theta(w_j)}, \tag{1.1}$$

and the multiplicative Sombor index [15] is defined as

$$\prod_{SO}(\Theta) = \prod_{w_i w_j \in E(\Theta)} \sqrt{(d^2_\Theta(w_i) + d^2_\Theta(w_j)}. \tag{1.2}$$

The Sombor index has the mathematical properties and chemical applications, see [1, 6, 9, 16, 21] and the references cited therein. Arif et al. [3] provided Sombor index on some graphs operation such as $C_n + C_m$, $K_n + K_m$, $P_n + P_m$, $C_n \odot C_m$, $P_n \odot P_m$, and $K_n \odot K_m$. Chanda and Iyer [5], studied the Sombor index of generalized Siperpiński graphs and generalized Mycielskian graphs and obtained some upper and lower bounds for them. Also, Liu [15] investigated multiplicative Sombor index on some graphs such as unicyclic graphs and trees. Shang [20] found the scaling constant of the Sombor index using network. Dehgardi and Shang [8] obtained some lower bounds of the first irregularity Sombor index of any tree. Also, see [14, 18, 19] for Sombor index under some graph products.
Motivated as the above works, we study the Sombor index and multiplicative Sombor index on some graphs.

2. Main results

In the following, we present and demonstrate our primary findings.

2.1. Cartesian product

The Cartesian product of Θ and Ω denoted by $\Theta \Box \Omega$ which is a graph, with

$$\begin{aligned} V(\Theta \Box \Omega) &= \{(x,x') | x \in V(\Theta) \ and \ x' \in V(\Omega)\}, \\ E(\Theta \Box \Omega) &= \{(x,x')(y,y') | x = y, \ x'y' \in E(\Omega), \ or \ xy \in E(\Theta), \ x' = y'\}. \end{aligned}$$

Graph products were studied in [13].

Theorem 1. *Let $\Theta = (V(\Theta), E(\Theta))$ and $\Omega = (V(\Omega), E(\Omega))$ are two graphs. We have*

$$SO(\Theta \square \Omega) \leq |V(\Theta)|SO(\Omega) + 4\sqrt{2}|E(\Theta)||E(\Omega)| + |V(\Omega)|SO(\Theta),$$

$$|V(\Theta)|SO(\Omega) + |V(\Omega)|SO(\Theta) \leq SO(\Theta \square \Omega),$$

and

$$\prod\nolimits_{SO}(\Theta \square \Omega) \geq (\prod\nolimits_{SO}(\Omega))^{|V(\Theta)|}(\prod\nolimits_{SO}(\Theta))^{|V(\Omega)|}.$$

Proof. By the definition of Cartesian product, we deduce $d_{\Theta\square\Omega}(a,b) = d_\Theta(a)+d_\Omega(b)$. Then

$$\begin{aligned} SO(\Theta \square \Omega) &= \sum_{(a,b)(c,d)\in E(\Theta\square\Omega)} \sqrt{(d_\Theta(a) + d_\Omega(b))^2 + (d_\Theta(c) + d_\Omega(d))^2} \\ &= \sum_{bd\in E(\Omega), a\in V(\Theta)} \sqrt{(d_\Theta(a) + d_\Omega(b))^2 + (d_\Theta(a) + d_\Omega(d))^2} \\ &\quad + \sum_{ac\in E(\Theta), b\in V(\Omega)} \sqrt{(d_\Theta(a) + d_\Omega(b))^2 + (d_\Theta(c) + d_\Omega(b))^2}. \end{aligned} \quad (2.1)$$

Using inequality $\sqrt{(x+z)^2 + (y+z)^2} \leq \sqrt{x^2+y^2} + \sqrt{2}z$ for $x, y, z \geq 0$, we obtain

$$\begin{aligned} SO(\Theta \square \Omega) \leq & \sum_{bd\in E(\Omega), a\in V(\Theta)} \left(\sqrt{d_\Omega^2(b) + d_\Omega^2(d)} + \sqrt{2}d_\Theta(a)\right) \\ & + \sum_{ac\in E(\Theta), b\in V(\Omega)} \left(\sqrt{(d_\Theta^2(a) + d_\Theta^2(c)} + \sqrt{2}d_\Omega(b)\right). \end{aligned}$$

Since

$$\begin{aligned} & \sum_{bd\in E(\Omega), a\in V(\Theta)} \left(\sqrt{d_\Omega^2(b) + d_\Omega^2(d)} + \sqrt{2}d_\Theta(a)\right) \\ &= \sum_{a\in V(\Theta)} \left(\sum_{bd\in E(\Omega)} \sqrt{d_\Omega^2(b) + d_\Omega^2(d)}\right) + \sum_{bd\in E(\Omega)} \left(\sum_{a\in V(\Theta)} \sqrt{2}d_\Theta(a)\right) \\ &= \sum_{a\in V(\Theta)} SO(\Omega) + \sum_{bd\in E(\Omega)} 2\sqrt{2}|E(\Theta)| \\ &= |V(\Theta)|SO(\Omega) + 2\sqrt{2}|E(\Theta)||E(\Omega)| \end{aligned}$$

and similarly,

$$\sum_{ac\in E(\Theta), b\in V(\Omega)} \left(\sqrt{(d_\Theta^2(a) + d_\Theta^2(c)} + \sqrt{2}d_\Omega(b)\right) = |V(\Omega)|SO(\Theta) + 2\sqrt{2}|E(\Theta)||E(\Omega)|,$$

we conclude

$$SO(\Theta\Box\Omega) \ \leq \ |V(\Theta)|SO(\Omega) + 4\sqrt{2}|E(\Theta)||E(\Omega)| + |V(\Omega)|SO(\Theta).$$

Inserting inequalities

$$\sum_{bd\in E(\Omega),a\in V(\Theta)} \sqrt{(d_\Theta(a)+d_\Omega(b))^2+(d_\Theta(a)+d_\Omega(d))^2} \geq \sum_{bd\in E(\Omega),a\in V(\Theta)} \sqrt{d_\Omega^2(b)+d_\Omega^2(d)}$$
$$\sum_{ac\in E(\Theta),b\in V(\Omega)} \sqrt{(d_\Theta(a)+d_\Omega(b))^2+(d_\Theta(d)+d_\Omega(b))^2} \geq \sum_{ac\in E(\Theta),b\in V(\Omega)} \sqrt{d_\Theta^2(a)+d_\Theta^2(c)}$$

in (2.1), gives $SO(\Theta\Box\Omega) \geq |V(\Theta)|SO(\Omega) + |V(\Omega)|SO(\Theta)$. Also

$$\begin{aligned}
\prod\nolimits_{SO}(\Theta\Box\Omega) &= \prod_{(a,b)(c,d)\in E(\Theta\Box\Omega)} \sqrt{(d_\Theta(a)+d_\Omega(b))^2+(d_\Theta(c)+d_\Omega(d))^2} \\
&= \prod_{bd\in E(\Omega),a\in V(\Theta)} \sqrt{(d_\Theta(a)+d_\Omega(b))^2+(d_\Theta(a)+d_\Omega(d))^2} \\
&\quad \times \prod_{ac\in E(\Theta),b\in V(\Omega)} \sqrt{(d_\Theta(a)+d_\Omega(b))^2+(d_\Theta(c)+d_\Omega(b))^2} \\
&\geq (\prod\nolimits_{SO}(\Omega))^{|V(\Theta)|}(\prod\nolimits_{SO}(\Theta))^{|V(\Omega)|}.
\end{aligned}$$

□

For an r-regular graph Θ, we have $SO(\Theta) = \sqrt{2}|E(\Theta)|r$ and $\prod_{SO}(\Theta) = (\sqrt{2}r)^{|E(\Theta)|}$. Hence, we can assert the subsequent corollary.

Corollary 1. *Suppose Θ is an r-regular graph and Ω is an r'-regular graph, then*

$$SO(\Theta\Box\Omega) = \frac{\sqrt{2}}{2}(r+r')^2|V(\Theta)||V(\Omega)| \ \ \textit{and} \ \ \prod\nolimits_{SO}(\Theta\Box\Omega) = (\sqrt{2}(r+r'))^{\frac{1}{2}(r+r')|V(\Theta)||V(\Omega)|}.$$

Especially,

$$\begin{aligned}
&SO(C_n\Box C_p) = 8np\sqrt{2}, && \prod\nolimits_{SO}(C_n\Box C_p) = (4\sqrt{2})^{2np}, \\
&SO(C_n\Box K_p) = \frac{\sqrt{2}}{2}(p+1)^2np, && \prod\nolimits_{SO}(C_n\Box K_p) = (\sqrt{2}(p+1))^{\frac{1}{2}(p+1)np}, \\
&SO(K_n\Box K_p) = \frac{\sqrt{2}}{2}(p+n-2)^2np, && \prod\nolimits_{SO}(K_n\Box K_p) = (\sqrt{2}(p+n-2))^{\frac{1}{2}(p+n-2)np}.
\end{aligned}$$

2.2. Direct product

The direct product of Θ and Ω is a graph, denoted by $\Theta \times \Omega$, and

$$\begin{aligned} V(\Theta \times \Omega) &= \{(x,y) | x \in V(\Theta) \text{ and } y \in V(\Omega)\}, \\ E(\Theta \times \Omega) &= \{(x,x')(y,y') | xy \in E(\Theta) \text{ and } x'y' \in E(\Omega)\}. \end{aligned}$$

Also, the direct product called Cartesian product, weak direct product, tensor product, cross product, Kronecker product.

Theorem 2. *Let Θ and Ω are two graphs. We have*

$$|E(\Omega)|\delta(\Omega)SO(\Theta) \le SO(\Theta \times \Omega) \le |E(\Omega)|\Delta(\Omega)SO(\Theta)$$

and

$$(\delta(\Omega)SO(\Theta))^{|E(\Omega)|} \le \prod_{SO}(\Theta \times \Omega) \le (\Delta(\Omega)SO(\Theta))^{|E(\Omega)|}.$$

Proof. We have $d_{\Theta\times\Omega}(a,b) = d_\Theta(a)d_\Omega(b)$. Then

$$SO(\Theta \times \Omega) = \sum_{(a,b)(c,d)\in E(\Theta\times\Omega)} \sqrt{(d_\Theta(a)d_\Omega(b))^2 + (d_\Theta(c)d_\Omega(d))^2}.$$

Inequality $\sqrt{(d_\Theta(a)d_\Omega(b))^2 + (d_\Theta(c)d_\Omega(d))^2} \ge \delta(\Omega)\sqrt{d_\Theta^2(a) + d_\Theta^2(c)}$ yields

$$SO(\Theta \times \Omega) \ge |E(\Omega)|\delta(\Omega)SO(\Theta).$$

Similarly Inequality $\sqrt{(d_\Theta(a)d_\Omega(b))^2 + (d_\Theta(c)d_\Omega(d))^2} \le \Delta(\Omega)\sqrt{d_\Theta^2(a) + d_\Theta^2(c)}$ leads to $SO(\Theta \times \Omega) \le |E(\Omega)|\Delta(\Omega)SO(\Theta)$. Also

$$\begin{aligned} \prod_{SO}(\Theta \times \Omega) &= \prod_{(a,b)(c,d)\in E(\Theta\times\Omega)} \sqrt{(d_\Theta(a)d_\Omega(b))^2 + (d_\Theta(c)d_\Omega(d))^2} \\ &\ge \prod_{(a,b)(c,\Omega' d)\in E(\Theta\times\Omega)} \delta(\Omega)\sqrt{d_\Theta^2(a) + d_\Theta^2(c)} \\ &= (\delta(\Omega)SO(\Theta))^{|E(\Omega)|}. \end{aligned}$$

Similarly $\prod_{SO}(\Theta \times \Omega) \le (\Delta(\Omega)SO(\Theta))^{|E(\Omega)|}$. □

By the last theorem, we have the following corollary.

Corollary 2. *If Ω is a r-regular graph then for any graph Θ we get $SO(\Theta \times \Omega) = rSO(\Theta)|E(\Omega)|$ and $\prod_{SO}(\Theta \times \Omega) = (r\prod_{SO}(\Theta))^{|E(\Omega)|}$. Especially*

$$SO(\Theta \times C_p) = 2pSO(\Theta), \qquad \prod_{SO}(\Theta \times C_p) = (2\prod_{SO}(\Theta))^p,$$

$$SO(\Theta \times K_p) = \frac{p(p-1)^2}{2}SO(\Theta), \qquad \prod_{SO}(\Theta \times K_p) = ((p-1)\prod_{SO}(\Theta))^{\frac{p(p-1)}{2}}.$$

2.3. Strong product

We denoted by $\Theta \boxtimes \Omega$ the strong product of two graphs Θ and Ω, which

$$\begin{aligned} V(\Theta \boxtimes \Omega) &= \{(x,y)|x \in V(\Theta) \text{ and } y \in V(\Omega)\}, \\ E(\Theta \boxtimes \Omega) &= E(\Theta \square \Omega) \bigcup E(\Theta \times \Omega). \end{aligned}$$

Theorem 3. *Let $\Theta = (V(\Theta), E(\Theta))$ and $\Omega = (V(\Omega), E(\Omega))$ are two graphs. We have*

$$\begin{aligned} SO(\Theta \boxtimes \Omega) \leq\ & \sqrt{2}(4+\Delta(\Omega))|E(\Theta)||E(\Omega)| + (2|E(\Theta)| + |V(\Theta)|)SO(\Omega) \\ & + ((3+\Delta(\Omega))|E(\Omega)| + |V(\Omega)|)\, SO(\Theta). \end{aligned}$$

Proof. We have $d_{\Theta\boxtimes\Omega}(a,b) = d_\Theta(a) + d_\Omega(b) + d_\Theta(a)d_\Omega(b)$. Then

$$\begin{aligned} SO(\Theta \boxtimes \Omega) =& \sum_{(a,b)(c,d)\in E(\Theta\boxtimes\Omega)} \sqrt{(d_\Theta(a)+d_\Omega(b)+d_\Theta(a)d_\Omega(b))^2 + (d_\Theta(c)+d_\Omega(d)+d_\Theta(c)d_\Omega(d))^2} \\ =& \sum_{bd\in E(\Omega), a\in V(\Theta)} \sqrt{(d_\Theta(a)+d_\Omega(b)+d_\Theta(a)d_\Omega(b))^2 + (d_\Theta(a)+d_\Omega(d)+d_\Theta(a)d_\Omega(d))^2} \\ +& \sum_{ac\in E(\Theta), b\in V(\Omega)} \sqrt{(d_\Theta(a)+d_\Omega(b)+d_\Theta(a)d_\Omega(b))^2 + (d_\Theta(c)+d_\Omega(b)+d_\Theta(c)d_\Omega(b))^2} \\ +& \sum_{ac\in E(\Theta), bd\in E(\Omega)} \sqrt{(d_\Theta(a)+d_\Omega(b)+d_\Theta(a)d_\Omega(b))^2 + (d_\Theta(c)+d_\Omega(d)+d_\Theta(c)d_\Omega(d))^2}. \end{aligned}$$

