Communications in Combinatorics and Optimization
Vol. 11, No. 3 (2026), pp. 923-939
https://doi.org/10.22049/cco.2024.29824.2173

CCO
Commun. Comb. Optim.

Research Article

A new approach for solving multi-objective interval-valued variational problems and its applications

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Received: 11 June 2024; Accepted: 19 November 2024
Published Online: 26 November 2024

Abstract: This study focuses on one of the methods for solving a nonlinear multiobjective convex interval-valued variational problem. Namely, the weighting method is used to find its weakly LU-efficient solution and LU-efficient solution. Therefore, the weighted variational problem is introduced for the given nonlinear multiobjective interval-valued variational problem. Then, under appropriate convexity assumptions, the equivalance between a (weakly) LU-efficient solution of the original nonlinear multiobjective interval-valued variational problem and an optimal solution of its associated weighting variational problem is established.

Keywords: multiobjective interval-valued variational problem; weighting method, (weakly) LU-efficient solution; convex interval-valued functional.

AMS Subject classification: 90C25, 90C29, 90C30, 34K35

1. Introduction

In the field of operations research, we typically address extremum problems, assuming that the components are deterministic real numbers. However, in the real world, there are extreme problems in many areas of human activity, such as industry, engineering, physics, machine learning, data analytics, finance and risk management, medicine, control theory, etc. In these cases, the data is much less accurate and vague, and optimization problems with unknown parameters make the modeling of those problems uncertain.

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There are numerous approaches to investigating and solving uncertain optimization problems. One of them is interval optimization, in which intervals are used rather than real numbers. Interval optimization problems specifically address problems when uncertain parameters are expressed as closed intervals in both the objective and constraint sets of functions. Using order relations on real numbers, significant results related to the interval optimization problems are obtained by Moore [23, 24], and later by Moore et al. [25]. Charnes et al. [9] considered the linear programming problems in which the right-hand sides of linear inequality constraints were taken as closed intervals. The interval optimization problem, addressed by Ishibuchi and Tanaka [15], has an interval-valued objective function that is free from interval uncertainty in the set of constraints. After developing the concept of interval analysis, many researchers discussed various approaches to solving interval-valued optimization problems (see, for example, [2, 3, 5, 6, 16, 21, 29, 35],and others). Very recently, Antczak [6] used the weighting method for finding solutions to the considered nonlinear vector optimization problem with the multiple interval-valued objective function. He demonstrated that an optimal solution to the associated noninterval scalar weighting optimization problem is equivalent to a (weak) Pareto solution to the original nonlinear interval-valued multiobjective programming problem, under the appropriate convexity assumptions. In the past few decades, variational problems have attracted the interest of many researchers due to their applications, for example, to engineering problems (see, for example, [27, 28]). Therefore, many authors proved optimality and duality results for various classes of variational problems (see, for example, [1, 7, 26, 30], and many others). The relationship between mathematical programming problems and variational problems was explored by Hanson [14]. Subsequently, there has been some interest in the literature regarding variational programming problems. A significant amount of literature has been dedicated to investigating variational problems. This research primarily focuses on finding solutions to optimization problems by examining optimality conditions, analyzing the properties of the classes of functionals involved, and exploring the connection between mathematical programming and variational problems. Arana et al. [8] introduced the concept of L-KT-pseudoinvexity for variational problems and proved that it is a necessary and sufficient condition for all Kuhn-Tucker points to be optimal solutions. Jayswal et al. [19] derived the concept of $(F, \alpha, \rho, \theta)$-convexity for variational problems and studied the second-order duality results for the same. Jayswal et al. [17] established an equivalence between a variational problem and its modified variational problem with the η-objective function under the invexity hypothesis. Using the so-called η-approximation method introduced by Antczak for differentiable vector optimization problems, Jha et al. [20] proved several duality results for a class of nonconvex variational problems. Very recently, Khatri and Prasad [22] also used the η-approximated method, this time to substantiate duality results for a class of nonconvex fractional variational problems.

In recent years, interval-valued optimization has also been applied for various variational problems. In [11], Debnath and Pokharna derived necessary and sufficient optimality conditions for the considered interval-valued variational optimization problem under B-(p, r)-invexity assumptions. Jayswal and Baranwal [18] introduced a new class of variational inequality with its weak and split forms to obtain an LU-optimal solution to the multi-dimensional interval-valued variational problem, which is a wider class of interval-valued programming problem in operations research. They demonstrated the connection between the optimal solutions and variational inequalities of the interval-valued variational problem for multi-dimensional problems by utilizing the idea of (strict) LU-convexity. In [32], Treanţă defined the KT-pseudoinvex variational control problems with interval-values that include multiple integral objective functionals. In [31], Treanţă investigated the connections between LU-optimal solutions of the considered interval-valued variational control problem and the saddle points associated with an interval-valued Lagrange functional corresponding to a modified interval-valued variational control problem. Ciontescu and Treanţă [10] developed the relationships between a class of interval-valued optimization problems and the associated inequalities. Treanţă and Ciontescu [33] have focused on results related to solutions for interval-valued optimal control problems with generalized invariant convex (invex) functionals.

Motivated by the above works, we use the weighting method introduced by Antczak [6], in the case of vector interval-valued optimization problems for solving the considered multiobjective interval-valued variational problem. Thus, for the aforesaid multiobjective interval-valued variational problem, we construct an associated (scalar) weighting variational problem. Then, under appropriate convexity hypotheses, we prove the equivalence between weakly LU-efficient solutions and LU-efficient solutions in the original multiobjective interval-valued variational problem and a minimizer in its associated scalar weighting variational problem constructed in the weighting method. The results established in the paper are illustrated by suitable examples of convex multiobjective interval-valued variational problems.

