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Research Article

Edge graceful labeling on neutrosophic graphs

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Abstract: In this article, the edge graceful labeling concept has been expanded from conventional fuzzy graphs to intuitionistic and neutrosophic graphs. There has been much discussion of the edge graceful labeling in intuitionistic and neutrosophic graphs with a certain sequence of edge labels(for each membership) in a clockwise or anticlockwise direction and the resultant vertices. Also, various irregular properties and applications of neutrosophic edge graceful labeling graphs have been discussed in detail

Keywords: fuzzy labeling graph, edge graceful labeling, intuitionistic fuzzy labeling graph, neutrosophic labeling graph, irregular property.

AMS Subject classification: 05C78

1. Introduction

Graph theory plays an important role in graphically addressing and resolving a wide range of physical difficulties. A graph $G = (V, E)$ is the combined output of vertices (V) and edges (E). Graph labeling is nothing more than assigning values to edges and/or vertices, which is important in obtaining solutions to real-world circumstances [17]. If a function is defined from V to some set of labels, then such graphs are called vertex-labeled graphs and if a function is defined from E to some set of labels, then it is called as an edge-labeled graph. Because of the occurrence of unclear and ambiguous outcomes in the graphical technique, certain innovative enhancements to past ideas are intended to support the accuracy.

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Following that, L.A. Zadeh [44] defined fuzzy sets and associated relations. A. Kaufmann [22] extended it to fuzzy graphs through which fuzzy graph models have been created by A. Rosenfeld [37]. Taking two functions from $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ for vertices and edges, a fuzzy graph $G = (\sigma, \mu)$ is one in which for every $u, v \in V$, we have $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ [20]. We acquired some unique outcomes and insights in fuzzy graphs owing to P. Bhattacharya [8] and K.R. Bhuttani [9]. A. Nagoor Gani et al. [27–33] established a solid foundation for fuzzy graphs and examined labeling features in fuzzy graphs. A fuzzy labeling is a bijective function represented by $\star$ from the set of all nodes and edges to $[0, 1]$ that assigns a membership value to each node $\sigma^\star(a), \sigma^\star(b)$ and edge $\mu^\star(a, b)$ such that $\mu^\star(a, b) \leq \sigma^\star(a) \wedge \sigma^\star(b)$, for all $a, b \in V$ [41]. A fuzzy labeling graph is defined as $G^\star$ when the label values are of a fuzzy kind. Other researchers enforce the development of theories and applications of fuzzy graphs regarding connectivity, nodes and arcs, fuzzy bridges, anti-fuzziness, magic labeling, graceful labeling, harmonious labeling, etc. The word "graceful labeling" is due to Solomon W. Golomb which was originally given the name $\beta - labeling$ by A. Rosa [36]. Let $G = (V, E)$ be a graph with vertex cardinality of p and edge cardinality of q. A graph with gracefully numbered labels is called graceful labeling. An injection $f : V(G) \to 0, 1, 2, \ldots, q$ that causes all of the edge labels to be distinct when each edge $xy \in E(G)$ is given by the label $|f(x) - f(y)|$ is a graceful labeling of a graph G with q edges [20]. Edge graceful labeling on graphs was introduced by Lo, Sheng-Ping [23]. According to the definition of an edge graceful labeling in a crisp graph, the vertex v of the graph is labeled by the sum of the labels of the edges incident to it, modulo p and this is expressed as $V(v) = \sum E(e) mod |V(G)|$, where $V(v)$ is the label for the vertex and $E(e)$ is the assigned value of an edge incident to v [27]. If a graph G allows edge graceful labeling, then it is said to be edge graceful. There may be many works on graceful graphs, but fuzzy graceful graphs have been done by R. Jahir Hussain et al. [19] at first. Then, graceful labeling has emerged and extended to fuzzy kind by R. Jebesty Shajila and S. Vimala [21]. Fuzzy graceful labeling is the recognition of fuzzy graceful labels in a graph. With prior knowledge from [23], A. Nagoor Gani et al. [27] elaborated edge graceful labeling to fuzzy kind and paved the way for our continuance to introduce intuitionistic and neutrosophic behaviors on edge graceful labeling graphs. Fuzzy edge graceful labeling of a graph deals with the assigned fuzzy edges and obtained vertices from the sum of edges incident to each vertex, which is denoted by $V(v) = \sum E(e)$.

Intuitionistic fuzzy sets with membership and non-membership functions of elements in a set that belong to the real unit interval $[0, 1]$ and whose sum belongs to the same interval have been established by Krassimir Atanassov [7] to forward the progress of fuzzy set theory. M. Akram [3, 4], A. Nagoor Gani [29], R. Parvathi and M.G. Karunambigai [34, 35] finalized supplemental significant works on intuitionistic fuzzy graphs (IFG) and its properties. Connectivity, degree, order and size, bipolar property, product on IFG, magic labeling on IFG, and anti-fuzziness were discussed and developed by other researchers working in this area. It was then extended to the interval-valued intuitionistic fuzzy sets and graphs to define discrepancies made by conventional fuzzy sets. Edge graceful labeling(EGL) didn't have any history on IFG,

so it was overviewed in this article.
The refinement of fuzzy sets and intuitionistic fuzzy sets into neutrosophic sets, which address the indeterminacy membership of set elements, was then put forth by Florentin Smarandache [38, 39]. The characterization of the truth membership function (T), indeterminacy membership function (I), and false membership function (F) is known as a neutrosophic set. Since each membership function is independent by nature, we explicitly estimate the indeterminacy here. To characterize the membership degree independently and to minimize the inconsistencies by monitoring real-world issues, Wang et al. [43] presented the novel notion of the single-valued neutrosophic set. Moreover, S. Broumi and Smarandache [13] negotiated with single valued neutrosophic graphs (SVNG) and their properties (degree, order, size, bipolar, and anti-fuzziness), which paved the route for the emergence of other neutrosophic sets such as interval-valued neutrosophic sets, bipolar neutrosophic sets, neutrosophic hesitant fuzzy sets, etc. Smarandache designed neutrosophic graphs and their conditions based on neutrosophic sets. Also, he deals with the new definitions and applications of soft sets and their extensions [40]. S. Broumi et al. [13] altered the constraints of neutrosophic graphs to obtain single-valued neutrosophic graphs and reviewed some aspects (degree, order, size, and bipolar property). Also, Broumi et al. [10–12, 14, 15] have a brief discussion on the specific properties of neutrosophic sets and neutrosophic graphs. The complexity analysis of neutrosophic graphs involves evaluating the computational effort required to process and interpret these graphs, which extend classical graph theory by incorporating degrees of truth, indeterminacy, and false associated with each element. This analysis typically examines various aspects including time complexity, space complexity, and the efficiency of algorithms designed for operations such as searching, sorting, and optimization on neutrosophic graphs. The added dimensions of truth, indeterminacy, and falsity increase both the computational and theoretical complexity when compared to traditional graph models. Mullai et al. [24–26] pioneered domination and accompanying creative research (dominating energy, split domination) in neutrosophic graphs. The multiple sorts of neutrosophic graphs and anti-neutrosophic behavior on SVNG were invented by R. Dhavaseelan et al. [16, 17]. M. Gomathi and V. Keerthika [18] recently discussed neutrosophic labeling graphs. D. Ajay et al. [1] recently presented fuzzy magic labeling on a neutrosophic path and star-related graphs. The existing labeling works on fuzzy graph theory are not carried on with its extensions like intuitionistic and neutrosophic graphs. The neutrosophic graph theory is an updated and refined version of fuzzy and intuitionistic fuzzy graph theory since it bears additional membership functions. Therefore, it produces new theoretical and practical results when applied to a real-life situation. It helps to deal with different criteria of an event at the same time and to analyze each membership output individually. This maximizes the accuracy of the proposed model since the concentration is given specifically to each criterion. This article bears an execution and detailed discussion of EGL on a neutrosophic graph.

