Communications in Combinatorics and Optimization
Vol. 11, No. 3 (2026), pp. 843-856
https://doi.org/10.22049/cco.2024.29283.1924

CCO
Commun. Comb. Optim.

Research Article

Dominated chromatic number of some kinds of the generalized Helm graphs

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Received: 20 December 2023; Accepted: 14 December 2024
Published Online: 25 December 2024

Abstract: Let G be a simple graph. The dominated coloring of a graph G is a proper coloring of G such that each color class is dominated by at least one vertex. The minimum number of colors needed for a dominated coloring of G is called the dominated chromatic number of G, denoted by $\chi_{dom}(G)$. The current study is devoted to investigate the dominated chromatic number of Helm graphs and some kinds of its the generalizations.

Keywords: dominated chromatic number, domination number, Helm graph.

AMS Subject classification: 05C15, 05C76

1. Introduction

As a challenging issue in graph theory, and its different kinds, including vertex coloring, edge coloring, total coloring, etc., have been investigated by many researchers so far (for more detail see [4]). On the other hand, a *proper vertex coloring* of a graph G contains an application of colors to the vertices of G individually, with the adjacent vertices are being colored differently. Therefore, a proper k-coloring can be considered as a function $c : V(G) \to [k]$ (where $[k] = \{1, 2, \ldots, k\}$), such that $c(u) \neq c(v)$ if u and v are adjacent in G. Moreover, a subset of vertices colored with the same color is called a color class. Thus, finding a proper k-coloring of a graph G contains

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the partitioning of its vertex set into k-independent sets $C_1, C_2, \ldots, C_k$. It should be noted that the chromatic number of G, $\chi(G)$, is the minimum number of colors required for a proper coloring of G.

The domination problem is another important subject of study in graph theory. Berge [3] was the first scholar to introduce the concept of domination number of a graph, referring to it as the coefficient of external, followed by Ore [13] (1962) who entitled the concept as the domination number.

Moreover, coloring and domination are often interrelated, exerting mutual influence on each other. In this regard, some relations between chromatic number and some domination parameters have been proven by Chellali and Volkmann in [5]. Therefore, the current study seeks to investigate a relationship between the dominated chromatic number and the domination number of graph.

Consider $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$ as a simple, undirected, finite and connected graph. A vertex v in a graph G, is assumed to dominate both itself and its neighbors. In other words, v dominates every vertex in its closed neighborhood $N[v]$. Therefore, v dominates $deg_G(v) + 1$ vertices of G. On the other hand, a set S of vertices in G is called a dominating set for (or of) G, if every vertex of G is dominated by some vertices in S. Equivalently, S is a dominating set for G if every vertex of G either belongs to S or is adjacent to some vertices in S. Moreover, the domination number $\gamma(G)$ of G stands as the minimum cardinality of a dominating set for G.

The concept of dominated coloring was introduced in 2015 by Boumediene Merouane et al. [11]. Accordingly, a dominated coloring of a graph G refers to a proper k-coloring with color classes $\{C_1, C_2, \ldots, C_k\}$ where for each $i \in \{1, 2, \ldots, k\}$, there exists a vertex $u \in V(G)$ such that $C_i \subseteq N(u)$. Thus, it could be said that u dominates C_i or to put it differently, C_i is dominated by u. On the other hand, the minimum number of colors required for a dominated coloring of a graph G is called the dominated chromatic number of G and is denoted by $\chi_{dom}(G)$.

Moreover, they showed that there is a relationship between the dominated coloring and total domination number and these two parameters are equal in a triangle-free graph. In 2018, Choopani and et al. studied dominated coloring of some graphs and its relation with some graph operations [7]. In [1, 2], the vertex and edge dominated coloring of some special graphs were studied. Chen proved an application of dominated coloring in social networks [6]. Also, the authors the vertex and edge dominated coloring of some families of $(n, n+1)$-graphs and some graph operations [16, 17].

Some basic definitions and notations of the graph theory which are used in the following sections are as follows: The set of neighbors of a vertex v is called the open neighborhood of v (or simply the neighborhood of v) and is denoted by $N(v)$. Finally, the set $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of v. We remind that two notations $\lceil k \rceil$ and $\lfloor k \rfloor$ stand as the least integer greater than or equal to k and the greatest integer less than or equal to k, respectively.

This paper is organized in three section. In Section 2, a series of preliminary def-

initions and results of some special graphs are presented. In Section 3, first, the domination number of generalized Helm graph is introduced and studied, followed by the investigation of a relationship between the domination number and the dominated coloring number. It should be noted that in some special graphs such as Helm graph, Web graph, Book graph and generalized Book graph, those two parameters are equal. The dominated chromatic number of Helm graph, closed Helm graph, prism graph, Web graph and some of generalized Helm graphs shall also be computed.
All notations and terminologies used in this study are standard and we refer to [18].