2. Preliminaries and notations

Let R^r be the Euclidean space of dimension r, and its nonnegative orthant is denoted by R^r_+. For any vectors $\rho = (\rho_1, \rho_2, ..., \rho_r)^T$ and $\delta = (\delta_1, \delta_2, ..., \delta_r)^T$ in R^r, where $\Lambda_r = \{1, 2, ..., r\}$ be an index set, then we define:

i. $\rho = \delta \Longleftrightarrow \rho_i = \delta_i, \forall i \in \Lambda_r;$

ii. $\rho < \delta \Longleftrightarrow \rho_i < \delta_i, \forall i \in \Lambda_r;$

iii. $\rho \leqq \delta \Longleftrightarrow \rho_i \leqq \delta_i, \forall i \in \Lambda_r;$

iv. $\rho \leq \delta \Longleftrightarrow \rho \leqq \delta$ and $\rho \neq \delta.$

This section focuses on basic definitions, noations, and basic calculus in interval mathematics, which are used in the following sections:

Consider $\mathcal{I}$ to be a closed interval in the real numbers, represented by $\left[\underline{r}^L, \overline{r}^U\right]$, and let X be a subset of $\mathbb{R}^n$. Furthermore, consider $\varphi = \left(\varphi^1, \ldots, \varphi^r\right) : \mathcal{I} \times X \times X \to \mathbb{R}^r$ and $g : \mathcal{I} \times X \times X \to \mathbb{R}^m$ as functions that possess continuous differentiability in relation to each of their inputs. The functional $\varphi\left(t, \kappa(t), \dot{\kappa}(t)\right)$ is defined for an independent variable t in the interval $\mathcal{I}$, where $\kappa : \mathcal{I} \to \mathbb{R}^n$ is a n-dimensional piecewise smooth function of t, and $\dot{\kappa}(t)$ represents the derivative of $\kappa(t)$ with respect to t in the interval $\left[\underline{r}^L, \overline{r}^U\right]$. In order to make the notation less complex, we will denote $\kappa(t)$ and $\dot{\kappa}(t)$ as κ and $\dot{\kappa}$, respectively. Let φ^i, where $i = 1, \ldots, k$, represent a function. The partial derivatives of φ^i with respect to t, κ, and $\dot{\kappa}$ are denoted as φ^i_t, φ^i_κ, and $\varphi^i_{\dot{\kappa}}$, respectively. Furthermore, φ^i_κ and $\varphi^i_{\dot{\kappa}}$ are defined as the vectors $\left(\frac{\partial \varphi^i}{\partial \kappa_1}, \ldots, \frac{\partial \varphi^i}{\partial \kappa_n}\right)$ and $\varphi^i_{\dot{\kappa}} = \left(\frac{\partial \varphi^i}{\partial \dot{\kappa}_1}, \ldots, \frac{\partial \varphi^i}{\partial \dot{\kappa}_n}\right)$ respectively. Similarly, the first-order partial derivatives g_κ and $g_{\dot{\kappa}}$ of the vector function g can be expressed using matrices with m rows instead of a single row.

Consider the set $I(\mathbb{R})$, which consists of all bounded and closed intervals in the real numbers. In this study, the term "closed interval" refers to a set $\mathcal{R}$ that is both closed and bounded in real numbers. When referring to a closed interval, we represent it as $\mathcal{R} = [\underline{r}^L, \overline{r}^U]$, where $\underline{r}^L$ and $\overline{r}^U$ represent the lower and upper boundaries of $\mathcal{R}$, respectively. To clarify, if $\mathcal{R} = [\underline{r}^L, \overline{r}^U] \in I(R)$, then $\mathcal{R} = [\underline{r}^L, \overline{r}^U] = \left\{u \in \mathbb{R} : \underline{r}^L \leqq u \leqq \overline{r}^U\right\}$. If $\underline{r}^L = \overline{r}^U = r$, then $\mathcal{R} = [r, r] = r$ is a real number. Let $\mathcal{R} = [\underline{r}^L, \overline{r}^U]$ and $\mathcal{P} = [\underline{p}^L, \overline{p}^U]$. Then, by definition, we have:

(a) $\mathcal{R} + \mathcal{P} = \{r + p : r \in \mathcal{R} \text{ and } p \in \mathcal{P}\} = [\underline{r}^L + \underline{p}^L, \overline{r}^U + \overline{p}^U]$,

(b) $-\mathcal{R} = \{-r : r \in \mathcal{R}\} = [-\overline{r}^U, -\underline{r}^L]$,

(c) $\mathcal{R} - \mathcal{P} = \mathcal{R} + (-\mathcal{P}) = \{r - p : r \in \mathcal{R} \text{ and } p \in \mathcal{P}\} = [\underline{r}^L - \overline{p}^U, \overline{r}^U - \underline{p}^L]$,

(d) $\tau + \mathcal{R} = \{\tau + r : r \in \mathcal{R}\} = [\tau + \underline{r}^L, k + \overline{p}^U]$, where τ is a real number,

(e) $\tau\mathcal{R} = \begin{cases} \left[\tau\underline{r}^L, \tau\overline{r}^U\right] & \text{if } \tau > 0, \\ \left[\tau\overline{r}^U, \tau\underline{r}^L\right] & \text{if } \tau \leqq 0, \end{cases}$ where τ is a real number.

For more details on the topic of interval analysis, we refer to Moore [23], Moore et al. [24], and Alefeld and Herzberger [4].

Interval mathematics commonly uses an order relation to establish a ranking among interval numbers. This connection indicates that one interval number is superior to another, but it does not imply that one is larger than the other. For $\mathcal{R} = [\underline{r}^L, \overline{r}^U]$ and $\mathcal{P} = [\underline{p}^L, \overline{p}^U]$, we write

$$\mathcal{R} \leqq_{LU} \mathcal{P} \text{ if and only if } \begin{cases} \underline{r}^L \leqq \underline{p}^L \\ \overline{r}^U \leqq \overline{p}^U \end{cases}.$$

It indicates that $\mathcal{R}$ is inferior to $\mathcal{P}$, or $\mathcal{P}$ is superior to $\mathcal{R}$. The fact that $\leqq_{LU}$ is a partial ordering on $I(\mathbb{R})$ is readily apparent. This implies that $\mathcal{R}$ is inferior to $\mathcal{P}$, or