1.1. Motivation of the article

The traditional graph labeling is designed based on the integer value assignment to the vertices and edges of a graph. Vast theoretical results and practical implications are observed and recorded to execute an application in the present world. It works well but the absolute output is not acquired. This issue gives rise to the establishment of fuzzy sets and systems, where assignment of fuzzy labeling values is used to enrich the output nearer to the accuracy. It produces finer results than the integer-valued and crisp type. However, the inaccuracy still exists and it is solved using the 2-valued intuitionistic set theory and based graphical labeling concepts. The discussion of the possibility and impossibility of an event in an application point-of-view is achieved but the indeterminate conditions are not analyzed. This query is resolved by the inauguration of a neutrosophic graphical approach, where the indeterminacy case is segregated as a membership function. This refined concept enhances the final output by reducing inaccuracies. There may be an introductory part on neutrosophic labeling [18] and its properties but the types of labeling are not yet explored. Graceful labeling has been studied profoundly in integer-based and fuzzy graphical systems. However, it is not extended to the intuitionistic and neutrosophic type of graphs. As an initiative, this article is dedicated to study the edge graceful labeling of graphs on fuzzy extensions. This article was written based on an inspirational fuzzy edge graceful work [27]. The fuzzy edge graceful labeling(FEGL) bears only one membership and it lacks accurate results during application, when compared to neutrosophic edge graceful labeling(NEGL). Since NEGL memberships are independent, the initiation of edge graceful labeling definition separately to each membership gives us best componential output to analyze the problem with different criterion.

1.2. Objectives and Contributions

Our objective is to improve the neutrosophic graphical environment by introducing conventional fuzzy graph labeling concepts, which shortens the research gap between conventional fuzzy and neutrosophic idea. It extends the area of application to real world since neutrosophic membership functions are more than the conventional fuzzy system, that can be used to assign different behaviours of a same event. The vision about graceful labeling is clear in case of fuzzy graphs since it is widely discussed by many researchers. But this labeling is not yet applied with the fuzzy extensions like intuitionistic and neutrosophic background. The authors of this article tends to extend their work to advanced intuitionistic and neutrosophic edge graceful labeling, which bridges the gap between fuzzy and neutrosophic labeling. Here, the edge graceful labeling has been applied newly to intuitionistic fuzzy graphs and neutrosophic graphs without violating the existing definitions and conditions, and as a result, we obtain distinct vertices satisfying the condition $V(v) = \sum E(e)$ for a vertex v of the graph G. The methodology is done by stating a general lemma to construct edge graceful labeling of intuitionistic and neutrosophic type of graphs. In addition, we investigate some irregular properties and applications with neutrosophic edge graceful labeling graphs. This work influences and enforce the various neutrosophic researchers to implement other labeling concepts like harmonious labeling, skolem labeling, cordial labeling, etc. in neutrosophic graphical background, through which more applications can be attained.

1.3. Structure of the article

This article is structured as mentioned below: The introductory section 1 emphasizes the origin, implementation and development of fuzzy graphical extensions along with the accomplishment done on graceful labeling. In section 2, EGL is applied as an initiative to deal with IFGs. The enforcement of EGL on a neutrosophic graphical environment is done in section 3. Various types of neutrosophic irregularities on EGL are inquired and listed in section 4. Section 5 holds an application of neutrosophic EGL on trade practice and an algorithm to proceed with. The final section 6 encloses the overall work done on each section of the paper and our future insights planned to deliver with neutrosophic graceful labeling, etc. All of the graphs used to demonstrate this notion are finite, simple, and undirected.

2. Some Results on Intuitionistic Edge Graceful Labeling Graphs

Graceful labeling has a unique place among all sorts of labeling since it is frequently studied and tested with some real-time fuzzy kind applications. However graceful labeling was not extensively illustrated in the subsequently improved classification of fuzzy graph (i.e.) intuitionistic fuzzy graph. Here, we incorporate edge graceful labeling of the intuitionistic fuzzy graph, and its implication enables us to reduce the inaccuracy of solutions in an application-oriented environment. The intuitionistic fuzzy labeling graphs in this section satisfy the edge graceful property in a sequence of clockwise or anti-clockwise configurations of edge labels.

Definition 1. An intuitionistic fuzzy graph is of the form $G = (V, E, \sigma, \mu)$, where $\sigma = (T_1, F_1)$ and $\mu = (T_2, F_2)$ with the following conditions,
(i) $V = v_1, v_2, \dots, v_n$ such that $T_1 : V \to [0, 1]$ and $F_1 : V \to [0, 1]$ denote the degree of membership and nonmembership of the element $v_i \in V$ respectively, and $0 \leq T_1(v_i) + F_1(v_i) \leq 1$ for every $v_i \in V$, $(i = 1, 2, \dots, n)$,
(ii) $E \subseteq V \times V$, where $T_2 : V \times V \to [0, 1]$ and $F_2 : V \times V \to [0, 1]$ are such that
$T_2(v_i, v_j) \leq min[T_1(v_i), T_1(v_j)]$,
$F_2(v_i, v_j) \leq max[F_1(v_i), F_1(v_j)]$ and
$0 \leq T_2(v_i, v_j) + F_2(v_i, v_j) \leq 1$, for every $(v_i, v_j) \in E$, $(i, j = 1, 2, \dots, n)$.

Definition 2. A graph $G = (V, E, \sigma, \mu)$, where $\sigma = (T_1, F_1)$ and $\mu = (T_2, F_2)$ is said to be intuitionistic fuzzy labeling graph if $T_1 : V \to [0, 1]$, $F_1 : V \to [0, 1]$, $T_2 : V \times V \to [0, 1]$ and $F_2 : V \times V \to [0, 1]$ are bijective such that $T_1(v_i)$,$F_1(v_i)$,$T_2(v_i, v_j)$,$F_2(v_i, v_j) \in [0, 1]$ all are distinct for each node and edge, where T_1 is the degree of membership and F_1 is the degree of non-membership of nodes. Similarly, T_2 and F_2 are the degrees of membership and non-membership of edges.

Definition 3. An intuitionistic fuzzy graph is said to be an edge graceful labeling graph if the following conditions hold for the bijective maps $\sigma = (T_1, F_1)$ and $\mu = (T_2, F_2)$,
(i) The functions $T_1 : V \to [0, 1]$ and $F_1 : V \to [0, 1]$ denote the truth and false memberships respectively for vertices with condition $0 \leq T_1(v_i) + F_1(v_i) \leq 1$, for all $v_i \in V$,
(ii) $T_2 : E \subseteq V \times V \to [0, 1]$ and $F_2 : E \subseteq V \times V \to [0, 1]$ denote the truth and false memberships respectively for edges with condition $0 \leq T_2(e_j) + F_2(e_j) \leq 1$ such that $\sigma = (T_1, F_1)$ and $\mu = (T_2, F_2)$ are defined by $\sigma(v_i) = \sum \mu(e_j)$=$(\sum T_2(e_j), \sum F_2(e_j))$, where $i = 1, 2, \ldots, n$. (i.e.) if the labelings of the edges are given, then the vertex v_i of the intuitionistic fuzzy graph is labeled by the sum of the labels of the edges incident to it.