2. Some definitions and basic results

In this section, some basic definitions and propositions used in the current study are briefly introduced. First, we start with one known product of two graphs.

Definition 1. [8] The Cartesian product $G\Box H$ is the graph with vertex set $V(G)\times V(H)$ and with edge set $E(G\times H)$ such that $(u_1,v_1)(u_2,v_2)\in E(G\times H)$, whenever $v_1=v_2$ and $u_1u_2\in E(G)$, or $u_1=u_2$ and $v_1v_2\in E(H)$.

Definition 2. A prism graph $Y_{n,m}$ refers to a simple graph produced by the Cartesian product graph $Y_{n,m}=C_n\Box P_m$, where C_n is a cycle with n vertices and P is a path with m vertices.

It should be noted that the *wheel graph* W_n consist of a cycle on n vertices such as $x_1,x_2,\ldots,x_n$ and a vertex z such that z is adjacent to each of $x_1,x_2,\ldots,x_n$.
On the other hand, some vertices and edges could be added to W_n, so that some new graphs could be found.
The Helm graph is one of known graphs which obtains from a wheel as the following definition.

Definition 3. The Helm graph H_n is obtained from the wheel graph W_n by attaching a pendent edge at each vertex of the n-cycle of the wheel.

Helm graphs for $n=3,4,5$ are presented in Figure 1. The following definition introduce a new graph obtained from the Helm graph.

Definition 4. A closed Helm graph CH_n is a graph obtained from a Helm graph H_n by joining each pendant vertex to form a cycle. The graph contains three types of vertices: a vertex of degree n, n vertices of degree 4 and n vertices of degree 3, and $V(CH_n)=V(H_n)$.

In this regard, the domination number of Cartesian products of cycles and paths has been investigated in some studies which are referred to in [12, 14, 15]. For instance, Mrinal Nandi et al. [12], found the following results:

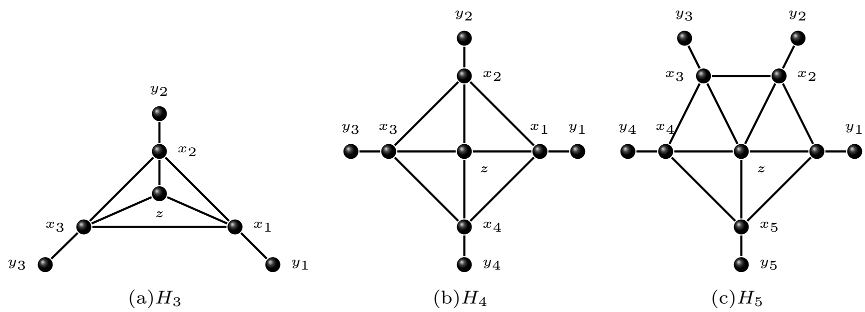


Figure 1. **Helm graphs H_n, for $n = 3, 4, 5$**

Proposition 1. *[12] For $n \geq 3$, we have*

(i) $\gamma(C_n \Box P_2) = \begin{cases} \frac{n}{2} & n \equiv 0 \pmod 4, \\ \lceil \frac{n+1}{2} \rceil & \text{otherwise.} \end{cases}$

(ii) $\gamma(C_n \Box P_3) = \lceil \dfrac{3n}{4} \rceil.$

(iii) $\gamma(C_n \Box P_4) = \begin{cases} n+1 & n = 3, 5, 9, \\ n & \text{otherwise.} \end{cases}$

(iv) $\gamma(C_3 \Box P_5) = 4$, $\gamma(C_4 \Box P_5) = 5$, *and* $\gamma(C_5 \Box P_5) = 7$. *Moreover, for* $n \geq 6$, $n + \lceil \frac{n}{5} \rceil \leq \gamma(C_n \Box P_5) \leq n + \lceil \frac{n}{4} \rceil$.

Also, Polona and Jans [15] determined the value of $\gamma(C_n \Box P_m)$ for $n \leq 11$ and $m \in N$. Furthermore, $\gamma(C_n \Box P_m)$ for $m \leq 7$ and $n \in N$, were investigated. On the other hand, exact values of $\gamma(C_n \Box P_5)$ were given for some n in [14]. The next proposition states the domination number of Helm graphs.

Proposition 2. *[9] If H_n be a Helm graph, then $\gamma(H_n) = n$.*

Here, we present a generalization of the Helm graph as the following.