$\mathcal{P}$ is superior to $\mathcal{R}$. It is evident that " $\leqq_{LU}$ " constitutes a partial ordering on $I(\mathbb{R})$. Moreover, $\mathcal{R} <_{LU} \mathcal{P}$ can be expressed if and only if $\mathcal{R} \leqq_{LU} \mathcal{P}$ and $\mathcal{R} \neq \mathcal{P}$. Similarly,

$$\mathcal{R} <_{LU} \mathcal{P} \text{ if and only if} \begin{cases} \underline{r}^L < \underline{p}^L \\ \overline{r}^U \leqq \overline{p}^U \end{cases} \text{or} \begin{cases} \underline{r}^L \leqq \underline{p}^L \\ \overline{r}^U < \overline{p}^U \end{cases} \text{or} \begin{cases} \underline{r}^L < \underline{p}^L \\ \overline{r}^U < \overline{p}^U \end{cases}.$$

An interval-valued vector, denoted as $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_p)$, is defined as a vector where each component $\mathcal{R}_i$ is an interval $[\underline{r}_i^L, \overline{r}_i^U]$. The interval for $i \in \Lambda_p$ is closed. Consider two interval-valued vectors, $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_p)$ and $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_p)$. We will use the notation $\mathcal{R} \leqq_{LU} \mathcal{P}$ to indicate that $\mathcal{R}_i \leqq_{LU} \mathcal{P}_i$, for all $i \in \Lambda_p$. Similarly, we will use $\mathcal{R} <_{LU} \mathcal{P}$ to indicate that $\mathcal{R}_i \leqq_{LU} \mathcal{P}_i$, for all $i \in \Lambda_p$, and $\mathcal{R}_{i^*} <_{LU} \mathcal{P}_{i^*}$, for at least one $i^* \in \Lambda_p$.
Now, we will investigate the convexity of an interval-valued functional. In particular, we use the very straightforward concept of convexity introduced by Wu [34].

Definition 1. Let $\varphi : \mathcal{I} \times X \times X \to I(\mathbb{R})$ be defined by $\int_a^b \varphi\left(t, \kappa, \dot{\kappa}\right) dt = \left[\int_a^b \varphi^L\left(t, \kappa, \dot{\kappa}\right) dt, \int_a^b \varphi^U\left(t, \kappa, \dot{\kappa}\right) dt\right]$ with φ^L, $\varphi^U : \mathcal{I} \times X \times X \to \mathbb{R}$ continuously differentiable functionals. Then, $\int_a^b \varphi(t, \cdot, \cdot)\, dt$ is said to be a convex interval-valued functional on X if the inequalities

$$\int_a^b \varphi^L\left(t, \kappa, \dot{\kappa}\right) dt - \int_a^b \varphi^L\left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt \geqq$$

$$\int_a^b \left\{ (\kappa - \overline{\kappa})^T \varphi_\kappa^L\left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) + \left(\dot{\kappa} - \dot{\overline{\kappa}}\right)^L \varphi_{\dot{\kappa}}^L\left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) \right\} dt,$$

$$\int_a^b \varphi^U\left(t, \kappa, \dot{\kappa}\right) dt - \int_a^b \varphi^U\left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt \geqq$$

$$\int_a^b \left\{ (\kappa - \overline{\kappa})^T \varphi_\kappa^U\left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) + \left(\dot{\kappa} - \dot{\overline{\kappa}}\right)^L \varphi_{\dot{\kappa}}^U\left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) \right\} dt$$

hold for all $\kappa, \overline{\kappa} \in X$.

The alternative theorem presented here is a specific case of the more generalized results established in [12, 13] for convex vector optimization problems.

Theorem 1. *Let $X \subset \mathbb{R}^n$ be a convex set and $\phi = [\phi^L, \phi^U] : \mathcal{I} \times X \times X \to \mathbb{R}^p$ and $\psi = [\psi^L, \psi^U] : \mathcal{I} \times X \times X \to \mathbb{R}^q$ be convex functional. If the system*

$$\int_a^b \phi_i(t, \kappa, \dot{\kappa})\, dt = \left[\int_a^b \phi_i^L(t, \kappa, \dot{\kappa})\, dt, \int_a^b \phi_i^U(t, \kappa, \dot{\kappa})\, dt\right] < 0, \quad \forall i \in \Lambda_p,$$

$$\int_a^b \psi_j(t, \kappa, \dot{\kappa})\, dt = \left[\int_a^b \psi_j^L(t, \kappa, \dot{\kappa})\, dt, \int_a^b \psi_j^U(t, \kappa, \dot{\kappa})\, dt\right] \leqq 0, \quad \forall j \in \Lambda_q,$$

$$\kappa \in X, t \in \mathcal{I}, \kappa(a) = \alpha, \kappa(b) = \beta,$$

has no solution then there exist $\delta = (\delta^L, \delta^U) \geqq 0$ *where* , $\delta^L, \delta^U \in \mathbb{R}^p$, *and* $\xi = (\xi^L, \xi^U) \geqq 0$ *where* , $\xi^L, \xi^U \in \mathbb{R}^q$ *such that*

$$\int_a^b \left(\sum_{i=1}^{p} \left(\delta_i^L \phi_i^L (t, \kappa, \dot{\kappa}) + \delta_i^U \phi_i^U (t, \kappa, \dot{\kappa}) \right) \right) dt + \int_a^b \left(\sum_{j=1}^{q} \left(\xi_j^L \psi_j^L (t, \kappa, \dot{\kappa}) + \xi_j^U \psi_j^U (t, \kappa, \dot{\kappa}) \right) \right) dt \geqq 0,$$

$$\forall \kappa \in X, t \in \mathcal{I}, \kappa(a) = \alpha, \kappa(b) = \beta.$$