Definition 4. A path graph is a sequence of edges joining the sequence of vertices in which the vertex of degree 3 doesn't exist. A path graph is said to be an intuitionistic fuzzy path graph if intuitionistic fuzzy labeling holds. A cycle graph is a closed chain formed by some number of vertices and edges and it is said to be an intuitionistic fuzzy cycle graph if intuitionistic fuzzy labeling holds in the cycle.

Lemma 1. *An intuitionistic fuzzy graph holds the edge graceful labeling if it satisfies the following condition for each edge of the graph. (i.e.)* $\mu = (T_2(e_j), F_2(e_j)) = (s_1(k_1 + 1), s_2(k_2+1))$*; where* $s_1, s_2 \in (0, 1)$ *and* $k_1, k_2 \in \mathbb{W}$*. To retain the range of each membership of edges and their total sum, minimize* s_1, s_2 *value of edge membership by* $\frac{s_1}{10^{m_1}}, \frac{s_2}{10^{m_2}}$*;* $m_1, m_2 \in \mathbb{W}$ *for consecutive order(either odd order or even order or both) of graphs, such that each vertex label must be distinct corresponding to the memberships.*

Note 1. The truth membership function (T) and false membership function (F) of an intuitionistic fuzzy graph are complement to each other (i.e.), $T = 1 - F$, which implies that both memberships are independent. Therefore, s_1 and s_2 values of edge memberships in the above Lemma 1 may or may not be minimized simultaneously, which regards the equal and unequal values of m_1 and m_2 respectively.

Theorem 1. *An intuitionistic fuzzy path graph* P_n*,* $n \geq 3$*, admits an intuitionistic fuzzy edge graceful labeling if* n *is odd.*

Proof. Let P_n, $n \geq 3$ be the path graph with intuitionistic fuzzy edge labels. For the vertex set $V = \{v_i \mid 1 \leq i \leq n\}$ of the path, the edge set is $E = \{e_j / 1 \leq j \leq n-1\} = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$ such that $\mu = (T_2(e_j), F_2(e_j)) > 0$. The label value for each vertex is obtained using the following expression, $\sigma(v_i)$=$\sum \mu(e_j)$, where $i = 1, 2, \cdots, n$, $j = 1, 2, \ldots, n-1$. If the intuitionistic fuzzy path graph is edge graceful, then the conditions given in Definition 3 hold here.
If n is odd and $n \geq 3$ for intuitionistic fuzzy path P_n, then the edge graceful labeling occurs for consecutive labels of edges starting from left to right or from right to left as per the Lemma 1. Each membership value of all the edges follows $\mu(e_j) > 0$, where j=1 to $n - 1$ and the truth & false memberships of each edge and vertex follow Definition 1, so that the graphs P_3 and P_5 in Figure 1 are said to be intuitionistic fuzzy

paths. Also the intuitionistic path graphs P_3 and P_5 satisfy the edge graceful labeling, since the given edges and the obtained vertices are distinct by each membership.
Hence, P_3 and P_5 are intuitionistic fuzzy edge graceful path graphs. In a similar way, intuitionistic fuzzy path graph of consecutive odd order are said to have an intuitionistic fuzzy edge graceful labeling. $\square$

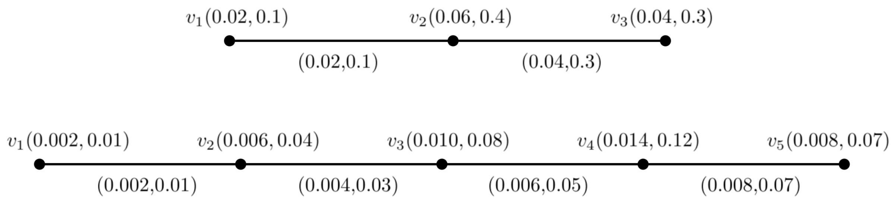

Figure 1. **Intuitionistic edge graceful P_3 & P_5 graph**

By considering the Definitions 1, 2, 3 and Lemma 1, the condition to label the edges of path P_n is defined as, $\mu = (T_2(e_{j=\frac{k_1+1}{2}}), F_2(e_{j=\frac{k_2+2}{2}}))=(s_1(k_1+1), s_2(k_2+1))$, for $k_1 = 1, 3, \ldots, 2n-3$ and $k_2 = 0, 2, \ldots, 2n-4$. Here, s_1 and s_2 values for P_3 is taken as 0.01 and 0.1. The graph P_5 yields the same result by minimizing s_1, s_2 of truth and false membership of edges by fixing $m_1 = m_2 = 1$ as per the Lemma 1. The intuitionistic fuzzy path graphs of consecutive odd order are proved to have an edge graceful labeling by increasing the m_1 and m_2 values successively.

Corollary 1. *An intuitionistic fuzzy path graph P_n, $n \geq 2$, does not admit an intuitionistic fuzzy edge graceful labeling if n is even.*

Proof. Suppose n is even for path P_n, $n \geq 2$, there exist some repeated membership values at some vertices which violate the condition for intuitionistic fuzzy labeling(in obtaining distinct vertices by each membership value correspondingly)fails.
Therefore, the path P_n, $n \geq 2$ with intuitionistic fuzzy labeling is not an intuitionistic fuzzy edge graceful graph, if n is even. $\square$

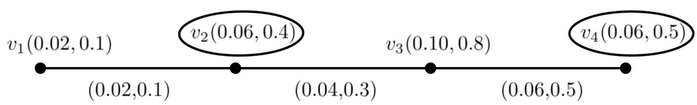

Figure 2. P_4

Example 1. The truth membership of the vertex labels are same at v_2 and v_4 in Figure 2, which implies that P_4 doesn't obey the edge graceful labeling property. Successively, every even path graph results in the same because of similar membership value occurs at different vertices.

Theorem 2. *An intuitionistic fuzzy cycle graph C_n, $n \geq 3$, admits an intuitionistic fuzzy edge graceful labeling if n is odd.*

Proof. Let C_n, $n \geq 3$ be the cycle graph with intuitionistic fuzzy edge labels. For the vertex set $V = \{v_i/1 \leq i \leq n\}$ of the cycle, the edge set is $E = \{e_j/1 \leq j \leq n\}$= $\{v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{v_nv_1\}$ such that $\mu = (T_2(e_j), F_2(e_j)) > 0$. The label value for each vertex is obtained by using the following expression, $\sigma(v_i)$=$\sum \mu(e_j)$, where $i, j = 1, 2 \cdots, n$. If the intuitionistic fuzzy cycle graph admits edge graceful labeling, then the conditions illustrated in Definition 3 hold here. If n is odd and $n \geq 3$ for intuitionistic fuzzy cycle C_n, then the edge graceful labeling occurs for clockwise consecutive labels of edges as per the Lemma 1. Each membership value of all the edges follows $\mu(e_j) > 0$, where j=1 to n and the truth & false memberships of each edge and vertex follow Definition 1, so that the graphs C_3 and C_5 in Figure 3 are said to be intuitionistic fuzzy cycles. Also, C_3 and C_5 graphs satisfy the edge graceful labeling since the given edges and the obtained vertices are distinct by each membership.
Hence the graphs C_3 and C_5 are declared to be intuitionistic fuzzy edge graceful cycle graphs. In a similar way, intuitionistic fuzzy cycle graphs of consecutive odd order are said to have an intuitionistic fuzzy edge graceful labeling. □

Example 2. The condition to label the edges for cycle C_n is defined as, $\mu = (T_2(e_{j=\frac{k_1+1}{2}}),$ $F_2(e_{j=\frac{k_2+2}{2}}))= (s_1(k_1 + 1), s_2(k_2 + 1))$ for $k_1 = 1, 3, \ldots, 2n-1$ and $k_2 = 0, 2, \ldots, 2(n-1)$ by considering the Definitions 1, 2, 3 and Lemma 1. Here, s_1 and s_2 values for C_3 is taken as 0.01 and 0.1. The graph C_5 yields the same result by minimizing s_1, s_2 of truth and false membership of edges by fixing $m_1 = m_2 = 1$ as per the Lemma 1. The intuitionistic fuzzy cycle graphs of consecutive odd order are proved to have an edge graceful labeling by increasing the m_1 and m_2 values successively.