Since $\sqrt{(x+y+xy)^2+(x+z+xz)^2} \leq \sqrt{2}x + (x+1)\sqrt{y^2+z^2}$ for $x,y,z \geq 0$, we arrive at

$$\begin{aligned} SO(\Theta \boxtimes \Omega) \leq\ & \sum_{bd\in E(\Omega), a\in V(\Theta)} \left(\sqrt{2}d_\Theta(a) + (d_\Theta(a)+1)\sqrt{d_\Omega^2(b)+d_\Omega^2(d)}\right) \\ & + \sum_{ac\in E(\Theta), b\in V(\Omega)} \left(\sqrt{2}d_\Theta(a) + (d_\Theta(b)+1)\sqrt{d_\Omega^2(a)+d_\Omega^2(c)}\right) \\ & + \sum_{ac\in E(\Theta), bd\in E(\Omega)} \left((1+\Delta(\Omega))\sqrt{d_\Theta^2(a)+d_\Theta^2(c)} + \sqrt{2}\Delta(\Omega)\right) \\ =\ & 4\sqrt{2}|E(\Theta)||E(\Omega)| + (2|E(\Theta)| + |V(\Theta)|)SO(\Omega) \\ & +(2|E(\Omega)| + |V(\Omega)|)SO(\Theta) + (1+\Delta(\Omega))|E(\Omega)|SO(\Theta) \\ & ++\sqrt{2}\Delta(\Omega)|E(\Theta)||E(\Omega)|. \end{aligned}$$

□

2.4. Lexicographic product

The lexicographic product of two graphs Θ and Ω is a graph which denoted by $\Theta \circ \Omega$ and

$$\begin{aligned} &V(\Theta \circ \Omega) = \{(x,y)|x \in V(\Theta), y \in V(\Omega)\}, \\ &E(\Theta \circ \Omega) = \{(x,x')(y,y')|xy \in E(\Theta), \text{or}\, x = y \,\text{and}\, x'y' \in E(\Omega)\}. \end{aligned}$$

Theorem 4. *Let $\Theta=(V(\Theta),E(\Theta))$ and $\Omega=(V(\Omega),E(\Omega))$ are two graphs. We have*

$$SO(\Theta\circ\Omega)\leq 2|E(\Theta)|\left(\sqrt{2}|E(\Omega)|+SO(\Omega)\right)+(1+\Delta(\Omega))|E(\Omega)|SO(\Theta),$$

$$SO(\Theta\circ\Omega)\geq 2|E(\Theta)|\left(|E(\Omega)|+SO(\Omega)\right)+(1+\delta(\Omega))|E(\Omega)|SO(\Theta),$$

and

$$\prod\nolimits_{SO}(\Theta\circ\Omega)\geq(\delta(\Theta))^{|V(\Theta)||E(\Theta)|}\left(\prod\nolimits_{SO}(\Omega)\right)^{|V(\Theta)|}(1+\delta(\Omega))^{|E(\Theta)||E(\Omega)|}\left(\prod\nolimits_{SO}(\Theta)\right)^{|E(\Omega)|}.$$

Proof. We have $d_{\Theta\circ\Omega}(a,b)=d_\Theta(a)+d_\Theta(a)d_\Omega(b)$. Then

$$\begin{aligned}SO(\Theta\circ\Omega)&=\sum_{(a,b)(c,d)\in E(\Theta\circ\Omega)}\sqrt{(d_\Theta(a)+d_\Theta(a)d_\Omega(b))^2+(d_\Theta(c)+d_\Theta(c)d_\Omega(d))^2}\\&=\sum_{bd\in E(\Omega),a\in V(\Theta)}\sqrt{(d_\Theta(a)+d_\Theta(a)d_\Omega(b))^2+(d_\Theta(a)+d_\Theta(a)d_\Omega(d))^2}\\&+\sum_{ac\in E(\Theta),bd\in E(\Omega)}\sqrt{(d_\Theta(a)+d_\Theta(a)d_\Omega(b))^2+(d_\Theta(c)+d_\Theta(c)d_\Omega(d))^2}.\end{aligned}\tag{2.2}$$

Applying inequalities

$$\begin{aligned}&\sum_{bd\in E(\Omega),a\in V(\Theta)}\sqrt{(d_\Theta(a)+d_\Theta(a)d_\Omega(b))^2+(d_\Theta(a)+d_\Theta(a)d_\Omega(d))^2}\\\leq\;&d_\Theta(a)\left(\sqrt{2}+\sqrt{d^2_\Omega(b)+d^2_\Omega(d)}\right),\end{aligned}$$

and

$$\begin{aligned}&\sum_{ac\in E(\Theta),bd\in E(\Omega)}\sqrt{(d_\Theta(a)+d_\Theta(a)d_\Omega(b))^2+(d_\Theta(c)+d_\Theta(c)d_\Omega(d))^2}\\\leq\;&(1+\Delta(\Omega))\sqrt{d^2_\Theta(a)+d^2_\Theta(c)}.\end{aligned}$$

We conclude that

$$\begin{aligned}SO(\Theta\circ\Omega)\;&\leq\;\sum_{bd\in E(\Omega),a\in V(\Theta)}d_\Theta(a)\left(\sqrt{2}+\sqrt{d^2_\Omega(b)+d^2_\Omega(d)}\right)\\&+\sum_{ac\in E(\Theta),bd\in E(\Omega)}\left((1+\Delta(\Omega))\sqrt{d^2_\Theta(a)+d^2_\Theta(c)}\right)\\&=\;2|E(\Theta)|\left(\sqrt{2}|E(\Omega)|+SO(\Omega)\right)+(1+\Delta(\Omega))|E(\Omega)|SO(\Theta).\end{aligned}$$

Also,

$$\begin{aligned} &\sum_{bd\in E(\Omega),a\in V(\Theta)} \sqrt{(d_\Theta(a)+d_\Theta(a)d_\Omega(b))^2+(d_\Theta(a)+d_\Theta(a)d_\Omega(d))^2} \\ \geq\ & d_\Theta(a)\left(1+\sqrt{d^2_\Omega(b)+d^2_\Omega(d)}\right), \end{aligned}$$

and

$$\begin{aligned} &\sum_{ac\in E(\Theta),bd\in E(\Omega)} \sqrt{(d_\Theta(a)+d_\Theta(a)d_\Omega(b))^2+(d_\Theta(c)+d_\Theta(c)d_\Omega(d))^2} \\ \geq\ & (1+\delta(\Omega))\sqrt{d^2_\Theta(a)+d^2_\Theta(c)}. \end{aligned}$$

Therefore,

$$\begin{aligned} SO(\Theta\circ\Omega) \geq & \sum_{bd\in E(\Omega),a\in V(\Theta)} d_\Theta(a)\left(1+\sqrt{d^2_\Omega(b)+d^2_\Omega(d)}\right) \\ & + \sum_{ac\in E(\Theta),bd\in E(\Omega)} \left((1+\delta(\Omega))\sqrt{d^2_\Theta(a)+d^2_\Theta(c)}\right) \\ = & \ 2|E(\Theta)|\left(|E(\Omega)|+SO(\Omega)\right)+(1+\delta(\Omega))|E(\Omega)|SO(\Theta). \end{aligned}$$

Morover, we can write

$$\begin{aligned} & \prod\nolimits_{SO}(\Theta\circ\Omega) \\ = & \prod_{bd\in E(\Omega),a\in V(\Theta)} \sqrt{(d_\Theta(a)+d_\Theta(a)d_\Omega(b))^2+(d_\Theta(a)+d_\Theta(a)d_\Omega(d))^2} \\ & \times \prod_{ac\in E(\Theta),bd\in E(\Omega)} \sqrt{(d_\Theta(a)+d_\Theta(a)d_\Omega(b))^2+(d_\Theta(c)+d_\Theta(c)d_\Omega(d))^2} \\ \geq & \prod_{bd\in E(\Omega),a\in V(\Theta)} \delta(\Theta)\left(\sqrt{d^2_\Omega(b)+d^2_\Omega(d)}\right) \\ & \times \prod_{ac\in E(\Theta),bd\in E(\Omega)} \left((1+\delta(\Omega))\sqrt{d^2_\Theta(a)+d^2_\Theta(c)}\right) \\ = & \ (\delta(\Theta))^{|V(\Theta)||E(\Theta)|}\left(\prod\nolimits_{SO}(\Omega)\right)^{|V(\Theta)|}(1+\delta(\Omega))^{|E(\Theta)||E(\Omega)|}\left(\prod\nolimits_{SO}(\Theta)\right)^{|E(\Omega)|}. \end{aligned}$$

□

Remark 1. If Ω is a k-regular graph then using (2.2) we get $SO(\Theta\circ\Omega) = 2\sqrt{2}(1+k)|E(\Theta)||E(\Omega)|+(1+k)|E(\Omega)|SO(\Theta)$.

2.5. Sum of graphs

We denote the sum of Θ and Ω is a graph by $\Theta + \Omega$, which is a graph and

$$V(\Theta + \Omega) = V(\Theta) \cup V(\Omega),$$
$$E(\Theta + \Omega) = \{ab|a \in V(\Theta) \text{and}\, b \in V(\Omega) \,\text{or}\, ab \in E(\Omega), \text{or}\, ab \in E(\Theta)\}.$$

Theorem 5. *Let Θ and H are two graphs. We have*

$$\begin{aligned} SO(\Theta + \Omega) \;\le\; & (2+\sqrt{2})|E(\Theta)||V(\Omega)| + |V(\Theta)||V(\Omega)|^2 + (2+\sqrt{2})|E(\Omega)||V(\Theta)| \\ & +|V(\Omega)||V(\Theta)|^2 + SO(\Theta) + SO(\Omega). \end{aligned}$$

and

$$\begin{aligned} SO(\Theta + \Omega) \ge & \sqrt{|V(\Omega)|^2 + |V(\Theta)|^2}|V(\Theta)||V(\Omega) + |E(\Theta)||V(\Omega)| + |E(\Omega)||V(\Theta)| \\ & + SO(\Theta) + SO(\Omega), \end{aligned}$$

Proof. We have $d_{\Theta+\Omega}(u) = \begin{cases} d_\Theta(u) + V(\Omega) & if \;\; u \in V(\Theta) \\ d_\Omega(u) + V(\Theta) & if \;\; u \in V(\Omega) \end{cases}$. Then by direct computation we obtain

$$\begin{aligned} SO(\Theta + \Omega) &= \sum_{ab \in E(\Theta+\Omega)} \sqrt{d^2_{\Theta+\Omega}(a) + d^2_{\Theta\cup\Omega}(b)} \\ &= \sum_{a \in V(\Theta), b \in V(\Omega)} \sqrt{(d_\Theta(a) + |V(\Omega)|)^2 + (d_\Omega(b) + |V(\Theta)|)^2} \\ &\quad + \sum_{ab \in E(\Theta)} \sqrt{(d_\Theta(a) + |V(\Omega)|)^2 + (d_\Theta(b) + |V(\Omega)|)^2} \qquad (2.3) \\ &\quad + \sum_{ab \in E(\Omega)} \sqrt{(d_\Omega(a) + |V(\Theta)|)^2 + (d_\Omega(b) + |V(\Theta)|)^2}. \end{aligned}$$

Thus,

$$\begin{aligned} SO(\Theta + \Omega) \;\le\; & \sum_{a \in V(\Theta), b \in V(\Omega)} (d_\Theta(a) + |V(\Omega)| + d_\Omega(b) + |V(\Theta)|) \\ & + \sum_{ab \in E(\Theta)} \left(\sqrt{d^2_\Theta(a) + d^2_\Theta(b)} + \sqrt{2}|V(\Omega)| \right) \\ & + \sum_{ab \in E(\Omega)} \left(\sqrt{d^2_\Omega(a) + d^2_\Omega(b)} + \sqrt{2}|V(\Theta)| \right) \\ = \; & (2+\sqrt{2})|E(\Theta)||V(\Omega)| + |V(\Theta)||V(\Omega)|^2 + (2+\sqrt{2})|E(\Omega)||V(\Theta)| \\ & +|V(\Omega)||V(\Theta)|^2 + SO(\Theta) + SO(\Omega). \end{aligned}$$

Also, we can write

$$\begin{aligned} SO(\Theta+\Omega) &\geq \sum_{a\in V(\Theta),b\in V(\Omega)} \sqrt{|V(\Omega)|^2+|V(\Theta)|^2} \\ &+ \sum_{ab\in E(\Theta)} \left(\sqrt{d^2_{\Theta}(a)+d^2_{\Theta}(b)}+|V(\Omega)|\right) \\ &+ \sum_{ab\in E(\Omega)} \left(\sqrt{d^2_{\Omega}(a)+d^2_{\Omega}(b)}+|V(\Theta)|\right) \\ &= \sqrt{|V(\Omega)|^2+|V(\Theta)|^2}|V(\Theta)||V(\Omega)+|E(\Theta)||V(\Omega)| \\ &+|E(\Omega)||V(\Theta)|+SO(\Theta)+SO(\Omega). \end{aligned}$$

□

Using (2.3), and

$$\begin{aligned} \prod\nolimits_{SO}(\Theta+\Omega) &= \prod_{a\in V(\Theta),b\in V(\Omega)} \sqrt{(d_{\Theta}(a)+|V(\Omega)|)^2+(d_{\Omega}(b)+|V(\Theta)|)^2} \\ &\times \prod_{ab\in E(\Theta)} \sqrt{(d_{\Theta}(a)+|V(\Omega)|)^2+(d_{\Theta}(b)+|V(\Omega)|)^2} \\ &\times \prod_{ab\in E(\Omega)} \sqrt{(d_{\Omega}(a)+|V(\Theta)|)^2+(d_{\Omega}(b)+|V(\Theta)|)^2}. \end{aligned}$$

We can state the following corollary.