3. The weighting method for solving convex interval-valued variational problems

In this study, we investigate following multiobjective interval-valued variational problem defined by

$$\textbf{(MIVP)} \ \min \to \int_a^b \varphi\left(t, \kappa, \dot{\kappa}\right) dt = \left(\int_a^b \varphi_1 \left(t, \kappa, \dot{\kappa}\right) dt, ..., \int_a^b \varphi_r \left(t, \kappa, \dot{\kappa}\right) dt \right)$$

$$\text{subject to } g\left(t, \kappa, \dot{\kappa}\right) \leqq 0, \ \forall t \in \mathcal{I},$$

$$\kappa(a) = \alpha, \ \kappa(b) = \beta,$$

where X is a nonempty subset of $\mathbb{R}^n$,
$\int_a^b \varphi_i \left(t, \kappa, \dot{\kappa}\right) dt = \left[\int_a^b \varphi_i^L \left(t, \kappa, \dot{\kappa}\right) dt, \int_a^b \varphi_i^U \left(t, \kappa, \dot{\kappa}\right) dt \right]$ for each $i \in \Lambda_r$,
$\varphi_i^L, \varphi_i^U : I \times X \times X \to \mathbb{R}$, $i \in \Lambda_r$, and $g = (g^1, ..., g^m) : I \times X \times X \to \mathbb{R}^m$. To simplify our presentation, we will frequently use certain notations in the following sections. Let

$$\Delta := \left\{ \kappa \in X : g\left(t, \kappa, \dot{\kappa}\right) \leqq 0, \ \forall t \in \mathcal{I}, \ \kappa(a) = \alpha, \ \kappa(b) = \beta \right\},$$

be the set of all feasible solutions in (MIVP). Further, we denote a set of active inequality constraints at point $\overline{\kappa} \in X$, that is
$J\left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) = \left\{ j \in J : g_j \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) = 0, \ \forall t \in \mathcal{I}, \ j = 1, ..., m \right\}$.
The optimal solutions for multiobjective interval-valued optimization problems are defined in terms of weakly LU-efficient and LU-efficient solutions, as defined below.

Definition 2. A feasible solution $\overline{\kappa}$ is said to be a weakly LU-efficient solution of (MIVP) if and only if there exists no other $\kappa \in \Delta$ such that

$$\int_a^b \varphi_i \left(t, \kappa, \dot{\kappa}\right) dt <_{LU} \int_a^b \varphi_i \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \ \forall i \in \Lambda_r.$$

Definition 3. A feasible solution $\overline{\kappa}$ is said to be a LU-efficient solution of (MIVP) if and only if there exists no other $\kappa \in \Delta$ such that

$$\int_a^b \varphi_i\left(t,\kappa,\dot{\kappa}\right)dt \leqq_{LU} \int_a^b \varphi_i\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right)dt,\ \forall i \in \Lambda_r,$$

$$\int_a^b \varphi_{i_0}\left(t,\kappa,\dot{\kappa}\right)dt <_{LU} \int_a^b \varphi_{i_0}\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right)dt,\ \text{for some } i_0 \in \Lambda_r.$$

In this section, for finding a weakly LU-efficient solution and a LU-efficient solution of (MIVP), we use the weighting approach for solving multiobjective interval-valued variational problems, which has been introduced by Antczak [6].
Therefore, for this purpose, a weighting variational problem is introduced for the considered multiobjective interval-valued variational problem as follows:

$$(\mathbf{WVP}_w)\ \min \to \Pi(\kappa) = \int_a^b \left\{\textstyle\sum_{i=1}^r w_i^L \varphi_i^L\left(t,\kappa,\dot{\kappa}\right) + \sum_{i=1}^r w_i^U \varphi_i^U\left(t,\kappa,\dot{\kappa}\right)\right\}dt$$

$$\text{subject to } g\left(t,\kappa,\dot{\kappa}\right) \leqq 0,\ \forall t \in \mathcal{I},$$

$$\kappa(a) = \alpha,\ \kappa(b) = \beta,$$

where $w^L = \left(w_1^L,...,w_r^L\right) \geqq 0$ and $w^U = \left(w_1^U,...,w_r^U\right) \geqq 0$.

Definition 4. A feasible solution $\overline{\kappa}$ is said to be an optimal solution of (WVP_w), if the inequality

$$\int_a^b \left\{\sum_{i=1}^r w_i^L \varphi_i^L\left(t,\kappa,\dot{\kappa}\right) + \sum_{i=1}^r w_i^U \varphi_i^U\left(t,\kappa,\dot{\kappa}\right)\right\}dt <$$

$$\int_a^b \left\{\sum_{i=1}^r w_i^L \varphi_i^L\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right) + \sum_{i=1}^r \overline{w}_i^U \varphi_i^U\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right)\right\}dt,$$

holds for all $\kappa \in \Delta$.

Theorem 2. *Let $\overline{\kappa} \in \Delta$ be an optimal solution of ($WVP_{\overline{w}}$). Further, assume that $\overline{w} = \left(\overline{w}^L,\overline{w}^U\right) = \left(\left(\overline{w}_1^L,...,\overline{w}_k^L\right),\left(\overline{w}_1^U,...,\overline{w}_k^U\right)\right) \geq 0$ with $\left(\overline{w}_{i_0}^L,\overline{w}_{i_0}^U\right) > 0$ for some $i_0 \in \Lambda_r$. Then $\overline{\kappa}$ is a weakly LU-efficient solution of the considered (MIVP) problem.*

Proof. By assumption, $\overline{\kappa}$ is an optimal solution of ($\text{WVP}_{\overline{w}}$), where $\overline{\kappa}$ belongs to Δ. We assume that $\overline{\kappa}$ is not a weakly LU-efficient solution to (MIVP). Therefore, according to the Definition 2, it follows that there exists another $\widetilde{\kappa} \in \Delta$ such that

$$\int_a^b \varphi_i\left(t,\widetilde{\kappa},\dot{\widetilde{\kappa}}\right)dt <_{LU} \int_a^b \varphi_i\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right)dt,\ \forall i \in \Lambda_r.$$