Definition 5. Let $n \geq 3$ and $m \geq 2$. Consider the Cartesian product $C_n \Box P_m$. Denoted by $H_{n,m}$ a generalized Helm graph is a graph obtained by adding n+1 new vertices to the Cartesian product $C_n \Box P_m$ such that the central vertex z joins all the vertices of the last layer of C_n and n pendant vertices y_i, for $1 \leq i \leq n$, join to the vertices of the first cycle. The vertex set of $H_{n,m}$ is $V(H_{n,m}) = \{x_i^j \mid 1 \leq i \leq n,\ 1 \leq j \leq m\} \cup \{y_i \mid 1 \leq i \leq n\} \cup \{z\}$, where x_i^j is the i^{th} vertex of the j^{th} layer of cycle C_n.

It can be seen that if $m = 1$, then $H_{n,1}$ is H_n. As an example of a generalized Helm graph, $H_{6,4}$ is shown in Figure 2.

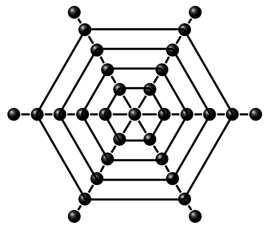


Figure 2. $H_{6,4}$

3. Main Results

In this section, First we obtain the domination number of graph $H_{n,m}$ and determine the dominated chromatic number of graphs H_n, CH_n, $Y_{n,m}$, $H_{n,m}$, Fl_n, H_n^k, Fl_n^k, $H_n^{k,r}$ and $Fl_n^{k,r}$, to be defined later. We start by computing the domination number for the generalized Helm graph.

Theorem 1. *For $n \geq 3$,*

$$\gamma(H_{n,m}) = \begin{cases} n+1, & m = 2,3 \\ 6, & m = n = 4 \\ n+1+\gamma(Y_{n,m-3}), & m \geq 4. \end{cases}$$

Proof. If $m = 2$ *or* 3, then according to definition 5, we have $V(H_{n,3}) = \{x_i^j,\ y_i,\ z \mid 1 \leq i \leq n,\ j = 1,2,3\}$. Let $S = \{x_i^1,\ z \mid 1 \leq i \leq n\}$ be a dominating set. Because S is a dominating set then $\gamma(H_{n,m}) \leq n+1$. Moreover, as the vertex z dominates vertex x_i^m, $1 \leq i \leq n$, S could be considered as a minimum dominating set. Therefore, the vertex z must belong to S. On the other hand, considering the fact only vertices x_i^1 and y_i, $1 \leq i \leq n$, dominate vertex y_i, at least one of the vertices x_i^1 or y_i, must belong to S. Furthermore, since vertex x_i^1 has the maximum degree respect to vertex y_i, therefore should necessarily belong to S. Thus $|S| \geq n+1$ and, consequently, $\gamma(H_{n,m}) \geq n+1$. Therefore, $\gamma(H_{n,m}) = n+1$.

As displayed in Figure 3, if $m = n = 4$, then $\gamma(H_{n,m}) = 6$. Now, let $m \geq 4$ and S is the dominating set as mentioned above. In this case, S_1 is the minimum dominating set of C_n and S_2 is the minimum dominating set of $Y_{n,m-3}$. Accordingly, if $m = 4$, then it can be seen that $S^* = S \cup S_1$ is the dominating set of $H_{n,4}$ and if $m > 4$, then $S^* = S \cup S_2$ is the dominating set of $H_{n,m}$. It should be noted that S^* is the minimum dominating set; otherwise, it contradicts being minimum S or S_1 or S_2. On the other hand, since S, S_1 and S, S_2 are disjointed, thus $|S^*| = |S| + |S_i|$, for $i = 1, 2$, and the proof is completed. We remind that the domination number $Y_{n,m}$ for some values of n and m are obtained in [12, 14, 15]. So we can replace the exact amount of $\gamma(Y_{n,m-3})$ depending on values m and n in the formula given as above for $\gamma(H_{n,m})$. □

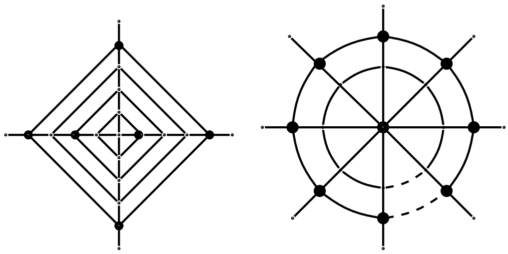


Figure 3. **A dominating set for $H_{4,4}$ and $H_{n,2}$, respectively.**

In the following lemma, we show that the domination number γ is a lower bound for the dominated chromatic number of arbitrary graphs.