Corollary 3. *If Θ and Ω are two r-regular and r'-regular graphs, respectively, then*

$$\begin{aligned} SO(\Theta+\Omega) &= \sqrt{(r+|V(\Omega)|)^2+(r'+|V(\Theta)|)^2}|V(\Theta)||V(\Omega)| \\ &+\frac{\sqrt{2}}{2}r(r+|V(\Omega)|)|V(\Theta)|+\frac{\sqrt{2}}{2}r'(r'+|V(\Theta)|)|V(\Omega)|. \end{aligned}$$

and

$$\begin{aligned} \prod\nolimits_{SO}(\Theta+\Omega) &= \left(\sqrt{(r+|V(\Omega)|)^2+(r'+|V(\Theta)|)^2}\right)^{|V(\Theta)||V(\Omega)|} \\ &\times \left(\sqrt{2}(r+|V(\Omega)|)\right)^{\frac{1}{2}r|V(\Theta)|}\left(\sqrt{2}(r'+|V(\Theta)|)\right)^{\frac{1}{2}r'|V(\Omega)|}. \end{aligned}$$

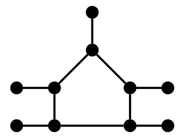

Figure 1. $C_5 \odot K_1$

Especially,

$$\begin{aligned}
SO(C_n + C_t) &= \sqrt{(t+2)^2 + (2+n)^2}np + \sqrt{2}n(2+t) + \sqrt{2}t(2+n),\\
\prod\nolimits_{SO}(C_n + C_t) &= (\sqrt{(2+t)^2 + (2+n)^2}nt)^{nt}\left(\sqrt{2}(t+2)\right)^n\left(\sqrt{2}(2+n)\right)^t,\\
SO(C_n + K_t) &= \sqrt{(2+t)^2 + (t+n-1)^2}np + \sqrt{2}n(t+2)\\
&\quad + \frac{\sqrt{2}}{2}t(t-1)(t+n-1),\\
\prod\nolimits_{SO}(C_n + K_t) &= \left(\sqrt{(2+t)^2 + (t+n-1)^2}\right)^{np}\left(\sqrt{2}(2+t)\right)^n\\
&\quad \times \left(\sqrt{2}(t+n-1)\right)^{\frac{1}{2}(t-1)t},\\
SO(K_n + K_t) &= \sqrt{(n+t-1)^2 + (t+n-1)^2}nt\\
&\quad + \frac{\sqrt{2}}{2}(n-1)(n-1+t)n + \frac{\sqrt{2}}{2}(t-1)(t+n-1)t,\\
\prod\nolimits_{SO}(K_n + K_t) &= \left(\sqrt{(n+t-1)^2 + (t+n-1)^2}\right)^{nt}\left(\sqrt{2}(n+t-1)\right)^{\frac{1}{2}(n-1)n}\\
&\quad \times \left(\sqrt{2}(t+n-1)\right)^{\frac{1}{2}(t-1)t}.
\end{aligned}$$

2.6. Corona of two graphs

We denote by $\Theta \odot H$ the corona of two graphs Θ and H which is obtained by union $|V(\Theta)|$ copies of H and one copy of Θ and joining the ith vertex of Θ with an edge to every vertex in the ith copy of H.

Theorem 6. *Let Θ and Ω are two graphs of order n and m, respectively. We have*

$$\begin{aligned}
SO(\Theta \odot \Omega) \;\leq\; & SO(\Theta) + \sqrt{2}m|E(\Theta)| + nSO(\Omega) + n\sqrt{2}|E(\Omega)| + 2nm|E(\Theta)|\\
& +2n^2|E(\Omega)| + n^2m\sqrt{m^2+1},
\end{aligned}$$

$$\begin{aligned}
SO(\Theta \odot \Omega) \;\geq\; & SO(\Theta) + m|E(\Theta)| + nSO(\Omega) + n|E(\Omega)|\\
& +n^2m\sqrt{(\delta(\Theta)) + m)^2 + (\delta(\Omega) + 1)^2},
\end{aligned}$$

$$\begin{aligned}\prod\nolimits_{SO}(\Theta\odot\Omega) \;\le\; & \left(\prod\nolimits_{SO}(\Theta)+\left(\sqrt{2}(\Delta(\Theta)+m)\right)^{|E(\Theta)|}-\left(\sqrt{2}\Delta(\Theta)\right)^{|E(\Theta)|}\right)\\ & \times\left(\prod\nolimits_{SO}(\Omega)+\left(\sqrt{2}(\Delta(\Omega)+1)\right)^{|E(\Omega)|}-\left(\sqrt{2}\Delta(\Omega)\right)^{|E(\Omega)|}\right)^{n}\\ & \times\left((\Delta(\Theta)+m)^2+(\Delta(\Omega)+1)^2\right)^{\frac{1}{2}n^2m},\end{aligned}$$

and

$$\begin{aligned}\prod\nolimits_{SO}(\Theta\odot\Omega) \;\ge\; & \left(\prod\nolimits_{SO}(\Theta)+\left(\sqrt{2}\delta(\Theta)+m\right)^{|E(\Theta)|}-\left(\sqrt{2}\delta(\Theta)\right)^{|E(\Theta)|}\right)\\ & \times\left(\prod\nolimits_{SO}(\Omega)+\left(\sqrt{2}\delta(\Omega)+1\right)^{|E(\Omega)|}-\left(\sqrt{2}\delta(\Omega)\right)^{|E(\Omega)|}\right)^{n}\\ & \times\left((\delta(\Theta)+m)^2+(\delta(\Omega)+1)^2\right)^{\frac{1}{2}n^2m}.\end{aligned}$$

Proof. Let $V(\Theta)=\{v_1,v_2,\ldots,v_n\}$, $V(\Omega)=\{u_1,u_2,\ldots,u_m\}$, $V(\Omega)^k=\{u_1^k,u_2^k,\ldots,u_m^k\}$ for $1\le k\le n$, and $V(\Theta\odot\Omega)=V(\Theta)\cup V(\Omega)^1\cup\cdots\cup V(\Omega)^n$. We have

$$d_{\Theta\odot\Omega}(v)=\begin{cases}d_\Theta(v)+m & \text{if } v\in V(\Theta),\\ d_\Omega(u_i)+1 & \text{if } v=u_i^k,\ 1\le k\le n.\end{cases}$$

For the Sombor index, we conclude

$$\begin{aligned}SO(\Theta\odot\Omega) &= \sum_{uv\in E(\Theta\odot\Omega)}\sqrt{d^2_{\Theta\odot\Omega}(u)+d^2_{\Theta\odot\Omega}(v)}\\ &= \sum_{v_iv_j\in E(\Theta)}\sqrt{(d_\Theta(v_i)+m)^2+(d_\Theta(v_j)+m)^2}\\ &\quad+\sum_{k=1}^{n}\sum_{u_i^ku_j^k\in E(\Theta\odot\Omega)}\sqrt{(d_\Omega(u_i)+1)^2+(d_\Omega(u_j)+1)^2}\\ &\quad+\sum_{k=1}^{n}\sum_{v_i\in V(\Theta),u_j^k\in V(\Omega)^k}\sqrt{(d_\Theta(v_i)+m)^2+(d_\Omega(u_j)+1)^2}.\end{aligned}$$

Using inequalities $\sqrt{(x+a)^2+(y+a)^2}\le\sqrt{x^2+y^2}+\sqrt{2}a$ for $x,y,a\ge 0$, and

$$\sqrt{(d_\Theta(v_i)+m)^2+(d_\Omega(u_j)+1)^2}\le d_\Theta(v_i)+d_\Omega(u_j)+\sqrt{m^2+1},$$

we find

$$\begin{aligned}SO(\Theta\odot\Omega)\le&\sum_{v_iv_j\in E(\Theta)}\left(\sqrt{d^2_\Theta(v_i)+d^2_\Theta(v_j)}+\sqrt{2}m\right)+n\sum_{u_iu_j\in E(\Omega)}\left(\sqrt{d^2_\Omega(u_i)+d^2_\Omega(u_j)}+\sqrt{2}\right)\\ &+n\sum_{v_i\in V(\Theta),u_j\in V(\Omega)}\left(d_\Theta(v_i)+d_\Omega(u_j)+\sqrt{m^2+1}\right)\\ &=SO(\Theta)+\sqrt{2}m|E(\Theta)|+nSO(\Omega)+n\sqrt{2}|E(\Omega)|+2nm|E(\Theta)|\\ &+2n^2|E(\Omega)|+n^2m\sqrt{m^2+1}.\end{aligned}$$

Also,

$$SO(\Theta \odot \Omega) \geq \sum_{v_iv_j \in E(\Theta)} \left(\sqrt{d^2_\Theta(v_i) + d^2_\Theta(v_j)} + m \right) + n \sum_{u_iu_j \in E(\Omega)} \left(\sqrt{d^2_\Omega(u_i) + d^2_\Omega(u_j)} + 1 \right)$$
$$+ n \sum_{v_i \in V(\Theta), u_j \in V(\Omega)} \sqrt{(\delta(\Theta) + m)^2 + (\delta(\Omega) + 1)^2}$$
$$= SO(\Theta) + m|E(\Theta)| + nSO(\Omega) + n|E(\Omega)|$$
$$+ n^2 m \sqrt{(\delta(\Theta) + m)^2 + (\delta(\Omega) + 1)^2}.$$

For computing the multiplicative Sombor index, we have

$$\begin{aligned} \prod_{SO}(\Theta \odot \Omega) &= \prod_{uv \in E(\Theta \odot \Omega)} \sqrt{d^2_{\Theta \odot \Omega}(u) + d^2_{\Theta \odot \Omega}(v)} \\ &= \prod_{v_iv_j \in E(\Theta)} \sqrt{(d_\Theta(v_i) + m)^2 + (d_\Theta(v_j) + m)^2} \\ &\quad \times \prod_{k=1}^{n} \prod_{u_i^k u_j^k \in E(\Theta \odot \Omega)} \sqrt{(d_\Omega(u_i) + 1)^2 + (d_\Omega(u_j) + 1)^2} \\ &\quad \times \prod_{k=1}^{n} \prod_{v_i \in V(\Theta), u_j^k \in V(\Omega)^k} \sqrt{(d_\Theta(v_i) + m)^2 + (d_\Omega(u_j) + 1)^2}, \end{aligned}$$

then

$$\prod_{SO}(\Theta \odot \Omega) \leq \prod_{v_iv_j \in E(\Theta)} \left(\sqrt{d^2_\Theta(v_i) + d^2_\Theta(v_j)} + \sqrt{2}m \right) \left(\prod_{u_iu_j \in E(\Omega)} \left(\sqrt{d^2_\Omega(u_i) + d^2_\Omega(u_j)} + \sqrt{2} \right) \right)^n$$
$$\times \left(\prod_{v_i \in V(\Theta), u_j \in V(\Omega)} \sqrt{(\Delta(\Theta) + m)^2 + (\Delta(\Omega) + 1)^2} \right)^n$$
$$\leq \left(\prod_{SO}(\Theta) + \left(\sqrt{2}(\Delta(\Theta) + m) \right)^{|E(\Theta)|} - \left(\sqrt{2}\Delta(\Theta) \right)^{|E(\Theta)|} \right)$$
$$\times \left(\prod_{SO}(\Omega) + \left(\sqrt{2}(\Delta(\Omega) + 1) \right)^{|E(\Omega)|} - \left(\sqrt{2}\Delta(\Omega) \right)^{|E(\Omega)|} \right)^n$$
$$\times \left((\Delta(\Theta) + m)^2 + (\Delta(\Omega) + 1)^2 \right)^{\frac{1}{2} n^2 m}.$$

For obtaining the lower bound for the multiplicative Sombor index, we get

$$\prod_{SO}(\Theta \odot \Omega) \geq \prod_{v_iv_j \in E(\Theta)} \left(\sqrt{d_\Theta^2(v_i) + d_\Theta^2(v_j)} + m\right) \left(\prod_{u_iu_j \in E(\Omega)} \left(\sqrt{d_\Omega^2(u_i) + d_\Omega^2(u_j)} + 1\right)\right)^n$$
$$\left(\prod_{v_i \in V(\Theta), u_j \in V(\Omega)} \sqrt{(\delta(\Theta) + m)^2 + (\delta(\Omega) + 1)^2}\right)^n$$
$$\geq \left(\prod_{SO}(\Theta) + \left(\sqrt{2}\delta(\Theta) + m\right)^{|E(\Theta)|} - \left(\sqrt{2}\delta(\Theta)\right)^{|E(\Theta)|}\right)$$
$$\times \left(\prod_{SO}(\Omega) + \left(\sqrt{2}\delta(\Omega) + 1\right)^{|E(\Omega)|} - \left(\sqrt{2}\delta(\Omega)\right)^{|E(\Omega)|}\right)^n$$
$$\times \left((\delta(\Theta) + m)^2 + (\delta(\Omega) + 1)^2\right)^{\frac{1}{2}n^2m}.$$

□

Corollary 4. *If Γ is an r-regular graph of order n and H is an r'-regular graph of order m, then*

$$SO(\Gamma \odot H) = \frac{\sqrt{2}}{2} rn(r+m) + \frac{\sqrt{2}}{2} r'nm(r'+1) + n^2m\sqrt{(r+m)^2 + (r'+1)^2},$$

$$\prod_{SO}(\Gamma \odot H) = \left(\sqrt{2}(m+r)\right)^{\frac{rn}{2}} \left(\sqrt{2}(r'+1)\right)^{\frac{r'mn}{2}} \left((m+r)^2 + (r'+1)^2\right)^{\frac{n^2m}{2}}.$$

Especially we get

$$SO(C_n \odot C_m) = \sqrt{2}n(2+m) + 3\sqrt{2}nm + n^2m\sqrt{(2+m)^2 + 9},$$
$$\prod_{SO}(C_n \odot C_m) = \left(3\sqrt{2}\right)^{nm} \left((2+m)\sqrt{2}\right)^n \left((2+m)^2 + 9\right)^{\frac{n^2m}{2}},$$
$$SO(C_n \odot K_m) = \sqrt{2}n(2+m) + \frac{\sqrt{2}}{2} nm^2(m-1) + n^2m\sqrt{(2+m)^2 + m^2},$$
$$\prod_{SO}(C_n \odot K_m) = \left(\sqrt{2}(2+m)\right)^n \left(\sqrt{2}m\right)^{\frac{(m-1)mn}{2}} \left((2+m)^2 + m^2\right)^{\frac{n^2m}{2}},$$
$$SO(K_n \odot K_m) = \frac{\sqrt{2}}{2} n(n+m-1)(n-1) + \frac{\sqrt{2}}{2} nm^2(m-1) + n^2m\sqrt{(n+m-1)^2 + m^2},$$
$$\prod_{SO}(K_n \odot K_m) = \left((n+m-1)\sqrt{2}\right)^{\frac{n(n-1)}{2}} \left(\sqrt{2}m\right)^{\frac{(m-1)mn}{2}} \left((n+m-1)^2 + m^2\right)^{\frac{n^2m}{2}}.$$

2.7. Crown graphs

A crown graph on $2p$ vertices is a simple graph with two sets of vertices $\{w_1, w_2, \ldots, w_p\}$ and $\{u_1, u_2, \ldots, u_p\}$ and with edges from w_i to u_j whenever $i \neq j$ for $1 \leq i, j \leq p$. Any crown graph Γ with $2p$ vertices is a $(p-1)$-regular graph, thus we get the following corollary.

Corollary 5. *For any crown graph Γ of order $2p$,*

$$SO(\Gamma) = \sqrt{2}(p-1)^2p, \quad \prod_{SO}(\Gamma) = \left(\sqrt{2}(p-1)\right)^{(p-1)p}.$$

Figure 2. **Crown graphs with six and eight vertices**

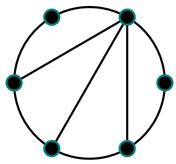

Figure 3. S_6

2.8. Shell graphs

The shell graph S_n is a graph constructed by adding $n-3$ concurrent chords incident with a common vertex in cycle C_n. An apex vertex is a vertex at which all the chords are concurrent.