According to the definition of the relation $<_{LU}$, it follows that, for any $i \in \Lambda_r$,

$$\begin{cases} \int_a^b \varphi_i^L \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt < \int_a^b \varphi_i^L \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \\ \int_a^b \varphi_i^U \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt \leqq \int_a^b \varphi_i^U \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \end{cases}$$

$$\text{or } \begin{cases} \int_a^b \varphi_i^L \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt \leqq \int_a^b \varphi_i^L \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \\ \int_a^b \varphi_i^U \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt < \int_a^b \varphi_i^U \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \end{cases}$$

$$\text{or } \begin{cases} \int_a^b \varphi_i^L \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt < \int_a^b \varphi_i^L \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \\ \int_a^b \varphi_i^U \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt < \int_a^b \varphi_i^U \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt. \end{cases}$$

Since $\overline{w} = \left(\overline{w}^L, \overline{w}^U\right) = \left(\left(\overline{w}_1^L, ..., \overline{w}_r^L\right), \left(\overline{w}_1^U, ..., \overline{w}_r^U\right)\right) \geq 0$ with $\left(\overline{w}_{i_0}^L, \overline{w}_{i_0}^U\right) > 0$ for some $i_0 \in \Lambda_r$, the above system of inequalities yields that the inequality

$$\int_a^b \left\{ \sum_{i=1}^r \overline{w}_i^L \varphi_i^L \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) + \sum_{i=1}^r \overline{w}_i^U \varphi_i^U \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) \right\} dt <$$

$$\int_a^b \left\{ \sum_{i=1}^r \overline{w}_i^L \varphi_i^L \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) + \sum_{i=1}^r \overline{w}_i^U \varphi_i^U \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) \right\} dt,$$

holds. This contradicts the assumption that $\overline{\kappa} \in \Delta$ is an optimal solution of (WVP$_{\overline{w}}$). Hence, $\overline{\kappa}$ is a weakly LU-efficient solution of the considered (MIVP) problem, which completes the proof of the theorem. □

Theorem 3. *Let $\overline{x} \in \Delta$ be an optimal solution of (WVP$_{\overline{w}}$). Further, assume that $\overline{w} = \left(\overline{w}^L, \overline{w}^U\right) = \left(\left(\overline{w}_1^L, ..., \overline{w}_r^L\right), \left(\overline{w}_1^U, ..., \overline{w}_r^U\right)\right) \geq 0$ with $\overline{w}^L \geq 0$ and $\overline{w}^U \geq 0$. Then $\overline{\kappa}$ is an LU-efficient solution of the considered (MIVP).*

Proof. By assumption, $\overline{\kappa}$ is an optimal solution of (WVP$_{\overline{w}}$), where $\overline{\kappa}$ belongs to Δ. We assume that $\overline{\kappa}$ is not an LU-efficient solution to (MIVP). Therefore, according to the Definition 3, it follows that there exists another $\widetilde{\kappa} \in \Delta$ such that

$$\int_a^b \varphi_i \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt \leqq_{LU} \int_a^b \varphi_i \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \ \forall i \in \Lambda_r,$$

$$\int_a^b \varphi_{i_0} \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt <_{LU} \int_a^b \varphi_{i_0} \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt \text{ for some } i_0 \in \Lambda_r.$$

According to the definition of the relation $\leqq_{LU}$, it follows that, for any $i \in \Lambda_r$,

$$\begin{cases} \int_a^b \varphi_i^L \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt \leqq \int_a^b \varphi_i^L \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \\ \int_a^b \varphi_i^U \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt \leqq \int_a^b \varphi_i^U \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt. \end{cases} \tag{3.1}$$

According to the definition of the relation $\leqq_{LU}$, it follows that, for any $i \in \Lambda_r$,

$$\begin{cases} \int_a^b \varphi_i^L \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt < \int_a^b \varphi_i^L \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \\ \int_a^b \varphi_i^U \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt \leqq \int_a^b \varphi_i^U \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \end{cases}$$

$$\text{or } \begin{cases} \int_a^b \varphi_i^L \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt \leqq \int_a^b \varphi_i^L \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \\ \int_a^b \varphi_i^U \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt < \int_a^b \varphi_i^U \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \end{cases} \tag{3.2}$$

$$\text{or } \begin{cases} \int_a^b \varphi_i^L \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt < \int_a^b \varphi_i^L \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt, \\ \int_a^b \varphi_i^U \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) dt < \int_a^b \varphi_i^U \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) dt. \end{cases}$$

Since $\overline{w} = \left(\overline{w}^L, \overline{w}^U\right) = \left(\left(\overline{w}_1^L, \ldots, \overline{w}_r^L\right), \left(\overline{w}_1^U, \ldots, \overline{w}_r^U\right)\right) \geq 0$ with $\overline{w}^L \geq 0$ and $\overline{w}^U \geq 0$, (3.1) and (3.2) imply that the inequality

$$\int_a^b \left\{ \sum_{i=1}^r \overline{w}_i^L \varphi_i^L \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) + \sum_{i=1}^r \overline{w}_i^U \varphi_i^U \left(t, \widetilde{\kappa}, \dot{\widetilde{\kappa}}\right) \right\} dt <$$

$$\int_a^b \left\{ \sum_{i=1}^k \overline{w}_i^L \varphi_i^L \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) + \sum_{i=1}^k \overline{w}_i^U \varphi_i^U \left(t, \overline{\kappa}, \dot{\overline{\kappa}}\right) \right\} dt,$$

holds. This contradicts the assumption that $\overline{\kappa} \in \Delta$ is an optimal solution of (WVP$_{\overline{w}}$). Hence, $\overline{\kappa}$ is an LU-efficient solution of the considered (MIVP) problem, which completes the proof of the theorem. □

Now, we give an example of a multiobjective interval-valued variational problem, which we solve using the weighting method to support our result established in Theorem 3.