Lemma 1. *If G be a graph, then $\chi_{dom}(G) \geq \gamma(G)$.*

Proof. Suppose $\chi_{dom}(G) = k$ and $C_1, C_2, ..., C_k$ are color classes related to dominated coloring. According to the definition of dominated coloring, every class C_i is dominated by a vertex. Therefore, we assume that $x_1, x_2, \ldots, x_k$ dominate the color classes $C_1, C_2, \ldots, C_k$, respectively. Consider $S = \{x_1, x_2, \ldots, x_k\}$. Since $x_1, x_2, \ldots, x_k$ may not all be distinct, so, $|S| \leq k$. On the other hand, every vertex in $V \backslash S$ is placed in a color class that is dominated at least by one vertex in S. Therefore, S could be regarded as a dominating set and $\gamma(G) \leq |S| \leq k = \chi_{dom}(G)$. As can be seen, this bound is sharp. □

There is a group of graphs such as Helm graph, book graph, Stacked Book graph, Web graph and Levi graph whose bound is sharp, the domination number of which is obtained in [9] and [10].

Let us review the definition of a book graph and a Stacked Book graph . The m-book graph is defined as the graph Cartesian product $B_m = S_{m+1} \square P_2$, where S_{m+1} is a star graph and P_2 is path graph on two vertices. The generalization of the book graph to n stacked is the (m, n)-Stacked book graph that denoted by $B_{m,n}$.

The following theorem, shows that the above bound is accurate for the mentioned graphs.

Theorem 2. *For $n \geq 4$, $\chi_{dom}(H_n) = \gamma(H_n) = n$.*

Proof. Let $V(H_n) = \{z, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$. According to Lemma 1, showing that $\chi_{dom}(H_n) \leq \gamma(H_n)$ would suffice. In this regard, a proper coloring with color classes is considered as followes:

$$C_1 = \{y_1, x_2, x_n\}, \quad C_2 = \{y_2, x_1, x_3\}$$
$$C_i = \{y_i, x_{i+1}\}, \quad i = 3, \ldots, n-2$$
$$C_{n-1} = \{y_{n-1}, z\}, \quad C_n = \{y_n\}.$$

It can be seen that every color class C_i is dominated by a vertex x_i, for $i = 1, 2, \ldots, n$. In other words, $C_i \subseteq N(x_i)$. Therefore, this coloring is dominated and we have $\chi_{dom}(H_n) \leq n$. Thus $\chi_{dom}(H_n) = \gamma(H_n) = n$ as required. □

Remark 1. As shown in Figure 4, it can be seen that $\chi_{dom}(H_3) = 4$.

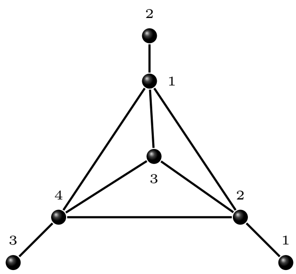


Figure 4. **Dominated coloring of** H_3.

In the following, we will introduce and determine the dominated coloring of some kinds of the Helm graphs, such as the flower graph Fl_n and generalizations of Flower and Helm graphs, including Fl_n^k, $Fl_n^{k,r}$, H_n^k, $H_n^{k,r}$, closed Helm graph CH_n and $H_{n,m}$, respectively.
First, we recall the definition of the flower graph. The flower graph, Fl_n, is obtained from the Helm graph H_n by joining each pendant vertex to the central vertex z. To determine the chromatic number of Fl_n, we state the following theorem.

Theorem 3. *For* $n \geq 3$, $\chi_{dom}(Fl_n) = \chi_{dom}(W_n)$ *+1.*

Proof. For dominated coloring, it is sufficient that we use $\chi_{dom}(W_n)$ colors for coloring of W_n and one new color for all of pendant vertices in the Helm graph. Therefore the result follows. □

Here, we are going to introduce a new generalization of Helm graph which can be formed by replacing the pendant edges in the Helm graph with paths of order k, denoted by H_n^k. The following theorem gives the dominated chromatic number of H_n^k.

Theorem 4. *For $n \geq 3$,*

$$\chi_{dom}(H_n^k) = \begin{cases} \chi_{dom}(W_n), & k = 1 \\ \chi_{dom}(H_n), & k = 2 \\ \chi_{dom}(H_n) + n\chi_{dom}(P_{k-2}), & k \geq 3. \end{cases}$$

Proof. We know $H_n^1 = W_n$ and $H_n^2 = H_n$. So, the result holds. For $k \geq 3$, we assign $\chi_{dom}(P_{k-2})$ colors as dominated coloring to all of paths of order $k-2$. □

Now, we want to determine the dominated chromatic number of the generalized flower graph. This graph is denoted by Fl_n^k and will be obtained from H_n^k by joining the last vertex of each path to the central vertex z. We have the following result about it. If $k = 1$, then $\chi_{dom}(Fl_n^k) = \chi_{dom}(W_n)$ and if $k = 2$, then $\chi_{dom}(Fl_n^k) = \chi_{dom}(Fl_n)$. Therefore for Fl_n^k, we may assume that $k \geq 3$.

Theorem 5. *For $n \geq 3$,*

$$\chi_{dom}(Fl_n^k) = \begin{cases} \chi_{dom}(H_n) + 1, & k = 3 \\ \chi_{dom}(H_n) + (n+1), & k = 4 \\ \chi_{dom}(H_n) + n\chi_{dom}(P_{k-2}), & k \geq 5. \end{cases}$$

Proof. We consider three cases.