Theorem 7. *For any shell graph S_n, we have*

$$SO(S_n) = 2\sqrt{5+2n+n^2} + 2\sqrt{13} + (n-4)\sqrt{18} + (n-3)\sqrt{n^2+2n+10},$$

and

$$\prod\nolimits_{SO}(S_n) = 13(5+2n+n^2)18^{\frac{n-4}{2}}(n^2+2n+10)^{\frac{n-3}{2}}.$$

Proof. Let S_n has vertex set $V(S_n) = \{v_1, \ldots, v_n\}$ such that v_1 is an apex vertex and $v_1v_2v_3\ldots v_{n-1}v_nv_1$ be a cycle in C_n. We have

$$d_{S_n}(v) = \begin{cases} n-1 & \text{if } v = v_1, \\ 2 & \text{if } v \in \{v_2, v_n\}, \\ 3 & \text{if } v = v_i,\ 3 \le i \le n-1. \end{cases}$$

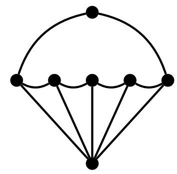

Figure 4. **Ice-cream graph I_6**

Then

$$\begin{aligned} SO(S_n) &= \sum_{uv\in E(S_n)} \sqrt{d^2_{S_n}(u)+d^2_{S_n}(v)} \\ &= \sqrt{d^2_{S_n}(v_1)+d^2_{S_n}(v_2)}+\sqrt{d^2_{S_n}(v_1)+d^2_{S_n}(v_n)} \\ &\quad +\sqrt{d^2_{S_n}(v_2)+d^2_{S_n}(v_3)}+\sqrt{d^2_{S_n}(v_n)+d^2_{S_n}(v_{n-1})} \\ &\quad +\sum_{\substack{v_iv_j\in E(S_n)\\ 3\le i,j\le n-1}} \sqrt{d^2_{S_n}(v_i)+d^2_{S_n}(v_j)}+\sum_{\substack{v_1v_i\in E(S_n)\\ 3\le i\le n-1}} \sqrt{d^2_{S_n}(v_1)+d^2_{S_n}(v_i)} \\ &= 2\sqrt{5+2n+n^2}+2\sqrt{13}+(n-4)\sqrt{18}+(n-3)\sqrt{n^2+2n+10}, \end{aligned}$$

and

$$\begin{aligned} \prod\nolimits_{SO}(S_n) &= \prod_{uv\in E(S_n)} \sqrt{d^2_{S_n}(u)+d^2_{S_n}(v)} \\ &= \sqrt{d^2_{S_n}(v_1)+d^2_{S_n}(v_2)}\times\sqrt{d^2_{S_n}(v_1)+d^2_{S_n}(v_n)} \\ &\quad \times\sqrt{d^2_{S_n}(v_2)+d^2_{S_n}(v_3)}\times\sqrt{d^2_{S_n}(v_n)+d^2_{S_n}(v_{n-1})} \\ &\quad \times\prod_{\substack{v_iv_j\in E(S_n)\\ 3\le i,j\le n-1}} \sqrt{d^2_{S_n}(v_i)+d^2_{S_n}(v_j)}\times\prod_{\substack{v_1v_i\in E(S_n)\\ 3\le i\le n-1}} \sqrt{d^2_{S_n}(v_1)+d^2_{S_n}(v_i)} \\ &= 13(5+2n+n^2)18^{\frac{n-4}{2}}(n^2+2n+10)^{\frac{n-3}{2}}. \end{aligned}$$

□

2.9. Ice-cream graphs

An Ice-cream graph is obtained by combining a shell graph and a path P_2 graph keeping v_1 and v_{n-1} common where $n > 3$ sharing common end point called the apex vertex v_0. It is denoted by I_n.

Theorem 8. *For Ice-cream graph I_n, we have*

$$SO(I_n) = 2\sqrt{13} + (n-2)\sqrt{18} + (n-1)\sqrt{9+(n-1)^2}$$

and

$$\prod\nolimits_{SO}(I_n) = 13(18)^{\frac{n-2}{2}}(9+(n-1)^2)^{\frac{n-1}{2}}.$$

Proof. Let I_n has vertex set $V(I_n) = \{v_0, v_1, v_2, \ldots, v_n\}$ such that v_0 is an apex vertex and $v_0v_1v_2\ldots v_{n-2}v_{n-1}v_0$ be a cycle in C_n. We have

$$d_{I_n}(v) = \begin{cases} n-1 & \text{if } v = v_0, \\ 3 & \text{if } v = v_i,\ 1 \le i \le n-1, \\ 2 & \text{if } v = v_n. \end{cases}$$

Therefore,

$$\begin{aligned} SO(I_n) &= \sum_{uv \in E(I_n)} \sqrt{d_{I_n}^2(u) + d_{I_n}^2(v)} \\ &= \sqrt{d_{I_n}^2(v_1) + d_{I_n}^2(v_n)} + \sqrt{d_{I_n}^2(v_{n-1}) + d_{I_n}^2(v_n)} \\ &\quad + \sum_{\substack{v_iv_j \in E(I_n) \\ 1 \le i,j \le n-1}} \sqrt{d_{I_n}^2(v_i) + d_{I_n}^2(v_j)} + \sum_{\substack{v_0v_i \in E(I_n) \\ 1 \le i \le n-1}} \sqrt{d_{I_n}^2(v_0) + d_{I_n}^2(v_i)} \\ &= 2\sqrt{13} + (n-2)\sqrt{18} + (n-1)\sqrt{9+(n-1)^2}, \end{aligned}$$

and

$$\begin{aligned} \prod\nolimits_{SO}(I_n) &= \prod_{uv \in E(I_n)} \sqrt{d_{I_n}^2(u) + d_{I_n}^2(v)} \\ &= \sqrt{d_{I_n}^2(v_1) + d_{I_n}^2(v_n)} \times \sqrt{d_{I_n}^2(v_{n-1}) + d_{I_n}^2(v_n)} \\ &\quad \times \prod_{\substack{v_iv_j \in E(I_n) \\ 1 \le i,j \le n-1}} \sqrt{d_{I_n}^2(v_i) + d_{I_n}^2(v_j)} \times \prod_{\substack{v_0v_i \in E(I_n) \\ 1 \le i \le n-1}} \sqrt{d_{I_n}^2(v_0) + d_{I_n}^2(v_i)} \\ &= 13(18)^{\frac{n-2}{2}}(10+2n+n^2)^{\frac{n-1}{2}}. \end{aligned}$$

□

2.10. Helm graphs

By attaching a pendent edge to each rim vertex of a wheel graph W_n, a graph is obtained, which is called Helm graph and denoted by H_n.

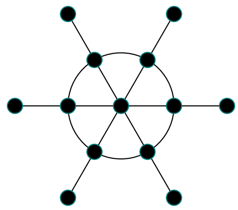

Figure 5. **Helm graph H_6**

Theorem 9. *For Helm graph H_n, we have*

$$SO(H_n) = n\sqrt{n^2+16} + n\sqrt{32} + n\sqrt{17}$$

and

$$\prod\nolimits_{SO}(H_n) = (544(n^2+16))^{\frac{n}{2}}.$$

Proof. Let H_n has vertex set $V(H_n) = \{v_0, v_1, \ldots, v_n, u_1, \ldots, u_n\}$ such that v_0 is an apex vertex and $\{v_1, v_2, \ldots v_{n-2}, v_{n-1}, v_n\}$ be a vertex set of C_n. We have

$$d_{H_n}(v) = \begin{cases} n & \text{if } v = v_0, \\ 4 & \text{if } v = v_i,\ 1 \leq i \leq n, \\ 1 & \text{if } v = u_i,\ 1 \leq i \leq n. \end{cases}$$

Then

$$\begin{aligned} SO(H_n) &= \sum_{uv \in E(H_n)} \sqrt{d^2_{H_n}(u) + d^2_{H_n}(v)} \\ &= \sum_{\substack{v_0v_i \in E(H_n) \\ 1 \leq i \leq n}} \sqrt{d^2_{H_n}(v_0) + d^2_{H_n}(v_i)} + \sum_{\substack{v_iv_j \in E(H_n) \\ 1 \leq i,j \leq n}} \sqrt{d^2_{H_n}(v_i) + d^2_{H_n}(v_j)} \\ &\quad + \sum_{\substack{u_iv_i \in E(H_n) \\ 1 \leq i \leq n}} \sqrt{d^2_{H_n}(u_i) + d^2_{H_n}(v_i)} \\ &= n\sqrt{n^2+16} + n\sqrt{32} + n\sqrt{17}, \end{aligned}$$

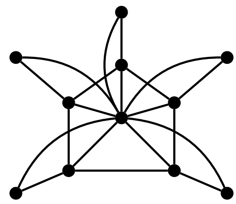

Figure 6. **Flower graph** Fl_5

and

$$\begin{aligned}\prod\nolimits_{SO}(H_n) &= \prod_{uv\in E(H_n)} \sqrt{d^2_{H_n}(u)+d^2_{H_n}(v)} \\ &= \prod_{\substack{v_0v_i\in E(H_n)\\1\le i\le n}} \sqrt{d^2_{H_n}(v_0)+d^2_{H_n}(v_i)} \times \prod_{\substack{v_iv_j\in E(H_n)\\1\le i,j\le n}} \sqrt{d^2_{H_n}(v_i)+d^2_{H_n}(v_j)} \\ &\quad \times \prod_{\substack{u_iv_i\in E(H_n)\\1\le i\le n}} \sqrt{d^2_{H_n}(u_i)+d^2_{H_n}(v_i)} \\ &= (544(n^2+16))^{\frac{n}{2}}.\end{aligned}$$

□

2.11. Flower graphs

By joining each pendent vertex of the Helm H_n to the apex vertex of H_n, a graph is obtained, which is called flower graph and it is denoted by Fl_n.

Theorem 10. *For flower graph Fl_n, we have*

$$SO(Fl_n) = 2n\sqrt{n^2+4} + 2n\sqrt{n^2+1} + n\sqrt{32} + n\sqrt{20}$$

and

$$\prod\nolimits_{SO}(Fl_n) = \left(10240(n^2+4)(n^2+1)\right)^{\frac{n}{2}}.$$

Proof. Let Fl_n has vertex set $V(Fl_n) = \{v_0, v_1, \ldots, v_n, u_1, \ldots, u_n\}$ such that v_0 is an apex vertex and $\{v_1, v_2, \ldots v_{n-2}, v_{n-1}, v_n\}$ be a vertex set of C_n. We have

$$d_{Fl_n}(v) = \begin{cases} 2n & \text{if } v = v_0, \\ 4 & \text{if } v = v_i,\ 1 \le i \le n, \\ 2 & \text{if } v = u_i,\ 1 \le i \le n. \end{cases}$$

Then

$$\begin{aligned}
SO(Fl_n) &= \sum_{uv\in E(Fl_n)} \sqrt{d^2_{Fl_n}(u)+d^2_{Fl_n}(v)} \\
&= \sum_{\substack{v_0v_i\in E(Fl_n)\\ 1\le i\le n}} \sqrt{d^2_{Fl_n}(v_0)+d^2_{Fl_n}(v_i)} + \sum_{\substack{v_0u_i\in E(Fl_n)\\ 1\le i\le n}} \sqrt{d^2_{Fl_n}(v_0)+d^2_{Fl_n}(u_i)} \\
&\quad + \sum_{\substack{v_iv_j\in E(Fl_n)\\ 1\le i,j\le n}} \sqrt{d^2_{Fl_n}(v_i)+d^2_{Fl_n}(v_j)} + \sum_{\substack{u_iv_i\in E(Fl_n)\\ 1\le i\le n}} \sqrt{d^2_{Fl_n}(u_i)+d^2_{Fl_n}(v_i)} \\
&= 2n\sqrt{n^2+4}+2n\sqrt{n^2+1}+n\sqrt{32}+n\sqrt{20},
\end{aligned}$$

and

$$\begin{aligned}
\prod\nolimits_{SO}(Fl_n) &= \prod_{uv\in E(Fl_n)} \sqrt{d^2_{Fl_n}(u)+d^2_{Fl_n}(v)} \\
&= \prod_{\substack{v_0v_i\in E(Fl_n)\\ 1\le i\le n}} \sqrt{d^2_{Fl_n}(v_0)+d^2_{Fl_n}(v_i)} \times \prod_{\substack{v_0u_i\in E(Fl_n)\\ 1\le i\le n}} \sqrt{d^2_{Fl_n}(v_0)+d^2_{Fl_n}(u_i)} \\
&\quad \times \prod_{\substack{v_iv_j\in E(Fl_n)\\ 1\le i,j\le n}} \sqrt{d^2_{Fl_n}(v_i)+d^2_{Fl_n}(v_j)} \times \prod_{\substack{u_iv_i\in E(Fl_n)\\ 1\le i\le n}} \sqrt{d^2_{Fl_n}(u_i)+d^2_{Fl_n}(v_i)} \\
&= \left(10240(n^2+4)(n^2+1)\right)^{\frac{n}{2}}.
\end{aligned}$$

□

2.12. Generalized Sierpiński graphs

Suppose t is an integer and Θ is a graph of order n. Let $V(\Theta)^t$ represent the set of all words $y_1y_2\ldots y_t$ of length t on alphabet $V(\Theta)$. The generalized Sierpiński graph [5, 11] $S(\Theta,t)$, is the graph with $V(\Theta)^t$ such that $E(S(\Theta,t))$ is defined as follows. For $x=x_1x_2\ldots x_t\in V(\Theta)^t$ and $y=y_1y_2\ldots y_t\in V(\Theta)^t$, $\{x,y\}\in E(S(\Theta,t))$ if there exist $i\in\{1,2,\ldots,t\}$ such that

$$(i)\, x_i=y_i, \text{for } j<i,\ \ (ii)\, x_i\neq y_i \text{ and } \{x_i,y_i\}\in E(\Theta)\ \ (iii)\, x_j=y_i \text{ and } y_j=x_i \text{ for } j>i.$$

For edge $\{x,y\}$ then there is a word $w=x_1x_2\ldots x_{i-1}$ and an edge $\{x_i,y_i\}\in E(\Theta)$ such that $x=wx_iy_i\ldots y_i$ and $y=wy_ix_i\ldots x_i$.
For every edge $x_iy_i\in E(\Theta)$, add an edge between vertex $x_iy_iy_i\ldots y_i$ and vertex $y_ix_ix_i\ldots x_i$. Vertices as $x_ix_i\ldots x_i$ are called extreme vertices of $S(\Theta,t)$. The number of extreme vertices of $S(\Theta,t)$ is $|V(\Theta)|$ and, $d_{S(\Theta,t)}(x_ix_i\ldots x_i)=d_\Theta(x_i)$. Also, the degrees of two vertices $y_ix_ix_i\ldots x_i$ and $x_iy_iy_i\ldots y_i$, are equal to $d_\Theta(x_i)+1$ and $d_\Theta(y_i)+1$, respectively. The copies of an edge $x_iy_i\in E(\Theta)$ in $S(\Theta,t)$ are edges

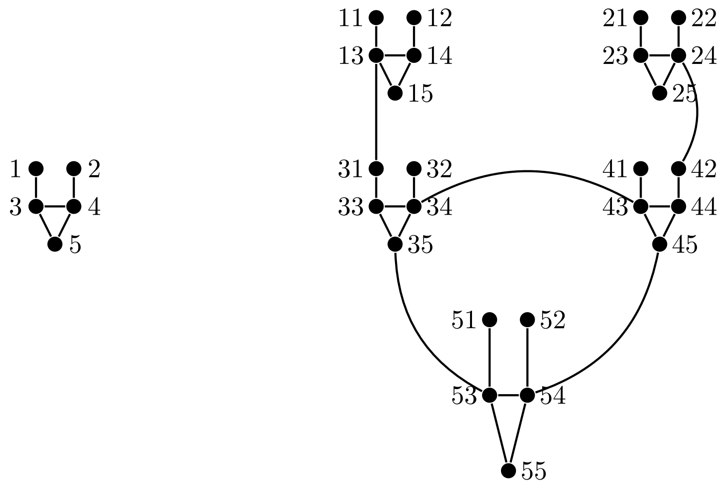

Figure 7. **A generalized Sierpiński graph** $S(\Theta, 2)$

$\{wx_iy_i^r, wy_ix_i^r\}$, where $r \in \{0, 1, \ldots, t-1\}$ and $w \in V(\Theta)^{t-1-r}$. We denote by $f_{S(\Theta,t)}(d_\Theta(x_i)+l, d_\Theta(y_i)+l')$ the number of copies $x_iy_i \in E(\Theta)$ in $S(\Theta, t)$, where vertex $wx_iy_i^r$ has degree $d_\Theta(x_i)+l$, vertex $wy_ix_i^r$ has degree $d_\Theta(y_i)+l'$, and $l, l' \in \{0, 1\}$.