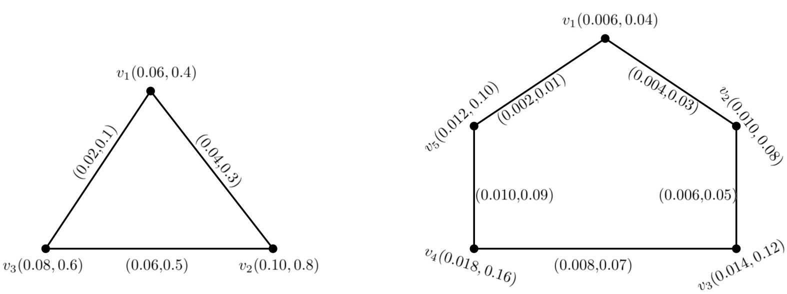

Figure 3. **Intuitionistic edge graceful C_3 and C_5**

Corollary 2. *An intuitionistic fuzzy cycle graph C_n, $n \geq 4$, does not admit an intuitionistic fuzzy edge graceful labeling if n is even.*

Proof. Suppose n is even for cycle C_n, $n \geq 4$, there exists some repeated membership values at some vertices which violate the condition for intuitionistic labeling(in obtaining distinct vertices by each membership value correspondingly) fails.
Therefore, cycle C_n, $n \geq 4$ with intuitionistic fuzzy labeling is not an intuitionistic edge graceful graph, if n is even. □

Example 3. The vertex labels are the same at v_1 and v_3 in Figure 4, which implies that C_4 doesn't obey the edge graceful labeling property. Successively, every even cycle graph results in the same because of similar vertex labels (by each membership) at different vertices.

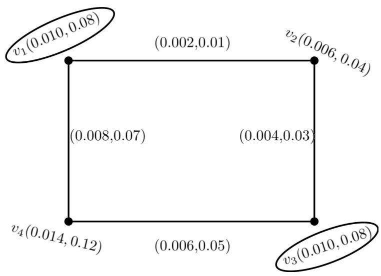

Figure 4. C_4

Definition 5. A wheel graph W_n is a graph with n vertices ($n \geq 4$) obtained by the union of a star graph S_n and a cycle with $n-1$ vertices (i.e),S_n+C_{n-1}. A wheel graph with intuitionistic fuzzy labeling is called an intuitionistic fuzzy wheel graph. An intuitionistic fuzzy wheel graph has a vertex set $V = \{v_c\} \cup \{v_i\}$ such that $\mu(v_c v_i) > 0$, where i= 1 to $n-1$ and $\mu(v_i v_{i+1}) > 0$, where i= 1 to $n-2$. If all edges of an intuitionistic fuzzy wheel graph are distinct, then it is called an intuitionistic fuzzy edge graceful wheel graph.

Theorem 3. *An intuitionistic fuzzy wheel graph W_n, $n \geq 4$, admits an intuitionistic fuzzy edge graceful labeling.*

Proof. Consider a wheel graph W_n with intuitionistic fuzzy edge labels. For the vertex set $V = \{v_c\} \cup \{v_i/1 \leq i \leq n-1\}$, the edge set is $E = \{e_j/1 \leq j \leq 2(n-1)\}$= $\{v_c v_i/1 \leq i \leq n-1\} \cup \{v_i v_{i+1}/1 \leq i \leq n-2\} \cup \{v_{n-1}v_1\}$ such that $\mu(v_c v_i) > 0$, for $1 \leq i \leq n-1$ and $\mu(v_i v_{i+1}) > 0$, for $1 \leq i \leq n-2$. The vertex labels of the wheel graph are given using the following expression, $\sigma(v) = \sum \mu(e_j)$, where $j = 1, 2, \ldots, 2(n-1)$. If the intuitionistic fuzzy wheel graph is edge graceful, then the conditions given in Definition 3 hold here. If $n \geq 4$ for intuitionistic fuzzy wheel W_n, then the edge graceful labeling occurs for consecutive anticlockwise labels for edges connected to central vertex v_c and consecutive clockwise labels for outer edges

connected to other vertices of V as per the Lemma 1. Since, each membership value of all the edges follows $\mu(e_j) > 0$, where j=1 to $2(n-1)$ and also the truth and false memberships of each edge and vertex follow Definition 1, the graphs W_4 & W_5 are declared as intuitionistic fuzzy wheel graph. Also, W_4 & W_5 graphs satisfy the edge graceful labeling since the given edges and the obtained vertices are distinct by each membership.
Hence, the graphs W_4 & W_5 are intuitionistic fuzzy edge graceful wheel graphs. In a similar way, intuitionistic fuzzy wheel graph of consecutive order are said to have an intuitionistic fuzzy edge graceful labeling. □

Example 4. By considering the Definitions 1, 2, 3 and Lemma 1, the condition to label the edges of W_n is defined as $\mu = (T_2(e_{j=\frac{k_1+1}{2}}), F_2(e_{j=\frac{k_2+2}{2}}))$=$(s_1(k_1+1), s_2(k_2+1))$, for $k_1 = 1, 3, \ldots, 4n-5$ and $k_2 = 0, 2, \cdots, 4n-6$. The graphs W_4 and W_5 shown in Figure 5 are declared as intuitionistic fuzzy wheel graph. Here, s_1 and s_2 values for W_4 is taken as 0.1 and 0.01. The graph W_5 yields the same result by minimizing s_1 and continuing with the same s_2 value of truth and false membership of edges respectively, by fixing $m_1 = 1$ and $m_2 = 0$ as per the Lemma 1. Intuitionistic fuzzy wheel graphs of successive order are proved to have an edge graceful labeling by increasing the values of m_1 and m_2 successively.