Example 1. Let $\mathcal{I} = [0,1]$. Let us consider the following multiobjective interval-valued variational problem (MIVP1), defined as

$$\textbf{(MIVP1)} \ \min \to \int_a^b \varphi\left(t,\kappa,\dot{\kappa}\right)dt = \left(\int_a^b \varphi_1\left(t,\kappa,\dot{\kappa}\right)dt\ ,\ \int_a^b \varphi_2\left(t,\kappa,\dot{\kappa}\right)dt\right)$$

$$= \left(\left[\int_0^1 (2\arctan\kappa(t) + \kappa(t))\,dt,\ \int_0^1 (2\arctan\kappa(t) + \kappa(t) + 1)\,dt\right],\right.$$
$$\left.\left[\int_0^1 \left(-e^{-\kappa(t)}\right)dt,\ \int_0^1 \left(-e^{-\kappa(t)} + \kappa(t)\right)dt\right]\right)$$

$$\text{subject to } g\left(t,\kappa,\dot{\kappa}\right) = -\kappa(t) + \kappa^2(t) \leqq 0,\ \forall t \in \mathcal{I} = [0,1],$$

$$\kappa(0) = 0\ ,\ \kappa(1) = 1.$$

As it follows from the formulation of (MIVP1), we have

$$\varphi_1^L\left(t,\kappa,\dot{\kappa}\right) = 2\arctan\kappa(t) + \kappa(t),\ \varphi_1^U\left(t,\kappa,\dot{\kappa}\right) = 2\arctan\kappa(t) + \kappa(t) + 1,$$

$$\varphi_2^L\left(t,\kappa,\dot{\kappa}\right) = -e^{-\kappa(t)},\ \varphi_2^U\left(t,\kappa,\dot{\kappa}\right) = -e^{-\kappa(t)} + \kappa(t).$$

For the graphs of the vector interval-valued functions φ_1 and φ_2, see Fig. 1(a) and 1(b).

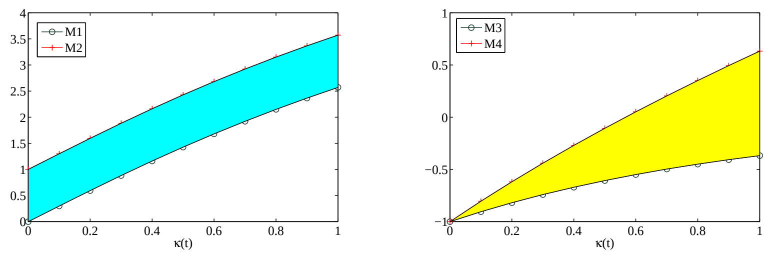

Figure 1. The graphs of the objective functions **(a)** $\varphi_1(t) = [\varphi_1^L(t), \varphi_1^U(t)] = [M_1, M_2]$,**(b)**$\varphi_2(t) = [\varphi_2^L(t), \varphi_2^U(t) = [M_3, M_4]$ in the multiobjective interval-valued variational problem (MIVP1).

We now use the weighting method for solving (MIVP1). Let $\overline{w}_1 = \left(\overline{w}_1^L, \overline{w}_1^U\right) = \left(\frac{1}{2}, \frac{1}{2}\right) > 0$ and $\overline{w}_2 = \left(\overline{w}_2^L, \overline{w}_2^U\right) = \left(\frac{1}{2}, 0\right) \geq 0$. Then, for the considered multiobjective interval-valued variational problem (MIVP1), we construct its associated weighting variational problem (WVP1$_{\overline{w}}$) defined by

$$\textbf{(WVP1}_{\overline{w}}\textbf{)} \ \min \to \Pi(\kappa) = \int_a^b \left\{\sum_{i=1}^2 w_i^L \varphi_i^L\left(t,\kappa,\dot{\kappa}\right) + \sum_{i=1}^2 w_i^U \varphi_i^U\left(t,\kappa,\dot{\kappa}\right)\right\}dt$$
$$= \int_0^1 \left(2\arctan\kappa(t) - \frac{1}{2}e^{-\kappa(t)} + \kappa(t) + \frac{1}{2}\right)dt$$

$$\text{subject to } \ g\left(t,\kappa,\dot{\kappa}\right) = -\kappa(t) + \kappa^2(t) \leqq 0,\ \ \forall t \in [0,1],$$
$$\kappa(0) = 0,\ \kappa(1) = 1.$$

The graph of the objective function $\Pi(\kappa)$ in $(\mathrm{WVP1}_{\overline{w}})$, see Fig.2. The set of all feasible solutions of $(\mathrm{WVP1}_{\overline{w}})$ is given by $\Delta = \{\kappa(t) \in \mathbb{R} : \kappa(0) = 0, \kappa(1) = 1, 0 \leqq \kappa(t) \leqq 1, \forall t \in [0,1]\}$ and $\overline{\kappa}(t) = 0$ is feasible in $(\mathrm{WVP1}_{\overline{w}})$. Further, since all the hypotheses of Theorem 3 are satisfied, $\overline{\kappa}(t) = 0$ is an optimal solution in the weighting variational problem $(\mathrm{WVP1}_{\overline{w}})$ which can also be seen in Fig.2.

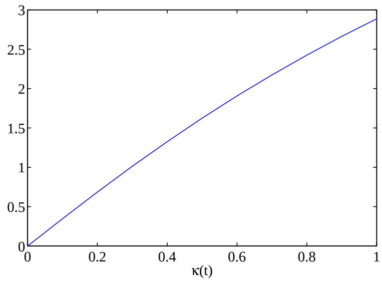

Figure 2. The graph of the objective function $\Pi(\kappa)$ in the weighting variational problem $(\mathrm{WVP1}_{\overline{w}})$.

Now, under appropriate convexity hypothesis, we prove the converse results to those ones established in Theorems 2 and 3.

Theorem 4. *Let each objective functional* $\int_a^b \varphi_i(t,\cdot,\cdot)\,dt, i \in \Lambda_r$, *be a convex interval-valued functional on* X. *If* $\overline{\kappa} \in \Delta$ *is a weakly LU-efficient solution in (MIVP), then there exists* $\overline{w} = (\overline{w}^L, \overline{w}^U) \geq 0$, *where* $\overline{w}^L = (\overline{w}_1^L, ..., \overline{w}_r^L), \overline{w}^U = (\overline{w}_1^U, ..., \overline{w}_r^U) \in \mathbb{R}^r$, *such that* $\overline{\kappa}$ *is an optimal solution of the weighting variational problem* ($WVP_{\overline{w}}$).