S. Chanda and R. Iyer [5] studied the Sombor index on generalized Sierpiński and generalized Mycielskian graphs. We investigate the multiplicative Sombor index on generalized Sierpiński graphs. We denote the set of neighbours $a \in V(\Theta)$ by $N(a)$, i.e., $N(a) = \{b \in V(\Theta) : ab \in E(\Theta)\}$. Let $\tau(a, b)$ denote the number of triangles of Θ containing a, b. For a graph Θ of order n, let $\psi(t) = 1 + n + n^2 + \cdots + n^{t-1} = \frac{n^t-1}{n-1}$. From [10], we get the following lemma.

Lemma 1 ([10]). *For any graph Θ of order n with edge xy and any integer $t \geq 1$, we have*

(i) $f_{S(\Theta,t)}(d_\Theta(x), d_\Theta(y)) = n^{t-2}(n - d_\Theta(x) - d_\Theta(y) + \tau(x, y))$.

(ii) $f_{S(\Theta,t)}(d_\Theta(x), d_\Theta(y)+1) = n^{t-2}(d_\Theta(y) - \tau(x, y)) - \psi(t-2)d_\Theta(x)$.

(ii) $f_{S(\Theta,t)}(d_\Theta(x)+1, d_\Theta(y)) = n^{t-2}(d_\Theta(x) - \tau(x, y)) - \psi(t-2)d_\Theta(y)$.

(iv) $f_{S(\Theta,t)}(d_\Theta(x)+1, d_\Theta(y)+1) = n^{t-2}(\tau(x, y)+1) + \psi(t-2)(d_\Theta(x) + d_\Theta(y) + 1)$.

Now we have:

Theorem 11. *For any integer $t \geq 2$ and any graph Θ of order $n \geq 3$,*

$$\prod\nolimits_{SO}(S(\Theta, t)) > \left(\prod\nolimits_{SO}(\Theta)\right)^{\psi(t)}.$$

Proof. The multiplicative Somobr index of $S(\Theta,t)$ can be computed as

$$\begin{aligned}
\prod\nolimits_{SO}(S(\Theta,t)) &= \prod_{ab\in E(S(\Theta,t))} \sqrt{d^2_{S(\Theta,t)}(a)+d^2_{S(\Theta,t)}(b)} \\
&= \prod_{ab\in E(\Theta)} \prod_{i=0}^{1}\prod_{j=0}^{1} \left((d_\Theta(a)+i)^2+(d_\Theta(b)+j)^2\right)^{\frac{1}{2}f_{S(\Theta,t))}(d_\Theta(a)+i,d_\Theta(b)+j)} \\
&= \prod_{ab\in E(\Theta)} \Big\{ \left((d_\Theta(a))^2+(d_\Theta(b))^2\right)^{\frac{1}{2}f_{S(\Theta,t))}(d_\Theta(a),d_\Theta(b))} \\
&\quad \times \left((d_\Theta(a)+1)^2+(d_\Theta(b))^2\right)^{\frac{1}{2}f_{S(\Theta,t))}(d_\Theta(a)+1,d_\Theta(b))} \\
&\quad \times \left((d_\Theta(a))^2+(d_\Theta(b)+1)^2\right)^{\frac{1}{2}f_{S(\Theta,t))}(d_\Theta(a),d_\Theta(b)+1)} \\
&\quad \times \left((d_\Theta(a)+1)^2+(d_\Theta(b)+1)^2\right)^{\frac{1}{2}f_{S(\Theta,t))}(d_\Theta(a)+1,d_\Theta(b)+1)} \Big\}.
\end{aligned}$$

Using Lemma 1, we get

$$\begin{aligned}
\prod\nolimits_{SO}(S(\Theta,t)) = \prod_{ab\in E(\Theta)} &\Big\{ \left((d_\Theta(a))^2+(d_\Theta(b))^2\right)^{\frac{1}{2}n^{t-2}(n-d_\Theta(a)-d_\Theta(b)+\tau(a,b))} \\
&\times \left((d_\Theta(a)+1)^2+(d_\Theta(b))^2\right)^{\frac{1}{2}\left(n^{t-2}(d_\Theta(b)-\tau(a,b))-\psi(t-2)d_\Theta(a)\right)} \\
&\times \left((d_\Theta(a))^2+(d_\Theta(b)+1)^2\right)^{\frac{1}{2}\left(n^{t-2}(d_\Theta(a)-\tau(a,b))-\psi(t-2)d_\Theta(b)\right)} \\
&\times \left((d_\Theta(a)+1)^2+(d_\Theta(b)+1)^2\right)^{\frac{1}{2}\left(n^{t-2}(\tau(a,b)+1)+\psi(t-2)(d_\Theta(a)+d_\Theta(b)+1)\right)} \Big\}.
\end{aligned}$$

Since

$$\begin{aligned}
(d_\Theta(a)+1)^2+(d_\Theta(b))^2 &> (d_\Theta(a))^2+(d_\Theta(b))^2, \\
(d_\Theta(a))^2+(d_\Theta(b)+1)^2 &> (d_\Theta(a))^2+(d_\Theta(b))^2, \\
(d_\Theta(a)+1)^2+(d_\Theta(b)+1)^2 &> (d_\Theta(a))^2+(d_\Theta(b))^2,
\end{aligned}$$

we conclude

$$\prod\nolimits_{SO}(S(\Theta,t)) \ > \prod_{ab\in E(\Theta)} \left((d_\Theta(a))^2+(d_\Theta(b))^2\right)^{\frac{1}{2}A}$$

where

$$\begin{aligned}
A &= \big\{n^{t-2}(n-d_\Theta(a)-d_\Theta(b)+\tau(a,b))+n^{t-2}(d_\Theta(b)-\tau(a,b))-\psi(t-2)d_\Theta(a) \\
&\quad +n^{t-2}(d_\Theta(a)-\tau(a,b))-\psi(t-2)d_\Theta(b) \\
&\quad +n^{t-2}(\tau(a,b)+1)+\psi(t-2)(d_\Theta(a)+d_\Theta(b)+1)\big\}. \\
&= n^{t-1}+n^{t-2}+\psi(t-2) \\
&= \psi(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}\prod\nolimits_{SO}(S(\Theta,t)) &> \prod_{ab\in E(\Theta)} \left((d_\Theta(a))^2+(d_\Theta(b))^2\right)^{\frac{1}{2}\psi(t)}\\ &= \left(\prod_{ab\in E(\Theta)} \left((d_\Theta(a))^2+(d_\Theta(b))^2\right)^{\frac{1}{2}}\right)^{\psi(t)}\\ &= \left(\prod\nolimits_{SO}(\Theta)\right)^{\psi(t)}.\end{aligned}$$

□

Theorem 12. *Let Θ be a triangle-free k-regular graph of order n. Then*

$$\begin{aligned}SO(S(\Theta,t)) &= \frac{kn}{2}\Big\{n^{t-2}(n-2k)\sqrt{2}k+2(n^{t-2}-\psi(t-2))k\sqrt{(k+1)^2+k^2}\\ &\quad+\sqrt{2}(n^{t-2}+\psi(t-2)(2k+1))(k+1)\Big\}\end{aligned}$$

and

$$\begin{aligned}\prod\nolimits_{SO}(S(\Theta,t)) &= \left(2k^2\right)^{\frac{k}{4}n^{t-1}(n-2k)}\left(2k^2+2k+1\right)^{\frac{1}{2}k^2n(n^{t-2}-\psi(t-2))}\\ &\quad\times\left(2k^2+4k+2\right)^{\frac{k}{4}n(n^{t-2}+\psi(t-2)(2k+1))}.\end{aligned}$$

Proof. If Θ is a triangle-free k-regular graph then for any $a,b\in V(\Theta)$ we have $\tau(a,b)=0$. Then from Lemma 1 we obtain

(i) $f_{S(\Theta,t)}(d_\Theta(a),d_\Theta(b))=n^{t-2}(n-2k)$.

(ii) $f_{S(\Theta,t)}(d_\Theta(a),d_\Theta(b)+1)=(n^{t-2}-\psi(t-2))k$.

(iii) $f_{S(\Theta,t)}(d_\Theta(a)+1,d_\Theta(b))=(n^{t-2}-\psi(t-2))k$.

(iv) $f_{S(\Theta,t)}(d_\Theta(a)+1,d_\Theta(b)+1)=n^{t-2}+\psi(t-2)(2k+1)$.

For the Sombor index, we deduce

$$\begin{aligned}SO(S(\Theta,t)) &= \sum_{ab\in E(S(\Theta,t))}\sqrt{d^2_{S(\Theta,t)}(a)+d^2_{S(\Theta,t)}(b)}\\ &= \sum_{ab\in E(\Theta)}\sum_{i=0}^{1}\sum_{j=0}^{1} f_{S(\Theta,t))}(d_\Theta(a)+i,d_\Theta(b)+j)\sqrt{(d_\Theta(a)+i)^2+(d_\Theta(b)+j)^2}\\ &= \sum_{ab\in E(\Theta)}\Big\{f_{S(\Theta,t))}(d_\Theta(a),d_\Theta(b))\sqrt{(d_\Theta(a))^2+(d_\Theta(b))^2}\\ &+ f_{S(\Theta,t))}(d_\Theta(a)+1,d_\Theta(b))\sqrt{(d_\Theta(a)+1)^2+(d_\Theta(b))^2}\\ &+ f_{S(\Theta,t))}(d_\Theta(a),d_\Theta(b)+1)\sqrt{(d_\Theta(a))^2+(d_\Theta(b)+1)^2}\\ &+ f_{S(\Theta,t))}(d_\Theta(a)+1,d_\Theta(b)+1)\sqrt{(d_\Theta(a)+1)^2+(d_\Theta(b)+1)^2}\Big\}.\end{aligned}$$

Inserting $(i)-(iv)$ in the last equation gives

$$\begin{aligned} SO(S(\Theta,t)) &= \sum_{ab\in E(\Theta)} \Big\{ n^{t-2}(n-2k)\sqrt{2}k + 2(n^{t-2}-\psi(t-2))k\sqrt{2k^2+2k+1} \\ &\quad + \sqrt{2}(n^{t-2}+\psi(t-2)(2k+1))(k+1) \Big\}. \\ &= \frac{kn}{2}\Big\{ \sqrt{2}kn^{t-2}(n-2k) + 2(n^{t-2}-\psi(t-2))k\sqrt{2k^2+2k+1} \\ &\quad + \sqrt{2}(n^{t-2}+\psi(t-2)(2k+1))(k+1) \Big\}. \end{aligned}$$

For the multiplicative Sombor index, we conclude

$$\begin{aligned} \prod\nolimits_{SO}(S(\Theta,t)) &= \prod_{ab\in E(S(\Theta,t))} \sqrt{d^2_{S(\Theta,t)}(a)+d^2_{S(\Theta,t)}(b)} \\ &= \prod_{ab\in E(\Theta)} \prod_{i=0}^{1}\prod_{j=0}^{1} \big((d_\Theta(a)+i)^2+(d_\Theta(b)+j)^2\big)^{\frac{1}{2}f_{S(\Theta,t))}(d_\Theta(a)+i,d_\Theta(b)+j)} \\ &= \prod_{ab\in E(\Theta)} \Big\{ \big((d_\Theta(a))^2+(d_\Theta(b))^2\big)^{\frac{1}{2}f_{S(\Theta,t))}(d_\Theta(a),d_\Theta(b))} \\ &\quad \times \big((d_\Theta(a)+1)^2+(d_\Theta(b))^2\big)^{\frac{1}{2}f_{S(\Theta,t))}(d_\Theta(a)+1,d_\Theta(b))} \\ &\quad \times \big((d_\Theta(a))^2+(d_\Theta(b)+1)^2\big)^{\frac{1}{2}f_{S(\Theta,t))}(d_\Theta(a),d_\Theta(b)+1)} \\ &\quad \times \big((d_\Theta(a)+1)^2+(d_\Theta(b)+1)^2\big)^{\frac{1}{2}f_{S(\Theta,t))}(d_\Theta(a)+1,d_\Theta(b)+1)} \Big\}. \end{aligned}$$

Using $(i)-(iv)$ in the above equation, we arrive at

$$\begin{aligned} \prod\nolimits_{SO}(S(\Theta,t)) &= \prod_{ab\in E(\Theta)} \Big\{ \big(2k^2\big)^{\frac{1}{2}n^{t-2}(n-2k)} \big(2k^2+2k+1\big)^{\frac{1}{2}k(n^{t-2}-\psi(t-2))} \\ &\quad \times \big(2k^2+2k+1\big)^{\frac{1}{2}k(n^{t-2}-\psi(t-2))} \\ &\quad \times \big(2k^2+4k+2\big)^{\frac{1}{2}(n^{t-2}+\psi(t-2)(2k+1))} \Big\} \\ &= \big(2k^2\big)^{\frac{k}{4}n^{t-1}(n-2k)} \big(2k^2+2k+1\big)^{\frac{1}{2}k^2n(n^{t-2}-\psi(t-2))} \\ &\quad \times \big(2k^2+4k+2\big)^{\frac{k}{4}n(n^{t-2}+\psi(t-2)(2k+1))}. \end{aligned}$$

□

Corollary 6. *For any $n \geq 4$ and any integer $k \geq 2$, we have*

$$SO(S(C_n,k)) = n\Big\{n^{k-2}(n-4)2\sqrt{2} + 4(n^{k-2}-\psi(k-2))\sqrt{13} + (n^{k-2}+5\psi(k-2))3\sqrt{2}\Big\}$$

and

$$\prod\nolimits_{SO}(S(C_n,k)) = (8)^{\frac{1}{2}n^{k-1}(n-4)}(13)^{2n(n^{k-2}-\psi(k-2))}(18)^{\frac{1}{2}n(n^{k-2}+5\psi(k-2))}.$$