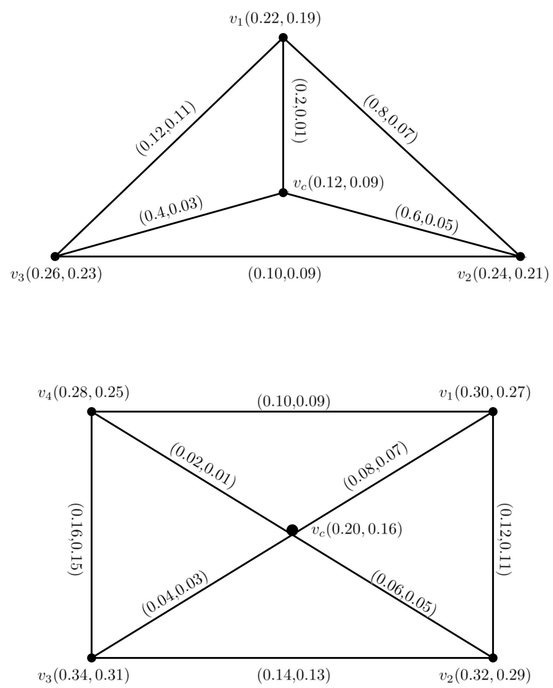

Figure 5. **Intuitionistic edge graceful W_4 and W_5**

3. Neutrosophic Edge Graceful Labeling Graphs

An edge graceful labeling in a neutrosophic environment is quite similar to the intuitionistic approach but neutrosophic edge graceful labeling tends to increase the accuracy in the final result using an additional membership function("indeterminacy"). With the knowledge about the intuitionistic fuzzy edge graceful graphs discussed in the previous section, the edge graceful labeling concept has been incorporated newly in neutrosophic graphs. To deal with the indeterminacy explicitly in a system or any other application-oriented problems, the edge graceful labeling concept has been extended from intuitionistic fuzzy graphs to neutrosophic graphs. In this section, some neutrosophic graphs will be considered with a set of edge labels given in a clockwise or anticlockwise manner to prove the edge graceful labeling and its applications.

Definition 6. ([18]) A neutrosophic graph is of the form $G = (V, \sigma, \mu)$, where $\sigma = (T_1, I_1, F_1)$ and $\mu = (T_2, I_2, F_2)$ with the following conditions,
(i) The functions $T_1 : V \to [0, 1]$, $I_1 : V \to [0, 1]$ and $F_1 : V \to [0, 1]$ denote the degree of truth, indeterminacy and false membership functions of the element $v_i \in V$, respectively and $0 \leq T_1(v_i) + I_1(v_i) + F_1(v_i) \leq 3$, for all $v_i \in V$.
(ii) The functions $T_2 : E \subseteq V \times V \to [0, 1]$, $I_2 : E \subseteq V \times V \to [0, 1]$ and $F_2 : E \subseteq V \times V \to [0, 1]$ denote the degree of truth, indeterminacy and false membership functions of the edge (v_i, v_j) respectively, such that
$T_2(v_i, v_j) \leq min[T_1(v_i), T_1(v_j)]$,
$I_2(v_i, v_j) \leq min[I_1(v_i), I_1(v_j)]$,
$F_2(v_i, v_j) \leq max[F_1(v_i), F_1(v_j)]$ and $0 \leq T_2(v_i, v_j) + I_2(v_i, v_j) + F_2(v_i, v_j) \leq 3$,
for every edge (v_i, v_j).

Definition 7. ([18]) A neutrosophic graph $G = (V, \sigma, \mu)$, where $\sigma = (T_1, I_1, F_1)$ and $\mu = (T_2, I_2, F_2)$ is said to be an neutrosophic labeling graph, if $T_1 : V \to [0, 1]$, $I_1 : V \to [0, 1]$ $F_1 : V \to [0, 1]$ and $T_2 : V \times V \to [0, 1]$, $I_2 : V \times V \to [0, 1]$, $F_2 : V \times V \to [0, 1]$ are bijective such that the truth, indeterminacy and false membership functions of the vertices and edges are distinct and
$T_2(v_i, v_j) \leq min[T_1(v_i), T_1(v_j)]$,
$I_2(v_i, v_j) \leq min[I_1(v_i), I_1(v_j)]$,
$F_2(v_i, v_j) \leq max[F_1(v_i), F_1(v_j)]$ and $0 \leq T_2(v_i, v_j) + I_2(v_i, v_j) + F_2(v_i, v_j) \leq 3$,
for every edge (v_i, v_j).

Definition 8. A neutrosophic graph is said to be an edge graceful labeling graph if the following conditions hold for the bijective maps $\sigma = (T_1, I_1, F_1)$ and $\mu = (T_2, I_2, F_2)$,
(i) The functions $T_1 : V \to [0, 1]$, $I_1 : V \to [0, 1]$ and $F_1 : V \to [0, 1]$ denote the truth, indeterminacy and false memberships respectively for vertices with condition $0 \leq T_1(v_i) + I_1(v_i) + F_1(v_i) \leq 3$, for all $v_i \in V$,
(ii) $T_2 : E \subseteq V \times V \to [0, 1]$, $I_2 : E \subseteq V \times V \to [0, 1]$ and $F_2 : E \subseteq V \times V \to [0, 1]$ denote the truth, indeterminacy and false memberships respectively for edges with condition $0 \leq T_2(e_j) + I_2(e_j) + F_2(e_j) \leq 3$ such that $\sigma = (T_1, I_1, F_1)$ and $\mu = (T_2, I_2, F_2)$ are defined by $\sigma(v_i) = \sum \mu(e_j)$=$(\sum T_2(e_j), \sum I_2(e_j), \sum F_2(e_j))$, where $i = 1, 2, \cdots, n$. (i.e.), if the labelings of the edges are given, then the vertex v_i of a neutrosophic graph is labeled by the sum of the labels of the edges incident to it.

Definition 9. A fan graph $F_{m,n}$ is defined as the graph join $K_m + P_n$, where K_m is the empty graph on m vertices and P_n is the path graph on n vertices. If $m = 1$ in K_m, then it corresponds to the normal fan graph and $m = 2$ corresponds to the double fan graph. A fan graph with neutrosophic labeling is called a neutrosophic fan graph. A neutrosophic fan graph has a vertex set $V = \{v_c\} \cup \{v_i\}$ such that $\mu(v_c v_i) > 0$, where i=1 to n and $\mu(v_i v_{i+1}) > 0$, where i=1 to $n-1$. If all edges of a neutrosophic fan graph are distinct, then it is called a neutrosophic edge graceful fan graph.

Lemma 2. *A neutrosophic graph holds the edge graceful labeling if it satisfies the following condition for each edge of the graph. (i.e), $\mu = (T_2(e_j), I_2(e_j), F_2(e_j)) = (s_1(k_1+1), s_2(k_2+1), s_3(k_3))$; where $s_1, s_2, s_3 \in (0,1)$ and $k_1, k_2, k_3 \in \mathbb{W}$. To retain the range of each membership of edges and their total sum, minimize s_1, s_2, s_3 values of edge membership by $\frac{s_1}{10^{m_1}}, \frac{s_2}{10^{m_2}}, \frac{s_3}{10^{m_3}}$; $m_1, m_2, m_3 \in \mathbb{W}$, for consecutive order(either odd order or even order or both) of graphs, such that each vertex label must be distinct corresponding to the memberships.*

Note 2. The truth membership function (T), indeterminacy membership function (I), and false membership function (F) of a neutrosophic graph are independent in nature. Therefore, s_1, s_2, and s_3 values of edge memberships in the above Lemma 2 may or may not be minimized simultaneously, which depends on equal or unequal values of m_1, m_2, and m_3 respectively.