Case 1. $k = 3$.
According to the structure of the generalized graph Fl_n^2, for the dominated coloring of this graph, $\chi_{dom}(H_n)$ colors are needed and one new color for the n pendant vertices connected to the central vertex. So the result holds.

Case 2. $k = 4$.
By similar methode in Case 1, we color Helm subgraph with $\chi_{dom}(H_n)$ and we use $(n+1)$ distinct colors for remain vertices. It is clear that this coloring is dominated coloring with minimum number of colors.

Case 3. For $k \geq 5$.
The method is similar to proof of Theorem 4. □

Now, if we put a path P_r instead of the edges adjacent to the centeral vertex z in the graph H_n^k, then another generalization of Helm graph will appear, which is denoted by $H_n^{k,r}$.

Theorem 6. *For $n \geq 3$,*

$$\chi_{dom}(H_n^{k,r}) = \begin{cases} n+1, & r = 3 \text{ and } k = 2 \\ n + \chi_{dom}(K_{1,n}), & r = 4 \text{ and } k = 2 \\ (n+1) + n\chi_{dom}(P_{k-2}), & r = 3 \text{ and } k \geq 3 \\ n + \chi_{dom}(K_{1,n}) + n\chi_{dom}(P_{k-2}), & r = 4 \text{ and } k \geq 3 \\ n + \chi_{dom}(K_{1,n}) + (n-1)\chi_{dom}(P_{r-3}) + & \\ \chi_{dom}(P_{r-4}) + n\chi_{dom}(P_{k-2}), & r \geq 5 \text{ or } k \geq 3. \end{cases}$$

Proof. We consider five cases.

Case 1. $r = 3$ and $k = 2$.
In this case, we color the graph similarly to the Helm graph. The only remaining vertex is the central vertex z, that we color it with one new color.

Case 2. $r = 4$ and $k = 2$.
In this case, the graph $H_n^{2,4}$ also contains a star graph. Therefore, According to Case 1, we color the remaining vertices of the graph with two new colors.

Case 3. $r = 3$ and $k \geq 3$.
Using Case 1 and two vertices of the n paths P_k are common to the Helm subgraph, the result is obtained.

Case 4. $r = 4$ and $k \geq 3$.
It is proved similarly to Case 3.

Case 5. $r \geq 5$ or $k \geq 3$.
According to Cases 1, 2, and Theorem 4, it is sufficient to determine the dominated chromatic number of n path P_r. Since $n - 1$ paths of P_r have three common vertices with the star graph and the cycle graph C_n, hence, we require $(n - 1)\chi_{dom}(P_{r-4})$ new colors for dominated coloring of this subgraph. Because in addition to the three common vertices, we assign the color of the central vertex z to one vertex of this path, so $\chi_{dom}(P_{r-4})$ new colors is needed for the dominanted coloring of this path. Thus, the result is obtained. □

By changing the path P_r instead of the edges adjacent to the center in the graph Fl_n^k, we can obtain another generalization of the flower graph, denote by $Fl_n^{k,r}$.

Since the method of coloring is very similar to Theorem 6, So the proof of the following theorem is omitted.

Theorem 7. *For $n \geq 3$,*

$$\chi_{dom}(Fl_n^{k,r}) = \begin{cases} n+1, & k=2,\ r=3 \\ n+2, & k=2,3,\ r=4 \\ (n+2)+(n-1)\chi_{dom}(P_{r-4})+ & \\ \chi_{dom}(P_{r-5})+n\chi_{dom}(P_{k-3}), & k\geq 3,\ r\geq 5. \end{cases}$$

Now, we consider the closed Helm graph and obtain its dominated chromatic number. For $n = 3, 4, 5, 6$ dominated coloring method of CH_n is shown in Figure 5. However, for $n \geq 7$, the following theorem is considered.

Theorem 8. *For $n \geq 7$,* $\chi_{dom}(CH_n) = \begin{cases} \lceil \frac{n-1}{2} \rceil + 3, & n \text{ is even} \\ \lfloor \frac{n-3}{2} \rfloor + 4, & n \text{ is odd.} \end{cases}$

Proof. The theorem is proved by considering two cases and some subcases as follows:

Case 1. n is even.
We color W_n with three colors such that the vertices x_i, $1 \leq i \leq n$, receive color 1 if i is odd and, color 2 if i is even, and vertex z receives color 3. For coloring the second layer of cycle C_n, two subcases are considered as follows.