From [5] for $uv \in S(K_p)$ we have

$$f_{S(K_p,n)}(d_{K_p})(u), d_{K_p})(v)+1) = 1,$$
$$f_{S(K_p,n)}(d_{K_p})(u)+1, d_{K_p})(v)) = 1,$$
$$f_{S(K_p,n)}(d_{K_p})(u), d_{K_p})(v)) = 0,$$
$$f_{S(K_p,n)}(d_{K_p})(u)+1, d_{K_p})(v)+1) = \frac{p^n-2p+1}{p-1}.$$

Since $d_{S(K_p,n)}(v) \in \{p-1, p\}$ for any $v \in V(S(K_p,n))$, we get

$$\begin{aligned}
\prod\nolimits_{SO}(S(K_p,n)) &= \prod_{uv\in E(S(K_p,n))} \sqrt{d^2_{S(K_p,n)}(u) + d^2_{S(K_p,n)}(v)} \\
&= \prod_{uv\in E(K_p)} \prod_{i=0}^{1}\prod_{j=0}^{1} \left((d_{K_p}(u)+i)^2 + (d_{K_p}(v)+j)^2\right)^{\frac{1}{2} f_{S(K_p,n)}(d_{K_p}(u)+i, d_{K_p}(v)+j)} \\
&= \prod_{uv\in E(K_p)} \left(2p^2-2p+1\right)\left(2p^2\right)^{\frac{p^n-2p+1}{2(p-1)}} \\
&= \left(2p^2-2p+1\right)^{\frac{(p-1)p}{2}} \left(2p^2\right)^{\frac{p(p^n-2p+1)}{4}}.
\end{aligned}$$

Thus we have:

Corollary 7. *For any p and any integer $n \geq 2$, we have*

$$\prod\nolimits_{SO}(S(K_p,n)) = \left((p-1)^2+p^2\right)^{\frac{(p-1)p}{2}} \left(2p^2\right)^{\frac{p(p^n-2p+1)}{4}}.$$

2.13. Generalized Mycielskian graph

Let $\Theta = (V(\Theta), E(\Theta))$ be a graph with $V(\Theta) = \{v_1, \ldots, v_n\}$. The Mycielskian graph $\mu(\Theta)$ of Θ is the graph with $V(\mu(\Theta)) = V(\Theta) \cup V'(\Theta) \cup \{w\}$ where $V'(\Theta) = \{u_i : v_i \in V(\Theta)\}$, that is,

$$V(\mu(\Theta)) = \{v_1, \ldots, v_n, u_1, \ldots, u_n, w\}$$

and

$$E(\mu(\Theta)) = E(\Theta) \cup \{u_i v_j, v_i u_j : v_i v_j \in E(\Theta)\} \cup \{u_i w : 1 \leq i \leq n\}.$$

Let t be any positive integer. For any integer k with $0 \leq k \leq t$, Suppose $V(\Theta)^k = \{v_1^k, v_2^k, \ldots, v_n^k\}$. The generalization Mycielskian or the t-Mycielskian $\mu_t(\Theta)$ of Θ is the graph with $V(\mu_t(\Theta)) = V(\Theta)^0 \cup (\bigcup_{i=1}^{t} V(\Theta)^i) \cup \{w\}$ and the edge set

$$\begin{aligned}
E(\mu_t(\Theta)) =& E(\Theta) \cup \left(\bigcup_{k=0}^{t-1} \{v_i^k v_j^{k+1} : v_i v_j \in E(\Theta)\}\right) \cup \left(\bigcup_{k=0}^{t-1} \{v_i^{k+1} v_j^k : v_i v_j \in E(\Theta)\}\right) \\
&\cup \{v_i^t w : v_i^t \in V(\Theta)^t\}.
\end{aligned}$$

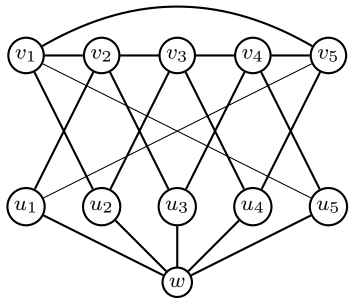

Figure 8. $\mu(C_5)$

Theorem 13. *Let Θ be a graph with $|V(\Theta)| = n$ and $t \geq 2$ be an integer, we have*

$$\prod\nolimits_{SO} \mu_t(\Theta) \leq 2^{(2t-1)|E(\Theta)|} \left(\prod\nolimits_{SO}(\Theta)\right)^{2t-1} \left(4\Delta(\Theta) + (\Delta(\Theta)+1)^2\right)^{|E(\Theta)|} \times \left((\Delta(\Theta)+1)^2 + n^2\right)^{\frac{n}{2}}$$

and

$$\prod\nolimits_{SO} \mu_t(\Theta) \geq 2^{(2t-1)|E(\Theta)|} \left(\prod\nolimits_{SO}(\Theta)\right)^{2t-1} \left(4\delta(\Theta) + (\delta(\Theta)+1)^2\right)^{|E(\Theta)|} \times \left((\delta(\Theta)+1)^2 + n^2\right)^{\frac{n}{2}}.$$

Proof. From [4], for each $v \in V(\mu_t(\Theta))$, we have

$$d_{\mu_t(\Theta)}(v) = \begin{cases} |V(\Theta)| & v = w, \\ 1 + d_\Theta(v_i) & v = v_i^t, \\ 2d_\Theta(v_i) & v = v_i^k, 0 \leq k \leq t-1. \end{cases}$$

Then

$$\prod\nolimits_{SO} \mu_t(\Theta) = \prod_{v_i v_j \in E(\Theta)} \sqrt{d^2_{\mu_t(\Theta)}(v_i^0) + d^2_{\mu_t(\Theta)}(v_j^0)} \times \prod_{k=0}^{t-2} \prod_{\{v_i^k v_j^{k+1} : v_i v_j \in E(\Theta)\}} \sqrt{d^2_{\mu_t(\Theta)}(v_i^k) + d^2_{\mu_t(\Theta)}(v_j^{k+1})} \times \prod_{k=0}^{t-2} \prod_{\{v_i^{k+1} v_j^k : v_i v_j \in E(\Theta)\}} \sqrt{d^2_{\mu_t(\Theta)}(v_i^{k+1}) + d^2_{\mu_t(\Theta)}(v_j^k)}$$

$$\times \prod_{\{v_i^{t-1}v_j^t : v_iv_j \in E(\Theta)\}} \sqrt{d^2_{\mu_t(\Theta)}(v_i^{t-1}) + d^2_{\mu_t(\Theta)}(v_j^t)}$$
$$\times \prod_{\{v_i^t v_j^{t-1} : v_iv_j \in E(\Theta)\}} \sqrt{d^2_{\mu_t(\Theta)}(v_i^t) + d^2_{\mu_t(\Theta)}(v_j^{t-1})}$$
$$\times \prod_{v_i \in V(\Theta)} \sqrt{d^2_{\mu_t(\Theta)}(v_i^t) + d^2_{\mu_t(\Theta)}(w)}.$$

The above equation becomes

$$\begin{aligned}
\prod\nolimits_{SO} \mu_t(\Theta) &= 2^{|E(\Theta)|} \prod_{v_iv_j \in E(\Theta)} \sqrt{d^2_\Theta(v_i) + d^2_\Theta(v_j)} \\
&\quad \times 2^{(t-1)|E(\Theta)|} \left(\prod_{v_iv_j \in E(\Theta)} \sqrt{d^2_\Theta(v_i) + d^2_\Theta(v_j)} \right)^{t-1} \\
&\quad \times 2^{(t-1)|E(\Theta)|} \left(\prod_{v_iv_j \in E(\Theta)} \sqrt{d^2_\Theta(v_i) + d^2_\Theta(v_j)} \right)^{t-1} \\
&\quad \times \prod_{\{v_i^{t-1}v_j^t : v_iv_j \in E(\Theta)\}} \sqrt{4d^2_\Theta(v_i) + (1 + d_\Theta(v_j))^2} \\
&\quad \times \prod_{\{v_i^t v_j^{t-1} : v_iv_j \in E(\Theta)\}} \sqrt{(1 + d_\Theta(v_i))^2 + 4d^2_\Theta(v_j)} \\
&\quad \times \prod_{v_i \in V(\Theta)} \sqrt{(1 + d_\Theta(v_i))^2 + n^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\prod\nolimits_{SO} \mu_t(\Theta) &\leq 2^{(2t-1)|E(\Theta)|} \left(\prod\nolimits_{SO}(\Theta)\right)^{2t-1} \left(4\Delta(\Theta) + (\Delta(\Theta)+1)^2\right)^{|E(\Theta)|} \\
&\quad \times \left((\Delta(\Theta)+1)^2 + n^2\right)^{\frac{n}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\prod\nolimits_{SO} \mu_t(\Theta) &\geq 2^{(2t-1)|E(\Theta)|} \left(\prod\nolimits_{SO}(\Theta)\right)^{2t-1} \left(4\delta(\Theta) + (1+\delta(\Theta))^2\right)^{|E(\Theta)|} \\
&\quad \times \left((1+\delta(\Theta))^2 + n^2\right)^{\frac{n}{2}}.
\end{aligned}$$

□

Corollary 8. *For any integer $t \geq 1$ and any k-regular graph Θ with n vertices,*

$$\prod\nolimits_{SO} \mu_t(\Theta) = (2\sqrt{2}k)^{(2t-1)\frac{kn}{2}} \left(4r + k^2 + 2k + 1\right)^{\frac{kn}{2}} \left(k^2 + 2k + 1 + n^2\right)^{\frac{n}{2}}.$$

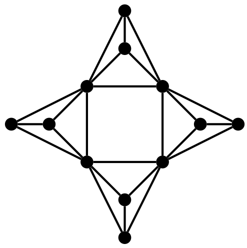

Figure 9. **Ciclo graph** $C_4(K_4)$

Especially,

$$\prod_{SO} \mu_t(C_p) \;=\; (4\sqrt{2})^{(2t-1)p}\,(17)^p\left(9+p^2\right)^{\frac{p}{2}}$$

and

$$\prod_{SO} \mu_t(K_s) \;=\; (2\sqrt{2}(s-1))^{(2t-1)\frac{s(s-1)}{2}}\left(4(s-1)+s^2\right)^{\frac{s(s-1)}{2}}\left(2s^2\right)^{\frac{s}{2}}.$$

2.14. Ciclo graphs

Suppose $\Gamma = (V(\Gamma), E(\Gamma))$ is a graph, $e \in E(\Gamma)$, and $t \geq 3$ is a integer. A t-ciclo graph $C_t(\Gamma, e)$ of Γ with e, is a graph obtained from t copies of Γ and a $t-$cycle C_t by identifying one copy of e with any edge of C_t. If there is no ambiguity, we will represent $C_t(\Gamma, e)$ with $C_t(\Gamma)$.

Theorem 14. *Let Γ be a graph, t be any positive integer, and $e = ab$ be an edge in Γ. We have*

$$tSO(\Gamma) < SO(C_t(\Gamma, e)) \leq tSO(\Gamma) + 2td_\Gamma(a)d_\Gamma(b)$$

and

$$\prod_{SO}(C_t(\Gamma, e)) > \left(\prod_{SO}(\Gamma)\right)^t.$$

Proof. Let t be any positive integer and Γ be a graph and $e = ab$ be an edge in Γ. Let $V(\Gamma) = \{a, b, v_1, \ldots, v_n\}$, $V(C_t) = \{u_1, u_2, \ldots, u_t\}$, $V(\Gamma)^k = \{v_1^k, v_2^k, \ldots, v_n^k\}$ for $1 \leq k \leq t$, and $V(C_t(\Gamma, e)) = V(C_t) \cup (\bigcup_{i=1}^t V(\Gamma)^i)$. We have

$$d_{C_t(\Gamma,e)}(v) = \begin{cases} d_\Gamma(a) + d_\Gamma(b) & \text{if } v = u_i \\ d_\Gamma(v_i) & \text{if } v = v_i^k, 1 \leq k \leq t. \end{cases} \tag{2.4}$$

Then

$$\begin{aligned} SO(C_t(\Gamma,e)) &= \sum_{u_iu_j \in E(C_t)} \sqrt{d^2_{C_t(\Gamma,e)}(u_i) + d^2_{C_t(\Gamma,e)}(u_j)} \\ &+ \sum_{u_iv_j^k \in E(C_t(\Gamma,e))} \sqrt{d^2_{C_t(\Gamma,e)}(u_i) + d^2_{C_t(\Gamma,e)}(v_j^k)} \\ &+ \sum_{v_i^kv_j^k \in E(C_t(\Gamma,e))} \sqrt{d^2_{C_t(\Gamma,e)}(v_i^k) + d^2_{C_t(\Gamma,e)}(v_j^k)}. \end{aligned}$$

Applying (2.4) in the last equation we obtain

$$\begin{aligned} SO(C_t(\Gamma,e)) &= t\sqrt{2}(d_\Gamma(a) + d_\Gamma(b)) \\ &+t \sum_{av_j \in E(\Gamma)} \sqrt{(d_\Gamma(a) + d_\Gamma(b))^2 + d^2_\Gamma(v_j)} \\ &+t \sum_{bv_j \in E(\Gamma)} \sqrt{(d_\Gamma(a) + d_\Gamma(b))^2 + d^2_\Gamma(v_j)} \\ &+t \sum_{v_iv_j \in E(\Gamma)} \sqrt{d^2_\Gamma(v_i) + d^2_\Gamma(v_j)} \end{aligned}$$

and

$$\begin{aligned} SO(C_t(\Gamma,e)) &> t\sqrt{d^2_\Gamma(a) + d^2_\Gamma(b)} + t \sum_{av_j \in E(\Gamma)} \sqrt{d^2_\Gamma(a) + d^2_\Gamma(v_j)} \\ &+t \sum_{bv_j \in E(\Gamma)} \sqrt{d^2_\Gamma(b) + d^2_\Gamma(v_j)} + t \sum_{v_iv_j \in E(\Gamma)} \sqrt{d^2_\Gamma(v_i) + d^2_\Gamma(v_j)} \\ &= tSO(\Gamma). \end{aligned}$$

Also, we have

$$\begin{aligned} SO(C_t(\Gamma,e)) &\leq t\sqrt{d^2_\Gamma(a) + d^2_\Gamma(b)} + t(d_\Gamma(a) + d_\Gamma(b)) \\ &+t \sum_{av_j \in E(\Gamma)} \left(\sqrt{d^2_\Gamma(a) + d^2_\Gamma(v_j)} + d_\Gamma(b)\right) \\ &+t \sum_{bv_j \in E(\Gamma)} \left(\sqrt{d^2_\Gamma(b) + d^2_\Gamma(v_j)} + d_\Gamma(a)\right) \\ &+t \sum_{v_iv_j \in E(\Gamma)} \sqrt{d^2_\Gamma(v_i) + d^2_\Gamma(v_j)} \\ &= tSO(\Gamma) + 2td_\Gamma(a)d_\Gamma(b). \end{aligned}$$