Theorem 4. *A neutrosophic fan graph $F_{1,n}$, $n \geq 2$, admits a neutrosophic edge graceful labeling.*

Proof. Consider a fan graph $F_{1,n}$, $n \geq 2$ with neutrosophic edge labels. For the vertex set $V(v) = \{v_c\} \cup \{v_i/1 \leq i \leq n\}$, the edge set is $E(e) = \{e_j/1 \leq j \leq 2n-1\} = \{v_c v_i/1 \leq i \leq n\} \cup \{v_i v_{i+1}/1 \leq i \leq n-1\}$ such that $\mu(v_c, v_i) > 0$, for $1 \leq i \leq n$ and $\mu(v_i v_{i+1}) > 0$, for $1 \leq i \leq n-1$. The label value for each vertex of the fan graph is given by the following expression, $\sigma(v) = \sum \mu(e_j)$, where $j = 1, 2, \cdots, 2n-1$. If the neutrosophic fan graph is edge graceful, then the conditions given in Definition 8 hold here. If n is even & $n \geq 2$, then for $F_{1,n}$, the edge graceful labeling occurs for consecutive clockwise labels of edges connected to central vertex v_c and consecutive clockwise labels for outer edges connected to other vertices of V and if n is odd and $n \geq 3$, then for $F_{1,n}$, the edge graceful labeling occurs for consecutive anticlockwise labels of edges connected to central vertex v_c and consecutive clockwise labels for outer edges connected to other vertices of V as per Lemma 2. Since, each membership value of all the edges follows $\mu(v_c v_i) > 0$, for $1 \leq i \leq n$ and $\mu(v_i v_{i+1}) > 0$, for $1 \leq i \leq n-1$ and also the truth, indeterminacy and false memberships of each edge and vertex follow Definition 6, the graphs shown in Figures 6 and 7 are said to be neutrosophic fan graphs. Fan graphs in Figures 6 and 7 also satisfy the edge graceful labeling, since the assigned edge labels and the resultant vertex labels are distinct by each membership. Hence, the graphs $F_{1,n}$ are neutrosophic edge graceful fan graphs if $n = 2, 3, 4, 5$. The same criteria are followed to prove this result for fan graphs of consecutive order. $\square$

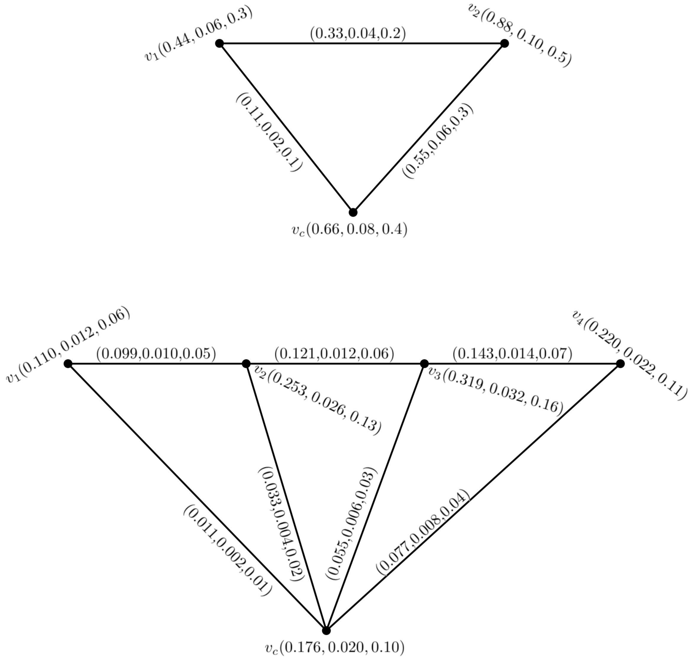

Figure 6. **Neutrosophic edge graceful $F_{1,2}$ & $F_{1,4}$ graphs**

Example 5. For a fan graph $F_{1,2}$, by considering the Definitions 6, 7, 8 and Lemma 2, the condition to label the edges is defined as $\mu = (T_2(e_{j=\frac{k_1+2}{2}}), I_2(e_{j=\frac{k_2+1}{2}}), F_2(e_{j=k_3}))= (s_1(k_1+1), s_2(k_2+1), s_3(k_3))$, where $s_1, s_2, s_3 \in (0, 1)$, $k_1 = 0, 2, \ldots, 4(n-1)$, $k_2 = 1, 3, \ldots, 4n-3$ and $k_3 = 1, 2, \ldots, 2n-1$. Here, s_1, s_2, and s_3 values for $F_{1,2}$ are taken as 0.11, 0.01, and 0.1 respectively. The graph $F_{1,4}$ yields the same result by minimizing s_1, s_2 and s_3 of truth, indeterminacy, and false membership of edges respectively, by fixing $m_1 = m_2 = m_3 = 1$ as per Lemma 2. The other neutrosophic fan graphs are proved to have an edge graceful labeling by increasing the m_1, m_2, and m_3 values successively. For a fan $F_{1,3}$, similar results can be obtained with the same condition to represent the edge labels but varying the values of s_1, s_2, and s_3 are needed. Here, s_1, s_2, and s_3 values for $F_{1,3}$ are taken as 0.011, 0.001, and 0.01 respectively. The graph $F_{1,4}$ yields the same result by maintaining the s_1, s_2 and s_3 of truth, indeterminacy, and false membership of edges respectively, by taking $m_1 = m_2 = m_3 = 0$ as per the Lemma 2. The other fan graphs are proved to have an edge graceful labeling by increasing the m_1, m_2, and m_3 values successively.

Corollary 3. *A neutrosophic double fan graph $F_{2,n}$,$n \geq 2$, also admits a neutrosophic edge graceful labeling.*

Definition 10. A friendship graph F_n,$n \geq 1$, is a graph that consists of n copies of cycles with a common vertex. A friendship graph with neutrosophic labeling is called a neutrosophic friendship graph which comprises a vertex set $V = \{v_c\} \cup \{v_i\}$ such that $\mu(v_c v_i) > 0$, where i= 1 to $n-1$ and $\mu(v_i v_{i+1}) > 0$, where i= 1 to $n-2$. If all edge values of a neutrosophic friendship graph are distinct, then it is called a neutrosophic edge graceful friendship graph.

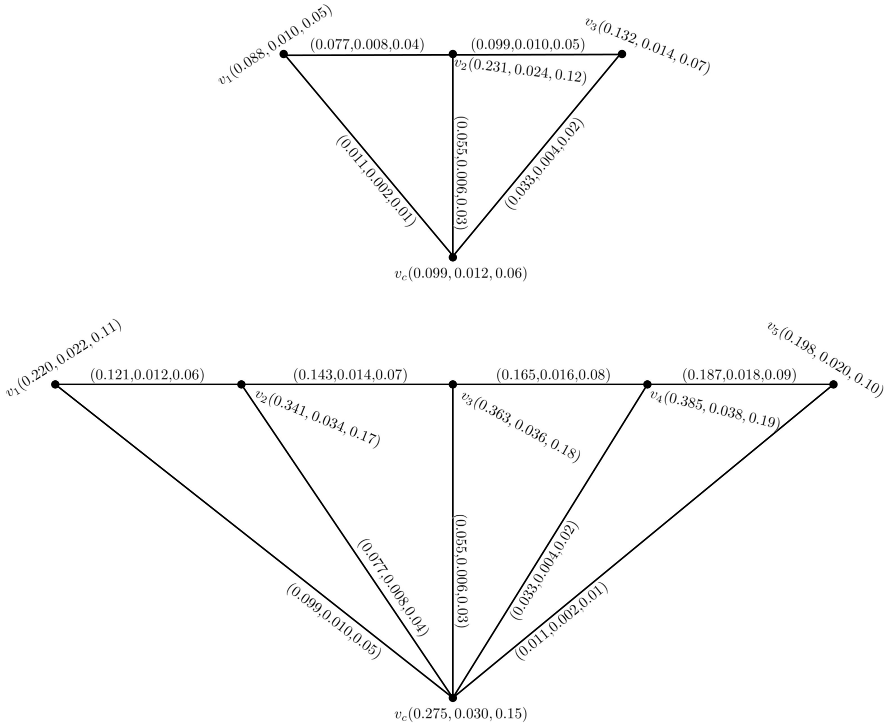

Figure 7. **Neutrosophic edge graceful $F_{1,3}$ & $F_{1,5}$ graphs**

Theorem 5. *A neutrosophic friendship graph F_n ,$n \geq 1$, admits a neutrosophic edge graceful labeling.*