Subcases 1. If $n \equiv 0 \pmod 4$, then color $2i+2$ is assigned to vertices y_{4i-3} and y_{4i-1}, color $2i+3$ is assigned to vertices y_{4i-2} and y_{4i}, $i = 1, 2, \ldots, \lfloor \frac{n}{4} \rfloor$, and color 3 is assigned to vertex y_n. For $1 \leq i \leq n$, the color classes 1, 2 are dominated by vertex z and the color class 3 is dominated by the vertex x_i. For $i = 1, 2, \ldots, \lfloor \frac{n}{4} \rfloor$, the color classes $2i+2$ and $2i+3$ are dominated by vertices x_{4i-2} and x_{4i-1}, respectively, and we obtain the result.
Subcase 2. If $n \equiv 2 \pmod 4$, then for $i = 1, 2, \ldots, \lfloor \frac{n-2}{4} \rfloor$, we color the vertices y_{4i-3}, y_{4i-1}, y_{4i-2} and y_{4i} similar to subcase 1 and we assign colors $\lceil \frac{n-1}{2} \rceil + 3$ and 3 to the remaining two vertices y_{n-1} and y_n, respectively. It is easy to check that this coloring is dominated and the result is obtained.

Case 2. n is odd.
For $1 \leq i \leq n-1$, we color the vertices x_i the same as case 1 and the vertex x_n with color 3 and the central vertex z with color 4. Now, for coloring the second layer of cycle C_n, we consider two subcases.

Subcase 1. If $n \equiv 1 \pmod 4$, then color 3 is assigned to the vertices y_1 and y_{n-1}, color 4 is assigned to the vertex y_{n-2} and color 5 is assigned to the vertices y_n and y_2. On the other hand, for $1 \leq i \leq \lfloor \frac{n}{4} \rfloor - 1$, color $4i+2$ is assigned to the vertices y_{4i-1}, y_{4i+1} and color $4i+3$ is assigned the vertices y_{4i} and y_{4i+2}.

Subcase 2. If $n \equiv 3 \pmod 4$, then colors 3 and 4 are assigned to the vertices y_1, y_{n-1} and y_n respectively. Also, $\lfloor \frac{n-3}{4} \rfloor$ new colors are required for coloring the remaining vertices of this cycle that for $1 \leq i \leq \lfloor \dfrac{n}{4} \rfloor$ color $2i+3$ is assigned to the vertices y_{4i-2} and y_{4i} and color $2i+4$ is assigned to the vertices y_{4i-1} and y_{4i+1}. Hence, the proof is completed. □

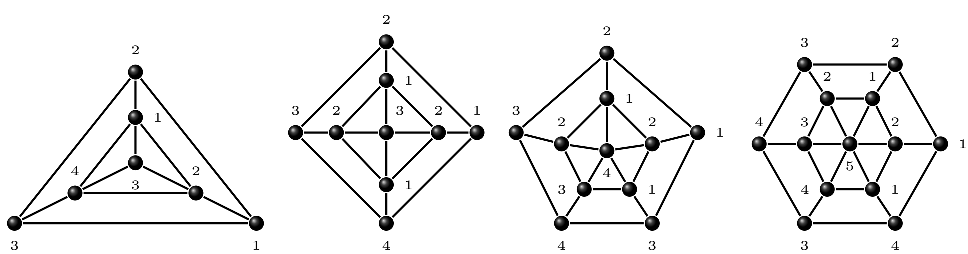


Figure 5. **Dominated colorings of CH_3, CH_4, CH_5 and CH_6, respectively.**

To obtain the dominated chromatic number of the generalized Helm, first, the dominated chromatic number of $Y_{n,m}$ is calculated which is stated through the following lemma.

Lemma 2. *(i) For $n \geq 4$,*

$$\chi_{dom}(Y_{n,2}) = \begin{cases} \lceil \frac{2n}{3} \rceil + 1, & n \equiv 1 \pmod 3 \\ \lceil \frac{2n}{3} \rceil, & \text{otherwise.} \end{cases}$$

(ii) For $n, m \geq 3$,

$$\chi_{dom}(Y_{n,m}) = \begin{cases} \lfloor \frac{m}{3} \rfloor n, & m \equiv 0 \pmod 3 \\ \lfloor \frac{m}{3} \rfloor n + \chi_{dom}(C_n), & m \equiv 1 \pmod 3 \\ \lfloor \frac{m}{3} \rfloor n + \chi_{dom}(Y_{n,2}), & \text{otherwise.} \end{cases}$$

Proof. (i) The way of dominated coloring of $Y_{n,2}$ is displayed in Figure 6. We should note that there is a formula for dominated chromatic number of $Y_{n,2}$ as given in [1] for all $n \geq 4$, but it seems that is not correct in general. For example, if $n = 13$, then formula does not work.

(ii) Let $m \equiv 0 \pmod 3$. Suppose $V(Y_{n,m}) = \{x_i^j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$, where x_i^j is the i^{th} vertex from the j^{th} layer of cycle C_n. Consider a proper coloring of $Y_{n,3}$ in which the color classes are as the following.