Proof. Let $\overline{\kappa} \in \Delta$ be an LU-efficient solution in (MIVP). Then, according to Definition 3, there is no other feasible solution κ such that

$$\int_a^b \varphi_i\left(t,\kappa,\dot{\kappa}\right) dt <_{LU} \int_a^b \varphi_i\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right) dt, \ \forall i \in \Lambda_r.$$

From the definition of the relation $<_{LU}$, it follows that, for every $i \in \Lambda_r$,

$$\begin{cases} \int_a^b \varphi_i^L\left(t,\kappa,\dot{\kappa}\right) dt < \int_a^b \varphi_i^L\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right) dt, \\ \int_a^b \varphi_i^U\left(t,\kappa,\dot{\kappa}\right) dt \leqq \int_a^b \varphi_i^U\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right) dt, \end{cases}$$

$$\text{or} \begin{cases} \int_a^b \varphi_i^L\left(t,\kappa,\dot{\kappa}\right) dt \leqq \int_a^b \varphi_i^L\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right) dt, \\ \int_a^b \varphi_i^U\left(t,\kappa,\dot{\kappa}\right) dt < \int_a^b \varphi_i^U\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right) dt, \end{cases} \tag{3.3}$$

$$\text{or} \begin{cases} \int_a^b \varphi_{i_0}^L\left(t,\kappa,\dot{\kappa}\right)dt < \int_a^b \varphi_{i_0}^L\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right)dt, \\ \int_a^b \varphi_i^U\left(t,\kappa,\dot{\kappa}\right)dt < \int_a^b \varphi_i^U\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right)dt. \end{cases}$$

By assumption, each objective functional $\int_a^b \varphi_i(t,\cdot,\cdot)\,dt, i \in \Lambda_r$, is a convex interval-valued functional on Δ. Then, by Definition 1, it follows that the functionals $\int_a^b \varphi_i^L(t,\cdot,\cdot)\,dt$, and $\int_a^b \varphi_i^U(t,\cdot,\cdot)\,dt$, $i \in \Lambda_r$, are convex on Δ. Since the system of inequalities (3.3) has no solution for $\kappa \in \Delta$. Then, by the theorem of alternative 1, there exist $\overline{w}^L, \overline{w}^U \in \mathbb{R}^r$ with $\left(\overline{w}^L, \overline{w}^U\right) = \left(\left(\overline{w}_1^L, ..., \overline{w}_r^L\right), \left(\overline{w}_1^U, ..., \overline{w}_r^U\right)\right) \geq 0$ such that the inequality

$$\int_a^b \left\{\sum_{i=1}^r \overline{w}_i^L \varphi_i^L\left(t,\kappa,\dot{\kappa}\right) + \sum_{i=1}^r \overline{w}_i^U \varphi_i^U\left(t,\kappa,\dot{\kappa}\right)\right\}dt \geqq$$

$$\int_a^b \left\{\sum_{i=1}^r \overline{w}_i^L \varphi_i^L\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right) + \sum_{i=1}^r \overline{w}_i^U \varphi_i^U\left(t,\overline{\kappa},\dot{\overline{\kappa}}\right)\right\}dt,$$

holds for all $\kappa \in \Delta$. This means, by Definition 4, that $\overline{\kappa}$ is an optimal solution of the weighting variational problem (WVP$_{\overline{w}}$), which completes the proof of this theorem. □

4. Application to manufacturing optimization problem

Example 2. A manufacturing company has two production units, and both produce different types of products. The company wants to minimize the cost of both products. We represent the range of total production costs for both products as an interval-valued function, which is given below.

$$\varphi\left(t,\kappa,\dot{\kappa}\right)dt = \left[\varphi_1\left(t,\kappa,\dot{\kappa}\right),\ \varphi_2\left(t,\kappa,\dot{\kappa}\right)\right]$$
$$= \left(\left[\left(\arctan\kappa^2(t) + 2\kappa(t) + 1\right),\ \left(\arctan\kappa^2(t) + 4\kappa(t) + 2\right)\right],\ \left[\left(\ln\left(\kappa^2(t)+1\right) + \kappa(t)\right)dt,\ \left(\ln\left(\kappa^2(t)+1\right) + 5\kappa(t)\right)\right]\right).$$

The costs of these two interval-valued functions should be minimized subject to the constraints $\kappa^2(t) - \kappa(t) \leqq 0$, where $t \in \mathcal{I} = [0,1]$ and the endpoint conditions are $\kappa(0) = 0$ and $\kappa(1) = 1$. Find an extremum that minimizes the production's cost for these two production units, which is an interval-valued function. The formulation of the multiobjective interval-valued variational problem is given below:

$$\textbf{(MIVP2)}\ \min \to \int_a^b \varphi\left(t,\kappa,\dot{\kappa}\right)dt = \left(\int_a^b \varphi_1\left(t,\kappa,\dot{\kappa}\right)dt\ ,\ \int_a^b \varphi_2\left(t,\kappa,\dot{\kappa}\right)dt\right)$$
$$= \left[\int_0^1 \left(\arctan\kappa^2(t) + 2\kappa(t) + 1\right)dt,\ \int_0^1 \left(\arctan\kappa^2(t) + 4\kappa(t) + 2\right)dt\right.$$
$$\left.\left(\int_0^1 \left(\ln\left(\kappa^2(t)+1\right) + \kappa(t)\right)dt\ ,\ \int_0^1 \left(\ln\left(\kappa^2(t)+1\right) + 5\kappa(t)\right)dt\right)\right]$$
$$\text{subject to } g\left(t,\kappa,\dot{\kappa}\right) = \kappa^2(t) - \kappa(t) \leqq 0,\ \forall t \in \mathcal{I} = [0,1],$$
$$\kappa(0) = 0,\ \kappa(1) = 1.$$

As it follows from the formulation of (MIVP2), we have

$$\varphi_1^L\left(t,\kappa,\dot{\kappa}\right)=\arctan\kappa^2\left(t\right)+2\kappa\left(t\right)+1,\ \varphi_1^U\left(t,\kappa,\dot{\kappa}\right)=\arctan\kappa^2\left(t\right)+4\kappa\left(t\right)+2,$$

$$\varphi_2^L\left(t,\kappa,\dot{\kappa}\right)=\ln\left(\kappa^2\left(t\right)+1\right)+\kappa\left(t\right),\ \varphi_2^U\left(t,\kappa,\dot{\kappa}\right)=\ln\left(\kappa^2\left(t\right)+1\right)+5\kappa\left(t\right).$$

For the graphs of the vector interval-valued functions φ_1 and φ_2, see Fig. 3(a) and 3(b).