Proof. Consider a friendship graph F_n,$n \geq 1$, with neutrosophic edge labels. For the vertex set $V(v) = \{v_c\} \cup \{v_i/1 \leq i \leq 2n\}$, the edge set is $E(e)= \{e_j/1 \leq j \leq 3n\}=\{v_cv_i/1 \leq i \leq 2n\} \cup \{v_iv_{i+1}/\text{i is odd}, 1 \leq i \leq 2n-1\}$ such that $\mu(v_cv_i) > 0$, for $1 \leq i \leq 2n$ and $\mu(v_iv_{i+1}) > 0$, for i is odd and $1 \leq i \leq 2n-1$. The vertex labels are obtained for friendship graphs, by the expression $\sigma(v) = \sum \mu(e_j)$, where $j = 1, 2, \cdots, 3n$. The neutrosophic friendship graph is edge graceful if Definition 8 holds. If $n \geq 1$, then for F_n, the edge graceful labeling occurs for consecutive clockwise or anticlockwise labels of all edges as per Lemma 2. The graphs shown in Figure 8 are neutrosophic friendship graphs, since each membership value of all the edges follows $\mu(v_cv_i) > 0$, for $1 \leq i \leq 2n$ and $\mu(v_iv_{i+1}) > 0$, for i is odd and $1 \leq i \leq 2n-1$ and also the truth, indeterminacy and false memberships of each edge and vertex follow Definition 6. Also, the friendship graphs in Figure 8 satisfy the edge graceful labeling, since the assigned value of edges and the resultant vertices are distinct by each membership.
Hence, the graphs F_n are neutrosophic edge graceful friendship graphs, if $n = 1, 2, 3$.

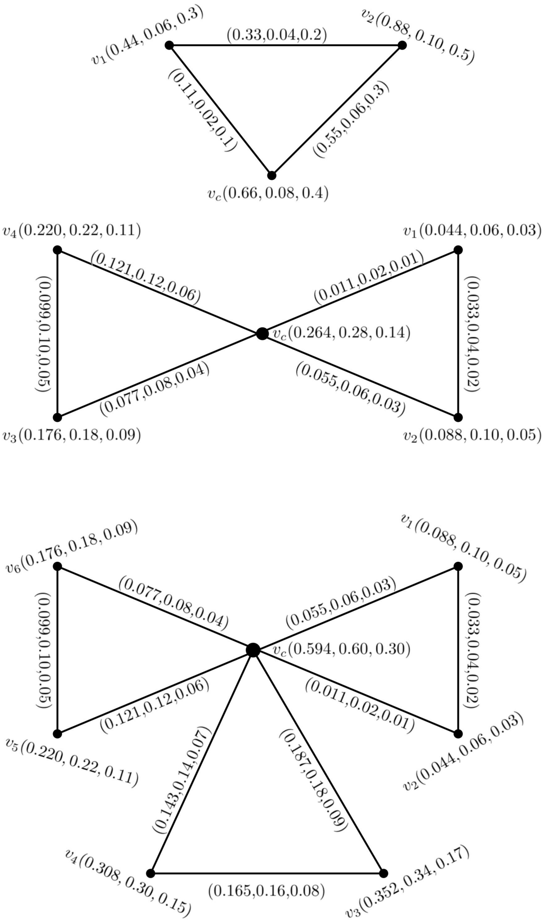

Figure 8. **Neutrosophic edge graceful F_1, F_2, F_3 graphs**

Similarly, other friendship graphs are proven to have an edge graceful labeling. □

Example 6. For F_n, $n \geq 1$, by considering the above stated Definitions 6, 7, 8 and Lemma 2, the condition to label the edges is defined as $\mu = (T_2(e_{j=\frac{k_1+2}{2}}), I_2(e_{j=\frac{k_2+1}{2}}), F_2(e_{j=k_3}))=$ $(s_1(k_1+1), s_2(k_2+1), s_3(k_3))$, where $s_1, s_2, s_3 \in (0,1)$, $k_1 = 0, 2, \cdots, 6n-2$, $k_2 = 1, 3, \cdots, 6n-1$ and $k_3 = 1, 2, \cdots, 3n$. Here, s_1, s_2, and s_3 values for F_1 are taken as 0.11, 0.01, and 0.1 respectively. The graph F_2 yields the same result by maintaining s_2 and minimizing s_1 & s_3 of truth and false memberships of edges respectively, by taking $m_1 = m_3 = 1$ as per Lemma 2. F_3 yields the same result by maintaining s_1, s_2 and s_3 values already used for F_2. In Figure 8, F_1 and F_2 are configured using consecutive clockwise edge labels and F_3 is configured by consecutive anticlockwise edge labels. The other neutrosophic friendship graphs are proved to have an edge graceful labeling by increasing the m_1, m_2, and m_3 values successively.

Definition 11. ([42])
A generalized butterfly graph $BF_n, n \geq 2$, obtained by inserting vertices to every wing with the assumption that the sum of inserting vertices to every wing is the same and it has $2n+1$ vertices and $4n-2$ edges. A generalized butterfly graph with neutrosophic labeling is called a generalized neutrosophic butterfly graph which comprises a vertex set $V(v) = \{v_c\} \cup \{v_i/1 \leq i \leq 2n\}$ and the edge set $E(e) = \{e_j/1 \leq j \leq 4n-2\} = \{v_c v_i/1 \leq i \leq 2n\} \cup \{v_i v_{i+1}/1 \leq i \leq 2n-1\}$ such that $\mu(v_c v_i) > 0$ for $1 \leq i \leq 2n$ and $\mu(v_i v_{i+1}) > 0$, for $1 \leq i \leq 2n-1$. If all the edge values of a neutrosophic generalized butterfly graph are distinct, then it is called a neutrosophic edge generalized butterfly graph.

Theorem 6. *A generalized neutrosophic butterfly graph BF_n, $n \geq 2$, admits a neutrosophic edge graceful labeling.*

Proof. Consider a generalized butterfly graph BF_n, $n \geq 2$, with neutrosophic edge labels. For the vertex set $V(v) = \{v_c\} \cup \{v_i/1 \leq i \leq 2n\}$, the set is $E(e) = \{e_j/1 \leq j \leq 4n-2\} = \{v_c v_i/1 \leq i \leq 2n\} \cup \{v_i v_{i+1}/1 \leq i \leq 2n-1\}$ such that $\mu(v_c v_i) > 0$, for $1 \leq i \leq 2n$ and $\mu(v_i v_{i+1}) > 0$, for $1 \leq i \leq 2n-1$. Each vertex is labeled using the following expression, $\sigma(v) = \sum \mu(e_j)$, where $j = 1, 2, \cdots, 4n-2$. If the neutrosophic generalized butterfly graph is edge graceful, then the Definition 8 hold here. In BF_n, $n \geq 2$, the edge graceful labeling occurs for consecutive clockwise