The definition of the multiplicative Sombor index leads us to the conclusion that

$$\begin{aligned}\prod\nolimits_{SO}(C_t(\Gamma,e)) &= \prod_{u_iu_j\in E(C_t)}\sqrt{d^2_{C_t(\Gamma,e)}(u_i)+d^2_{C_t(\Gamma,e)}(u_j)}\\ &\times\prod_{u_iv_j^k\in E(C_t(\Gamma,e))}\sqrt{d^2_{C_t(\Gamma,e)}(u_i)+d^2_{C_t(\Gamma,e)}(v_j^k)}\\ &\times\prod_{v_i^kv_j^k\in E(C_t(\Gamma,e))}\sqrt{d^2_{C_t(\Gamma,e)}(v_i^k)+d^2_{C_t(\Gamma,e)}(v_j^k)}.\end{aligned}$$

Inserting (2.4) in the above equation, we conclude that

$$\begin{aligned}\prod\nolimits_{SO}(C_t(\Gamma,e)) &= \left(\sqrt{2}(d_\Gamma(a)+d_\Gamma(b))\right)^t\\ &\times\left(\prod_{av_j\in E(\Gamma)}\sqrt{(d_\Gamma(a)+d_\Gamma(b))^2+d^2_\Gamma(v_j)}\right)^t\\ &\times\left(\prod_{bv_j\in E(\Gamma)}\sqrt{(d_\Gamma(a)+d_\Gamma(b))^2+d^2_\Gamma(v_j)}\right)^t\\ &\times\left(\prod_{v_iv_j\in E(\Gamma)}\sqrt{d^2_\Gamma(v_i)+d^2_\Gamma(v_j)}\right)^t\\ &> \left(\prod\nolimits_{SO}(\Gamma)\right)^t.\end{aligned}$$

□

Corollary 9. *Suppose Γ is a k-regular graph of order p. Then*

$$SO(C_t(\Gamma)) = t\left(2\sqrt{5}k(k-1)+(\frac{kp}{2}+1)\sqrt{2}\right),$$

and

$$\prod\nolimits_{SO}(C_t(\Gamma)) = \left(2\sqrt{2}k\right)^t\left(\sqrt{5}r\right)^{2(k-1)t}\left(\sqrt{2}k\right)^{(\frac{kp}{2}-2k+1)t}.$$

Especially,

$$\begin{aligned}SO(C_t(C_p)) &= t\left(4\sqrt{2}+4\sqrt{5}+(p-3)\sqrt{2}\right),\\ \prod\nolimits_{SO}(C_t(C_p)) &= \left(4\sqrt{2}\right)^t\left(2\sqrt{5}\right)^{2t}\left(2\sqrt{2}\right)^{(p-3)t},\\ SO(C_t(K_p)) &= t\left(2(p-1)(\sqrt{2}+(p-2)\sqrt{5})+(\frac{p(p-1)}{2}-2p+3)\sqrt{2}\right),\\ \prod\nolimits_{SO}(C_t(K_p)) &= \left(\sqrt{5}\right)^{2(p-2)t}\left(\sqrt{2}\right)^{(\frac{p(p-1)}{2}-2p+6)t}(p-1)^{\frac{p(p-1)}{2}t}.\end{aligned}$$

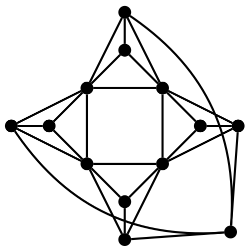

Figure 10. **Estrella graph** $S_4(K_4)$

2.15. Estrella graphs

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph, $e \in E(\Gamma)$, $v \in V(\Gamma)$ such that v is not a vertex of edge e, and t is an positive integer. The union of the complete bipartite graph $K_{1,t}$ and a t-ciclo $C_t(\Gamma, e)$ such that any copy of v is adjacent with each degree one vertex of $K_{1,t}$, is called a t-estrella graph of Γ with edge e and vertex v presented by $S_t(\Gamma, e, v)$. If there is no ambiguity, we will represent $S_t(\Gamma, e, v)$ with $S_t(\Gamma)$.

Theorem 15. *Let Γ be a graph, $e = ab \in E(\Gamma)$, $c \in V(\Gamma)$ such that c is not an endpoint of e, and t is a positive integer. We have*

$$SO(S_t(\Gamma, e, c)) > tSO(\Gamma) + t\sqrt{t^2 + (1 + d_\Gamma(c))^2},$$

$$SO(S_t(\Gamma, e, c)) \leq tSO(\Gamma) + 2td_\Gamma(a)d_\Gamma(b) + td_\Gamma(c) + t\sqrt{t^2 + (1 + d_\Gamma(c))^2},$$

and

$$\prod\nolimits_{SO}(S_t(\Gamma, e, c)) > \left(\prod\nolimits_{SO}(\Gamma)\right)^t \left(\sqrt{t^2 + (1 + d_\Gamma(c))^2}\right)^t.$$

Proof. Let t be any positive integer and Γ be a graph, $e = ab$ be a edge of Γ, and $c \in V(\Gamma)$. Let $V(\Gamma) = \{a, b, c, v_1, v_2, \ldots, v_{n-3}\}$, $V(C_t) = \{u_1, u_2, \ldots, u_t\}$, $V(\Gamma)^k = \{c^k, v_1^k, v_2^k, \ldots, v_{n-3}^k\}$ for $1 \leq k \leq t$, and $V(S_t(\Gamma, e, c)) = \{w\} \cup V(C_t) \cup V(\Gamma)^1 \cup V(\Gamma)^2 \cup \cdots \cup V(\Gamma)^t$ where w is the star vertex and vertices c^k are starneighbor vertices. We get

$$d_{S_t(\Gamma,e,c)}(v) = \begin{cases} d_\Gamma(a) + d_\Gamma(b) & \text{if } v = u_i \\ d_\Gamma(v_i) & \text{if } v = v_i^k, 1 \leq k \leq t, \\ d_\Gamma(c) + 1 & \text{if } v = c^k, 1 \leq k \leq t, \\ t & \text{if } v = w. \end{cases} \quad (2.5)$$

Suppose $c \notin N(a)$ and $c \notin N(b)$. In this case,

$$\begin{aligned} SO(S_t(\Gamma,e,c)) &= \sum_{u_iu_j\in E(C_t)} \sqrt{d^2_{C_t(\Gamma,e)}(u_i) + d^2_{C_t(\Gamma,e)}(u_j)} \\ &+ \sum_{u_iv_j^k\in E(S_t(\Gamma,e,c))} \sqrt{d^2_{C_t(\Gamma,e)}(u_i) + d^2_{C_t(\Gamma,e)}(v_j^k)} \\ &+ \sum_{v_i^kv_j^k\in E(S_t(\Gamma,e,c))} \sqrt{d^2_{C_t(\Gamma,e)}(v_i^k) + d^2_{C_t(\Gamma,e)}(v_j^k)} \\ &+ \sum_{v_i^kc^k\in E(S_t(\Gamma,e,c))} \sqrt{d^2_{C_t(\Gamma,e)}(v_i^k) + d^2_{C_t(\Gamma,e)}(c^k)} \\ &+ \sum_{wc^k\in E(S_t(\Gamma,e,c))} \sqrt{d^2_{C_t(\Gamma,e)}(w) + d^2_{C_t(\Gamma,e)}(c^k)}. \end{aligned}$$

Substituting (2.5) in the last equation, we arrive at

$$\begin{aligned} & SO(S_t(\Gamma,e,c)) \\ = \ & t\sqrt{2}(d_\Gamma(a)+d_\Gamma(b)) + t\sum_{av_j\in E(\Gamma)} \sqrt{(d_\Gamma(a)+d_\Gamma(b))^2 + d^2_\Gamma(v_j)} \\ & +t\sum_{bv_j\in E(\Gamma)} \sqrt{(d_\Gamma(a)+d_\Gamma(b))^2 + d^2_\Gamma(v_j)} + t\sum_{v_iv_j\in E(\Gamma)} \sqrt{d^2_\Gamma(v_i) + d^2_\Gamma(v_j)} \\ & +t\sum_{v_ic\in E(\Gamma)} \sqrt{d^2_\Gamma(v_i) + (1+d_\Gamma(c))^2} + t\sqrt{t^2 + (1+d_\Gamma(c))^2}. \end{aligned}$$

Then

$$\begin{aligned} SO(S_t(\Gamma,e,c)) &> t\sqrt{d^2_\Gamma(a) + d^2_\Gamma(b)} + t\sum_{av_j\in E(\Gamma)} \sqrt{d^2_\Gamma(a) + d^2_\Gamma(v_j)} \\ &+t\sum_{bv_j\in E(\Gamma)} \sqrt{d^2_\Gamma(b) + d^2_\Gamma(v_j)} + t\sum_{v_iv_j\in E(\Gamma)} \sqrt{d^2_\Gamma(v_i) + d^2_\Gamma(v_j)} \\ &+t\sum_{v_ic\in E(\Gamma)} \sqrt{d^2_\Gamma(v_i) + d^2_\Gamma(c)} + t\sqrt{t^2 + (1+d_\Gamma(c))^2} \\ &= tSO(\Gamma) + t\sqrt{t^2 + (1+d_\Gamma(c))^2}. \end{aligned}$$

Also,

$$\begin{aligned}
& SO(S_t(\Gamma,e,c)) \\
\leq\ & t\sqrt{d_\Gamma^2(a)+d_\Gamma^2(b)} + t(d_\Gamma(a)+d_\Gamma(b)) + t\sum_{av_j\in E(\Gamma)}\left(\sqrt{d_\Gamma^2(a)+d_\Gamma^2(v_j)}+d_\Gamma(b)\right) \\
& +t\sum_{bv_j\in E(\Gamma)}\left(\sqrt{d_\Gamma^2(b)+d_\Gamma^2(v_j)}+d_\Gamma(a)\right) + t\sum_{v_iv_j\in E(\Gamma)}\sqrt{d_\Gamma^2(v_i)+d_\Gamma^2(v_j)} \\
& +t\sum_{v_ic\in E(\Gamma)}\left(\sqrt{d_\Gamma^2(v_i)+d_\Gamma^2(c)}+1\right) + t\sqrt{t^2+(1+d_\Gamma(c))^2} \\
=\ & tSO(\Gamma)+2td_\Gamma(a)d_\Gamma(b)+td_\Gamma(c)+t\sqrt{t^2+(1+d_\Gamma(c))^2}.
\end{aligned}$$

For computing the multiplicative Sombor index, we have

$$\begin{aligned}
\prod\nolimits_{SO}(S_t(\Gamma,e,c)) =\ & \prod_{u_iu_j\in E(C_t)}\sqrt{d^2_{C_t(\Gamma,e)}(u_i)+d^2_{C_t(\Gamma,e)}(u_j)} \\
& \times \prod_{u_iv_j^k\in E(S_t(\Gamma,e,c))}\sqrt{d^2_{C_t(\Gamma,e)}(u_i)+d^2_{C_t(\Gamma,e)}(v_j^k)} \\
& \times \prod_{v_i^kv_j^k\in E(S_t(\Gamma,e,c))}\sqrt{d^2_{C_t(\Gamma,e)}(v_i^k)+d^2_{C_t(\Gamma,e)}(v_j^k)} \\
& \times \prod_{v_i^kc^k\in E(S_t(\Gamma,e,c))}\sqrt{d^2_{C_t(\Gamma,e)}(v_i^k)+d^2_{C_t(\Gamma,e)}(c^k)} \\
& \times \prod_{wc^k\in E(S_t(\Gamma,e,c))}\sqrt{d^2_{C_t(\Gamma,e)}(w)+d^2_{C_t(\Gamma,e)}(c^k)}.
\end{aligned}$$

Applying (2.5) in the above equation, yields

$$\begin{aligned}
& \prod\nolimits_{SO}(S_t(\Gamma,e,c)) \\
=\ & \left(\sqrt{2}(d_\Gamma(a)+d_\Gamma(b))\right)^t \times \left(\prod_{av_j\in E(\Gamma)}\sqrt{(d_\Gamma(a)+d_\Gamma(b))^2+d_\Gamma^2(v_j)}\right)^t \\
& \times\left(\prod_{bv_j\in E(\Gamma)}\sqrt{(d_\Gamma(a)+d_\Gamma(b))^2+d_\Gamma^2(v_j)}\right)^t \times \left(\prod_{v_iv_j\in E(\Gamma)}\sqrt{d_\Gamma^2(v_i)+d_\Gamma^2(v_j)}\right)^t \\
& \times\left(\prod_{v_ic\in E(\Gamma)}\sqrt{d_\Gamma^2(v_i)+(1+d_\Gamma(c))^2}\right)^t \times\left(\sqrt{t^2+(1+d_\Gamma(c))^2}\right)^t \\
>\ & \left(\prod\nolimits_{SO}(\Gamma)\right)^t\left(\sqrt{t^2+(1+d_\Gamma(c))^2}\right)^t.
\end{aligned}$$

Suppose $c \in N(a)$ and $c \notin N(b)$ or $c \notin N(a)$ and $c \in N(b)$. We obtain

$$\begin{aligned}
SO(S_t(\Gamma,e,c)) &= \sum_{u_iu_j\in E(C_t)} \sqrt{d^2_{C_t(\Gamma,e)}(u_i)+d^2_{C_t(\Gamma,e)}(u_j)} \\
&+ \sum_{u_iv_j^k\in E(S_t(\Gamma,e,c))} \sqrt{d^2_{C_t(\Gamma,e)}(u_i)+d^2_{C_t(\Gamma,e)}(v_j^k)} \\
&+ \sum_{u_ic^k\in E(S_t(\Gamma,e,c))} \sqrt{d^2_{C_t(\Gamma,e)}(u_i)+d^2_{C_t(\Gamma,e)}(c^k)} \\
&+ \sum_{v_i^kv_j^k\in E(S_t(\Gamma,e,c))} \sqrt{d^2_{C_t(\Gamma,e)}(v_i^k)+d^2_{C_t(\Gamma,e)}(v_j^k)} \\
&+ \sum_{v_i^kc^k\in E(S_t(\Gamma,e,c))} \sqrt{d^2_{C_t(\Gamma,e)}(v_i^k)+d^2_{C_t(\Gamma,e)}(c^k)} \\
&+ \sum_{wc^k\in E(S_t(\Gamma,e,c))} \sqrt{d^2_{C_t(\Gamma,e)}(w)+d^2_{C_t(\Gamma,e)}(c^k)}.
\end{aligned}$$