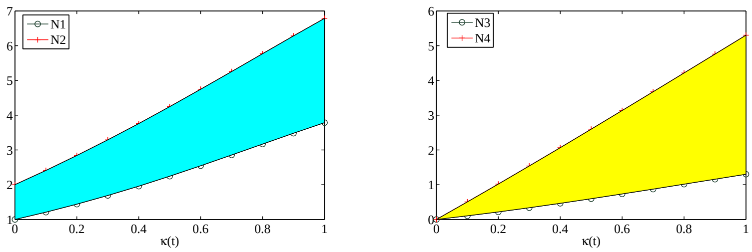

Figure 3. The graphs of the objective functions **(a)**$\varphi_1(t)=[\varphi_1^L(t),\varphi_1^U(t)]=[N_1,N_2]$,**(b)**$\varphi_2(t)=[\varphi_2^L(t),\varphi_2^U(t)=[N_3,N_4]$ in the multiobjective interval-valued variational problem (MIVP2).

Further, the set of all feasible solutions in (MIVP2) is given by
$\Delta=\{\kappa(t)\in\mathbb{R}:\kappa(0)=0,\ \kappa(1)=1,\ 0\leqq\kappa(t)\leqq 1,\ \forall t\in[0,1]\}$. Note that both objective functionals in (MIVP2) are convex interval-valued functionals, and $\overline{\kappa}(t)$ is an LU-efficient solution of (MIVP2).
We now use the weighting method to prove the considered multiobjective interval-valued variational problem (MIVP2). Let its weights be defined by $\overline{w}_1=\left(\overline{w}_1^L,\overline{w}_1^U\right)=\left(\frac{1}{2},\frac{1}{2}\right)>0$ and $\overline{w}_2=\left(\overline{w}_2^L,\overline{w}_2^U\right)=\left(\frac{1}{2},\frac{1}{2}\right)>0$. Then, for the considered multiobjective interval-valued variational problem (MIVP2), we construct its associated weighting variational problem (WVP2$_{\overline{w}}$) as follows:

$$\begin{aligned}(\mathbf{WVP2}_{\overline{w}})\ \min\to\Pi(\kappa)&=\int_a^b\left\{\sum_{i=1}^2 w_i^L\varphi_i^L\left(t,\kappa,\dot{\kappa}\right)+\sum_{i=1}^2 w_i^U\varphi_i^U\left(t,\kappa,\dot{\kappa}\right)\right\}dt\\&=\int_0^1\left(\arctan\kappa^2(t)+\ln\left(\kappa^2(t)+1\right)+6\kappa(t)+\frac{3}{2}\right)dt\\ \text{subject to }\ g\left(t,\kappa,\dot{\kappa}\right)&=\kappa^2(t)-\kappa(t)\leqq 0,\ \ \forall t\in[0,1],\\ \kappa(0)&=0,\ \kappa(1)=1.\end{aligned}$$

The graph of the objective function $\Pi(\kappa)$ in (WVP2$_{\overline{w}}$), see Fig. 4. The set of all feasible solutions of (WVP2$_{\overline{w}}$) is given by
$\Delta=\{\kappa(t)\in\mathbb{R}:\kappa(0)=0,\kappa(1)=1,0\leqq\kappa(t)\leqq 1,\ \forall t\in[0,1]\}$. Since all the hypotheses of Theorem 4 are satisfied. Then $\overline{\kappa}(t)=0$ is an optimal solution in the weighting variational problem (WVP2$_{\overline{w}}$), which can also be seen in Fig 4, and can minimize the production cost.

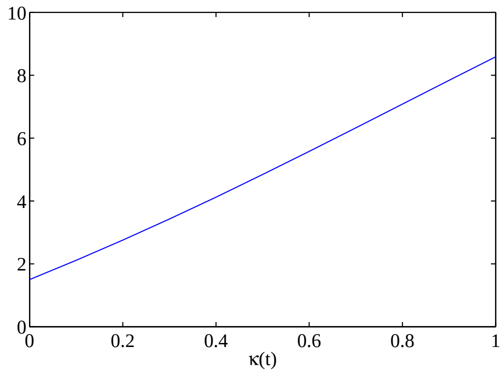

Figure 4. The graph of the objective function $\Pi(\kappa)$ in the weighting variational problem (WVP2$_{\overline{w}}$).

5. Conclusions

In the paper, we have used the weighting method for solving the given multiobjective interval-valued variational problem. As a result, we have constructed the associated weighting variational problem using this method. Then, under appropriate convexity hypotheses, we have demonstrated the correlation between an LU-efficient solution (a weakly LU-efficient solution) of the considered multiobjective interval-valued variational problem and an optimal solution of its corresponding weighting variational problem constructed in the used method. It is known from the literature that the weighting method is one of methods for solving vector optimization problems (in the considered case, for solving the considered multiobjective interval-valued variational problem). Moreover, the methods employed in this study seem to yield comparable results for other types of multiobjective interval-valued variational problems. For future work, we will investigate these findings.

Acknowledgements: Authors are thankful to VIT-AP University, Amaravati, for providing financial support to doctoral students under the research scheme. We also extend our gratitude to the editor and the esteemed referee for their valuable suggestions, which helped improve the quality of the paper.

Author contributions: S. Singh contributed to methodology, conceptualization, manuscript preparation, writing, review, and editing; Shalini contributed to methodology, conceptualization, writing, review, editing, and supervision.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: The current study did not generate or alter any datasets.

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