$C_1 = \{x_n^2, x_1^1, x_1^3, x_2^2\}$, $C_2 = \{x_1^2, x_2^1, x_2^3, x_3^2\}$, $C_i = \{x_i^j, x_{i+1}^2 \mid j = 1, 3\}$, $3 \leq i \leq n-2$ and $C_i = \{x_i^j \mid j = 1, 3\}$, $i = n-1, n$.

It can be seen that the color classes C_i are dominated by vertex x_i, for $1 \leq i \leq n$. The result is obtained by continuing this process. We shall show that we cannot have a dominated coloring with less than n. According to the given dominated coloring, we have $|C_1| = |C_2| = 4$, $|C_i| = 3$, $3 \leq i \leq n-2$, and $|C_{n-1}| = |C_n| = 2$. Now, it is enough to show that there is no coloring with less than classes. Because otherwise $|C_i| > 4$, $i = 1, 2$ or $|C_i| > 3$, $i = 3, \ldots, n-2$ or $|C_i| > 2$, $i = n-1, n$, and this is a contradiction, Note that two adjacent vertices cannot be in the same color class or there is no vertex that dominated these color classes. Hence, $\chi_{dom}(Y_{n,3}) \geq n$. If $m \equiv 1$ *or* $2 \pmod 3$, then by adding one or two cycle of C_n to the above graph, we need $\chi_{dom}(C_n)$ or $\chi_{dom}(Y_{n,2})$ new colors for dominated coloring of this graph, and by continuing this process, the proof is completed. □

Denoted by $\mathbf{W}_n$, a Web graph is a prism graph $Y_{n,3}$ where the edges of the outer cycle are removed. As an example of a Web graph, $\mathbf{W}_6$ is shown in Figure 7. The following result can be obtained for the general case, $\mathbf{W}_n$, $n \geq 3$.

Proposition 3. *For $n \geq 3$, $\chi_{dom}(Y_{n,3}) = \chi_{dom}(\mathbf{W}_n) = n$.*

Proof. It is very similar to the proof of the previous lemma of part (ii). □

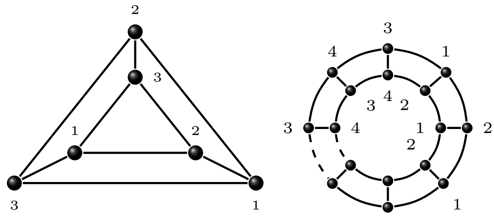
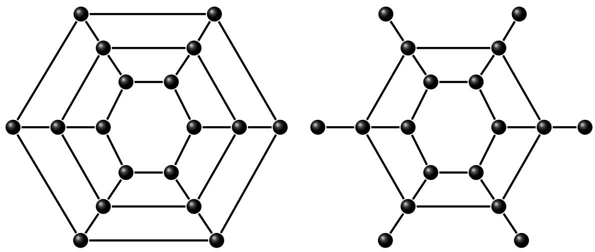


Figure 6. **dominated colorings of $Y_{n,2}$ for $n = 3$ and $n \geq 4$ respectively.**



(a)prism graph $Y_{6,3}$ (b)Web graph $\mathbf{W}_6$.

Figure 7. **Dominated colorings of $Y_{6,3}$ and W_6**

Here, the dominated coloring of the generalized Helm graph is computed. As previously mentioned in Section 1, $H_{n,m}$ refers to m cycles of length n such that each vertex of each cycle joins the corresponding vertex of the later cycle with a path of length 1. Moreover, a pendant edge joins each vertex of the first cycle, and a central vertex such as z joins the vertices of the last cycle.

Lemma 3. *For $n \geq 3$,*

$$\chi_{dom}(H_{n,m}) = \begin{cases} n+1, & m = 2 \\ n+1+\chi(C_n), & m = 3. \end{cases}$$

Proof. Let $V(H_{n,2}) = V(H_n) \cup V(C_n^2)$, where C_n^2 is the second layer from cycle C_n. If $m = 2$, according to the dominated color classes introduced in Theorem 2, the dominated color classes of Helm graph $H_{n,2}$ would be obtained as follows by adding a cycle to the Helm graph H_n:

$$C_1 \cup \{x_1^2\},\ C_2 \cup \{x_2^2\},\ C_i \cup \{x_i^2\},\ \ 3 \leq i \leq n-2,$$

$$C_i = \{y_i, x_i^2\},\ i = n-1, n \textit{ and } C_{n+1} = \{z\}.$$

Similar to Theorem 2, it is proved that this coloring is dominated. Now, if $m = 3$, then by adding two cycles to the Helm graph H_n, we need $\chi(C_n)$ new colors for its dominated coloring. □