Equation (2.5) implies that

$$\begin{aligned}
& SO(S_t(\Gamma,e,c)) \\
= {} & t\sqrt{2}(d_\Gamma(a)+d_\Gamma(b)) + t\sum_{av_j\in E(\Gamma)} \sqrt{(d_\Gamma(a)+d_\Gamma(b))^2+d^2_\Gamma(v_j)} \\
& +t\sum_{bv_j\in E(\Gamma)} \sqrt{(d_\Gamma(a)+d_\Gamma(b))^2+d^2_\Gamma(v_j)} \\
& +t\sum_{ac\in E(\Gamma)} \sqrt{(d_\Gamma(a)+d_\Gamma(b))^2+(1+d_\Gamma(c))^2} \\
& +t\sum_{v_iv_j\in E(\Gamma)} \sqrt{d^2_\Gamma(v_i)+d^2_\Gamma(v_j)} + t\sum_{v_ic\in E(\Gamma)} \sqrt{d^2_\Gamma(v_i)+(1+d_\Gamma(c))^2} \\
& +t\sqrt{t^2+(1+d_\Gamma(c))^2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
SO(S_t(\Gamma,e,c)) &> t\sqrt{d^2_\Gamma(a)+d^2_\Gamma(b)} + t\sum_{av_j\in E(\Gamma)} \sqrt{d^2_\Gamma(a)+d^2_\Gamma(v_j)} \\
&+t\sum_{bv_j\in E(\Gamma)} \sqrt{d^2_\Gamma(b)+d^2_\Gamma(v_j)} + t\sum_{ac\in E(\Gamma)} \sqrt{d^2_\Gamma(a)+d^2_\Gamma(c)} \\
&+t\sum_{v_iv_j\in E(\Gamma)} \sqrt{d^2_\Gamma(v_i)+d^2_\Gamma(v_j)} + t\sum_{v_ic\in E(\Gamma)} \sqrt{d^2_\Gamma(v_i)+d^2_\Gamma(c)} \\
&+t\sqrt{t^2+(1+d_\Gamma(c))^2} \\
&= tSO(\Gamma) + t\sqrt{t^2+(1+d_\Gamma(c))^2}
\end{aligned}$$

and

$$\begin{aligned}
& SO(S_t(\Gamma,e,c)) \\
\leq\ & t\sqrt{d_\Gamma^2(a)+d_\Gamma^2(b)}+t(d_\Gamma(a)+d_\Gamma(b))+t\sum_{av_j\in E(\Gamma)}\left(\sqrt{d_\Gamma^2(a)+d_\Gamma^2(v_j)}+d_\Gamma(b)\right) \\
& +t\sum_{bv_j\in E(\Gamma)}\left(\sqrt{d_\Gamma^2(b)+d_\Gamma^2(v_j)}+d_\Gamma(a)\right)+t\sum_{ac\in E(\Gamma)}\left(\sqrt{d_\Gamma^2(a)+d_\Gamma^2(c)}+d_\Gamma(b)+1\right) \\
& +t\sum_{v_iv_j\in E(\Gamma)}\sqrt{d_\Gamma^2(v_i)+d_\Gamma^2(v_j)}+t\sum_{v_ic\in E(\Gamma)}\left(\sqrt{d_\Gamma^2(v_i)+d_\Gamma^2(c)}+1\right) \\
& +t\sqrt{t^2+(1+d_\Gamma(c))^2} \\
=\ & tSO(\Gamma)+2td_\Gamma(a)d_\Gamma(b)+td_\Gamma(c)+t\sqrt{t^2+(1+d_\Gamma(c))^2}.
\end{aligned}$$

Also,

$$\prod\nolimits_{SO}(S_t(\Gamma,e,c)) \ > \ \left(\prod\nolimits_{SO}(\Gamma)\right)^t\left(\sqrt{t^2+(1+d_\Gamma(c))^2}\right)^t.$$

Suppose $c\in N(a)$ and $c\in N(b)$. We conclude

$$\begin{aligned}
SO(S_t(\Gamma,e,c)) \ =\ & \sum_{u_iu_j\in E(C_t)}\sqrt{d^2_{C_t(\Gamma,e)}(u_i)+d^2_{C_t(\Gamma,e)}(u_j)} \\
& +\sum_{u_iv_j^k\in E(S_t(\Gamma,e,c))}\sqrt{d^2_{C_t(\Gamma,e)}(u_i)+d^2_{C_t(\Gamma,e)}(v_j^k)} \\
& +\sum_{u_ic^k\in E(S_t(\Gamma,e,c))}\sqrt{d^2_{C_t(\Gamma,e)}(u_i)+d^2_{C_t(\Gamma,e)}(c^k)} \\
& +\sum_{v_i^kv_j^k\in E(S_t(\Gamma,e,c))}\sqrt{d^2_{C_t(\Gamma,e)}(v_i^k)+d^2_{C_t(\Gamma,e)}(v_j^k)} \\
& +\sum_{v_i^kc^k\in E(S_t(\Gamma,e,c))}\sqrt{d^2_{C_t(\Gamma,e)}(v_i^k)+d^2_{C_t(\Gamma,e)}(c^k)} \\
& +\sum_{wc^k\in E(S_t(\Gamma,e,c))}\sqrt{d^2_{C_t(\Gamma,e)}(w)+d^2_{C_t(\Gamma,e)}(c^k)}
\end{aligned}$$

Applying (2.5) we obtain

$$\begin{aligned}
& SO(S_t(\Gamma,e,c)) \\
= & \; t\sqrt{2}(d_\Gamma(a)+d_\Gamma(b)) + t\sum_{av_j\in E(\Gamma)}\sqrt{(d_\Gamma(a)+d_\Gamma(b))^2+d_\Gamma^2(v_j)} \\
& + t\sum_{bv_j\in E(\Gamma)}\sqrt{(d_\Gamma(a)+d_\Gamma(b))^2+d_\Gamma^2(v_j)} \\
& + 2t\sum_{ac\in E(\Gamma)}\sqrt{(d_\Gamma(a)+d_\Gamma(b))^2+(1+d_\Gamma(c))^2} \\
& + t\sum_{v_iv_j\in E(\Gamma)}\sqrt{d_\Gamma^2(v_i)+d_\Gamma^2(v_j)} + t\sum_{v_ic\in E(\Gamma)}\sqrt{d_\Gamma^2(v_i)+(1+d_\Gamma(c))^2} \\
& + t\sqrt{t^2+(1+d_\Gamma(c))^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
SO(S_t(\Gamma,e,c)) \; > \; & t\sqrt{d_\Gamma^2(a)+d_\Gamma^2(b)} + t\sum_{av_j\in E(\Gamma)}\sqrt{d_\Gamma^2(a)+d_\Gamma^2(v_j)} \\
& + t\sum_{bv_j\in E(\Gamma)}\sqrt{d_\Gamma^2(b)+d_\Gamma^2(v_j)} + t\sum_{ac\in E(\Gamma)}\sqrt{d_\Gamma^2(a)+d_\Gamma^2(c)} \\
& + t\sum_{v_iv_j\in E(\Gamma)}\sqrt{d_\Gamma^2(v_i)+d_\Gamma^2(v_j)} + t\sum_{v_ic\in E(\Gamma)}\sqrt{d_\Gamma^2(v_i)+d_\Gamma^2(c)} \\
& + t\sum_{bc\in E(\Gamma)}\sqrt{d_\Gamma^2(b)+d_\Gamma^2(c)} + t\sqrt{t^2+(1+d_\Gamma(c))^2} \\
= \; & tSO(\Gamma) + t\sqrt{t^2+(1+d_\Gamma(c))^2}
\end{aligned}$$

and

$$\begin{aligned}
& SO(S_t(\Gamma,e,c)) \\
\leq & \; t\sqrt{d_\Gamma^2(a)+d_\Gamma^2(b)} + t(d_\Gamma(a)+d_\Gamma(b)) + t\sum_{av_j\in E(\Gamma)}\left(\sqrt{d_\Gamma^2(a)+d_\Gamma^2(v_j)}+d_\Gamma(b)\right) \\
& + t\sum_{bv_j\in E(\Gamma)}\left(\sqrt{d_\Gamma^2(b)+d_\Gamma^2(v_j)}+d_\Gamma(a)\right) + t\sum_{ac\in E(\Gamma)}\left(\sqrt{d_\Gamma^2(a)+d_\Gamma^2(c)}+d_\Gamma(b)+1\right) \\
& + t\sum_{bc\in E(\Gamma)}\left(\sqrt{d_\Gamma^2(b)+d_\Gamma^2(c)}+d_\Gamma(a)+1\right) + t\sum_{v_iv_j\in E(\Gamma)}\sqrt{d_\Gamma^2(v_i)+d_\Gamma^2(v_j)} \\
& + t\sum_{v_ic\in E(\Gamma)}\left(\sqrt{d_\Gamma^2(v_i)+d_\Gamma^2(c)}+1\right) + t\sqrt{t^2+(1+d_\Gamma(c))^2} \\
= & \; tSO(\Gamma) + 2td_\Gamma(a)d_\Gamma(b) + td_\Gamma(c) + t\sqrt{t^2+(1+d_\Gamma(c))^2}.
\end{aligned}$$

□

Corollary 10. *Suppose Γ is a q-regular graph of order m, $e = ab \in E(\Gamma)$, $c \in V(\Gamma)$ such that c is not an endpoint of e, and t is an positive integer. If $c \notin N(a)$ and $c \notin N(b)$ then*

$$\begin{aligned} SO(S_t(\Gamma, e, c)) &= t\Big(2\sqrt{5}q(q-1) + (\frac{qm}{2} - 3q + 3)\sqrt{2}q \\ &\quad + q\sqrt{2q^2 + 2q + 1} + \sqrt{t^2 + (1+q)^2}\Big) \end{aligned}$$

and

$$\begin{aligned} \prod\nolimits_{SO}(S_t(\Gamma, e, c)) &= \left(2\sqrt{2}q\right)^t \left(\sqrt{5}q\right)^{2t(q-1)} \left(\sqrt{2}q\right)^{t(\frac{qm}{2} - 3q + 1)} \\ &\quad \times \left(\sqrt{2q^2 + 2q + 1}\right)^{tq} \left(\sqrt{t^2 + (1+q)^2}\right)^t. \end{aligned}$$

If $c \in N(a)$ and $c \notin N(b)$ or $c \notin N(a)$ and $c \in N(b)$ then

$$\begin{aligned} SO(S_t(\Gamma, e, c)) &= t\Big(\sqrt{5}q(2q-3) + (\frac{qm}{2} - 3q + 3)\sqrt{2}q + \sqrt{5q^2 + 2q + 1} \\ &\quad + (q-1)\sqrt{2q^2 + 2q + 1} + \sqrt{t^2 + (1+q)^2}\Big) \end{aligned}$$

and

$$\begin{aligned} \prod\nolimits_{SO}(S_t(\Gamma, e, c)) &= \left(2\sqrt{2}q\right)^t \left(\sqrt{5}q\right)^{t(2q-3)} \left(\sqrt{2}q\right)^{t(\frac{qm}{2} - 3q + 1)} \left(\sqrt{5q^2 + 2q + 1}\right)^t \\ &\quad \times \left(\sqrt{2q^2 + 2q + 1}\right)^{t(q-1)} \left(\sqrt{t^2 + (1+q)^2}\right)^t. \end{aligned}$$

If $c \in N(a)$ and $c \in N(b)$ then

$$\begin{aligned} SO(S_t(\Gamma, e, c)) &= t\Big(2\sqrt{5}q(q-2) + (\frac{qm}{2} - 3q + 3)\sqrt{2}q + 2\sqrt{5q^2 + 2q + 1} \\ &\quad + (q-2)\sqrt{q^2 + (1+q)^2} + \sqrt{t^2 + (1+q)^2}\Big) \end{aligned}$$

and

$$\begin{aligned} \prod\nolimits_{SO}(S_t(\Gamma, e, c)) &= \left(2\sqrt{2}q\right)^t \left(\sqrt{5}q\right)^{2t(q-2)} \left(\sqrt{2}q\right)^{t(\frac{qm}{2} - 3q + 1)} \left(\sqrt{4q^2 + (1+q)^2}\right)^{2t} \\ &\quad \times \left(\sqrt{2q^2 + 2q + 1}\right)^{t(q-2)} \left(\sqrt{t^2 + (1+q)^2}\right)^t. \end{aligned}$$

Corollary 11. *For any complete graph of order q we have*

$$\begin{aligned} SO(S_t(K_q)) &= t\Bigg(2\sqrt{5}(q-1)(q-3) + (\frac{(q-1)q}{2} - 3p + 6)\sqrt{2}(q-1) \\ &\quad + 2\sqrt{5q^2 - 8q + 1} + (r-2)\sqrt{2q^2 - 2q + 1} + \sqrt{t^2 + q^2}\Bigg) \end{aligned}$$

and

$$\begin{aligned} \prod\nolimits_{SO}(S_t(K_q)) &= \left(2\sqrt{2}(q-1)\right)^t \left(\sqrt{5}(q-1)\right)^{2t(q-3)} \left(\sqrt{2}r\right)^{t(\frac{(q-1)q}{2} - 3p + 6)} \\ &\quad \times \left(\sqrt{5q^2 - 8q + 1}\right)^{2t} \left(\sqrt{2q^2 - 2q + 1}\right)^{t(q-3)} \left(\sqrt{t^2 + q^2}\right)^t. \end{aligned}$$

3. Conclusion

In this paper, we considered the Sombor index and multiplicative Sombor index on some product of graphs. We study these indices on Cartesian product of two graphs Θ, Ω and we obtained some bounds for Sombor index of Cartesian product of Θ and Ω in terms of the Sombor indices of Θ and Ω. Also, we provided a lower bound for the multiplicative Sombor index of Cartesian product of Θ and Ω in terms of the multiplicative Sombor indeices of Θ and Ω. In special case, when Θ is an r-regular graph and Ω is an r'-regular graph, we finded the Sombor index and multiplicative Sombor index of Cartesian product of Θ and Ω in terms of $r, r', |V(\Theta)|$ and $|V(\Omega)|$. Then, we presented upper and lower bounds for the Sombor index and multiplicative Sombor index of Direct product of Θ and Ω. Especially, we computed $SO(\Theta \times C_p), \prod_{SO}(\Theta \times C_p), SO(\Theta \times K_p), \prod_{SO}(\Theta \times K_p)$ for an arbitrary graph Θ where C_p is a cycle graph of order p and K_p is a complete graph of order p. In following, we gave some bounds for the Sombor index and multiplicative Sombor index of strong product, lexicographic product, sum, corona of Θ and Ω. In the specific case, we calculated $SO(C_n \odot C_m), \prod_{SO}(C_n \odot C_m), SO(C_n \odot K_m), \prod_{SO}(C_n \odot K_m), SO(K_n \odot K_m), \prod_{SO}(K_n \odot K_m)$. Also, we found Sombor index and multiplicative Sombor index for crown graphs, shell graphs, Ice-cream graphs, Helm graphs, flower graphs.
We proved a lower bound for the multiplicative Sombor index of generalized Sierpiński graph of an arbitrary graph Θ in terms of the multiplicative Sombor index of Θ. In particular, when Θ is a triangle-free k-regular graph of order n, we obtained the Sombor index of generalized Sierpiński graph of Θ. Lower and upper bounds for the multiplicative Sombor index of generalized Mycielskian graph of an arbitrary graph Θ are obtained. Especially $\prod_{SO} \mu_t(C_p)$ and $\prod_{SO} \mu_t(K_p)$ were calculated. We showed upper and lower bounds for Sombor index and lower bound for multiplicative Sombor index of t-ciclo graph and t-estrella graph of an arbitrary graph Θ. Especially, we calculated $SO(C_t(C_p)), \prod_{SO}(C_t(C_p)), SO(C_t(K_p)), \prod_{SO}(C_t(K_p))$, $SO(S_t(K_p))$, and $\prod_{SO}(S_t(K_p))$.

Conflict of Interest: The author declares that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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