Finally, the following result is suggested by considering Theorems 2, Lemma 2 and Lemma 3, we state the following result:

Theorem 9. *For $n \geq 3$ and $m \geq 4$,*

$$\chi_{dom}(H_{n,m}) = \begin{cases} n + \chi_{dom}(Y_{n,m-2}), & n = 4,5 \text{ and } m \equiv 1 \pmod 3 \\ n + 1 + \chi_{dom}(Y_{n,m-2}), & \text{otherwise.} \end{cases}$$

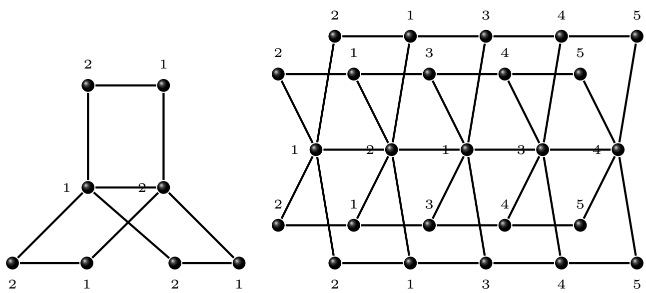


Figure 8. Dominated colorings of book graph B_3 and Stacked book graph $B_{4,5}$, respectively.

According to Figures 8 the following result is obtained.

Theorem 10. *For $m \geq 3$ $n \geq 2$, $\chi_{dom}(B_m) = 2$ and $\chi_{dom}(B_{m,n}) = n$.*

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] S. Alikhani and M.R. Piri, *On the dominated chromatic number of certain graphs*, Trans. Comb. **9** (2020), no. 4, 217–230. https://doi.org/10.22108/toc.2020.119361.1675.

[2] ______, *Introduction to dominated edge chromatic number of a graph*, Opuscula Math. **41** (2021), no. 2, 245–257. https://doi.org/10.7494/OpMath.2021.41.2.245.

[3] C. Berge, *The Theory of Graphs*, Courier Corporation, 2001.
[4] G. Chartrand and P. Zhang, *Chromatic Graph Theory*, Chapman and Hall/CRC, 2019.
[5] M. Chellali and L. Volkmann, *Relations between the lower domination parameters and the chromatic number of a graph*, Discrete Math. **274** (2004), no. 1-3, 1–8. https://doi.org/10.1016/S0012-365X(03)00093-1.
[6] Y.H. Chen, *The dominated coloring problem and its application*, International Conference on Computational Science and Its Applications (B. Murgante, S. Misra, A.M.A. C. Rocha, C. Torre, J.G. Rocha, M.I. Falcão, D. Taniar, B.O. Apduhan, and O. Gervasi, eds.), Springer, 2014, pp. 132–145.
[7] F. Choopani, A. Jafarzadeh, A. Erfanian, and D.A. Mojdeh, *On dominated coloring of graphs and some Nordhaus–Gaddum-type relations*, Turkish J. Math. **42** (2018), no. 5, 2148–2156.
https://doi.org/10.3906/mat-1710-97.
[8] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, Wiley, 2000.
[9] A.A. Khalil, *Determination and testing the domination numbers of Helm graph, web graph and Levi graph using matlab*, J. Educ. Sci. **24** (1999), no. 2, 103–116.
[10] A.A. Khalil and O.A. Khalil, *Determination and testing the domination numbers of tadpole graph, book graph and stacked book graph using MATLAB*, College Of Basic Education Researches Journal **10** (2010), no. 1, 491–506.
[11] H.B. Merouane, M. Haddad, M. Chellali, and H. Kheddouci, *Dominated colorings of graphs*, Graphs Combin. **31** (2015), no. 3, 713–727.
https://doi.org/10.1007/s00373-014-1407-3.
[12] M. Nandi, S. Parui, and A. Adhikari, *The domination numbers of cylindrical grid graphs*, Appl. Math. Comput. **217** (2011), no. 10, 4879–4889.
https://doi.org/10.1016/j.amc.2010.11.019.
[13] O. Ore, *Theory of Graphs*, American Mathematical Society Colloquium Publications, 1962.
[14] P. Pavlič and J. Žerovnik, *Roman domination number of the Cartesian products of paths and cycles*, Electron. J. Comb. **19** (2012), no. 3, Article Number: P19 https://doi.org/10.37236/2595.
[15] ———, *A note on the domination number of the Cartesian products of paths and cycles*, Kragujevac J. Math. **37** (2013), no. 2, 275–285.
[16] F. Poryousefi, A. Erfanian, and M. Nasiri, *Dominated chromatic number of semi-strong, strong and modular product of graphs*, Submitted.
[17] ———, *Dominated chromatic number of some kinds of $(n, n+1)$ graphs*, Submitted.
[18] D.B. West, *Introduction to Graph Theory*, Prentice hall Upper Saddle River, 2001.