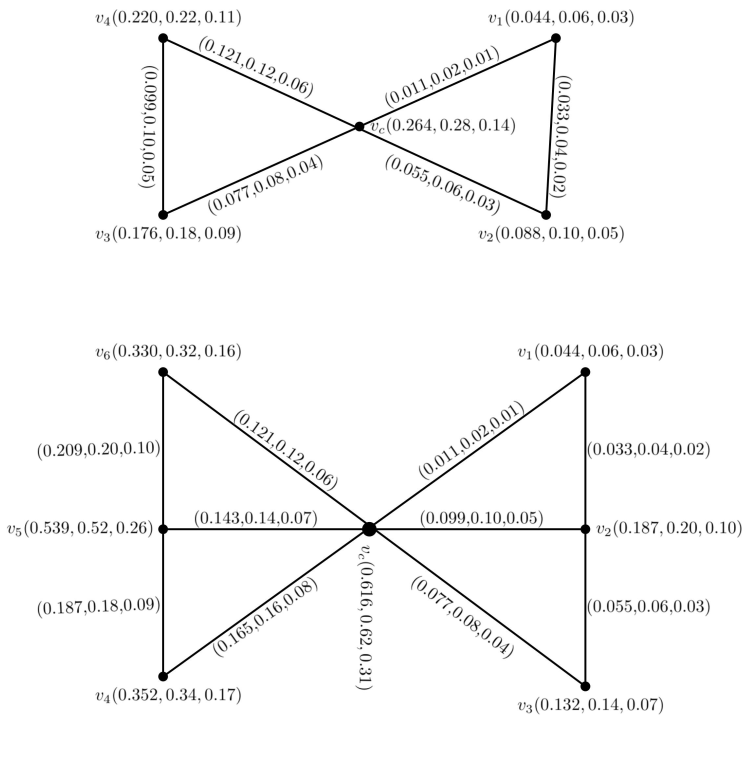

Figure 9. **Neutrosophic edge graceful butterfly graphs**

edge labels given in both wings of butterfly graph. Since, each membership value of all the edges follows $\mu(v_c v_i) > 0$, for $1 \leq i \leq 2n$ and $\mu(v_i v_{i+1}) > 0$, for $1 \leq i \leq 2n-1$ and also the truth, indeterminacy and false memberships of each edge and vertex follows Definition 6, the graphs shown in Figure 9 are said to be neutrosophic butterfly graphs. Also, the butterfly graphs in Figure 9 satisfy the edge graceful labeling since the given edges and the obtained vertices are distinct by each membership.
Hence, the graphs BF_n are neutrosophic edge graceful butterfly graphs if $n = 2, 3$. The other butterfly graphs are proved to be edge graceful labeling graph in a similar manner. □

Example 7. For BF_2, consider the above mentioned Definitions 6, 7, 8 and Lemma 2, to label the edges as $\mu = (T_2(e_{j=\frac{k_1+2}{2}}), I_2(e_{j=\frac{k_2+1}{2}}), F_2(e_{j=k_3}))$=$(s_1(k_1+1), s_2(k_2+1), s_3(k_3))$, where $s_1, s_2, s_3 \in (0,1)$, $k_1 = 0, 2, \cdots, 8n-6$, $k_2 = 1, 3, \cdots, 8n-5$ and $k_3 = 1, 2, \cdots, 4n-2$. Here, s_1, s_2, and s_3 values for BF_2 are taken as 0.011, 0.01, and 0.01 respectively. The graph BF_3 yields the same result with the unchanged s_1, s_2, and s_3 values of truth, indeterminacy, and false memberships of edges respectively, by taking $m_1 = m_2 = m_3 = 0$ as per the Lemma 2. The other neutrosophic butterfly graphs are proved to have an edge graceful labeling by increasing the m_1, m_2, and m_3 values successively.

4. Irregular Properties Of Neutrosophic Edge Graceful Labeling Graphs

Regular and irregular properties in fuzzy graph structure are nothing but the consideration of vertex with the same and different degrees in their adjacent vertices. However, in the intuitionistic fuzzy and neutrosophic concept of graphs, the degree should be evaluated separately for each membership, and we have to check its unique and varying degrees individually for each membership. The complexity is not much in finding irregularity for each membership of a vertex since the neutrosophic edge graceful labeling(NEGL) graph automatically generates the vertex membership values by using the sum of edge memberships and checks for distinctness in an individual manner. This complexity allows modeling more nuanced and uncertain relationships compared to traditional graphs.

Definition 12. ([13]) Let $G = (V, E)$ be a neutrosophic graph. The degree of a vertex $v_i \in G$ is sum of degrees of truth, indeterminacy and false memberships of all those edges which are incident on vertex v_i denoted by $d_G(v_i) = (d_T(v_i), d_I(v_i), d_F(v_i))$, where $d_T(v_i)$= $\sum_{v_i \neq v_j} T_2(v_i, v_j)$, $d_I(v_i)$= $\sum_{v_j \neq v_j} I_2(v_i, v_j)$ and $d_F(v_i)$= $\sum_{v_i \neq v_j} F_2(v_i, v_j)$ denote the degree of truth, indeterminacy and false memberships of vertex, for $v_i, v_j \in V$ and $T_2(v_i, v_j) = 0, I_2(v_i, v_j) = 0, F_2(v_i, v_j) = 0$, for $v_i, v_j \notin V$.

Definition 13. ([13]) Let $G = (V, E)$ be a neutrosophic graph. The total degree of a vertex $v_i \in G$ is defined by $td(v_i) = (td_T(v_i), td_I(v_i), td_F(v_i))$, where $td_T(v_i)$= $\sum_{v_i \neq v_j} T_2(v_i, v_j) + T_1(v_i)$, $td_I(v_i)$= $\sum_{v_i \neq v_j} I_2(v_i, v_j) + I_1(v_i)$, $td_F(v_i)$= $\sum_{v_i \neq v_j} F_2(v_i, v_j) + F_1(v_i)$ denote the total degree of truth, indeterminacy and false memberships of vertex, for $v_i, v_j \in V$.

Definition 14. A neutrosophic graph $G = (V, E)$ is said to be $(K_1, K_2, K_3) - regular$, if $d_G(v_i)= (d_T(v_i), d_I(v_i), d_F(v_i))= (K_1, K_2, K_3)$, for all $v_i \in V$ and also G is said to be regular neutrosophic graph of degree (K_1, K_2, K_3).

Definition 15. A neutrosophic graph $G = (V, E)$ is said to be irregular, if there is a vertex which is adjacent to vertices with distinct degrees (i.e.) each membership degree of adjacent vertices is not constant.

Definition 16. Let $G = (V, E)$ be a connected neutrosophic graph and G is said to be a neighbourly irregular neutrosophic graph, if every two adjacent vertices of G have distinct degrees.

Definition 17. Let $G = (V, E)$ be a neutrosophic graph and G is said to be totally irregular, if there is a vertex v which is adjacent to vertices with distinct total degrees.

Definition 18. If every two adjacent vertices of a neutrosophic graph $G = (V, E)$ have distinct total degree, then G is said to be neighbourly total irregular neutrosophic graph.

Definition 19. Let $G = (V, E)$ be a connected neutrosophic graph and G is said to be highly irregular neutrosophic graph, if every vertex v_i of G is adjacent to vertices v_j $(i \neq j)$ with distinct degrees.

Irregularity Properties:
In general, NEGL graphs have distinct edge labels corresponding to each membership and each vertex label will be obtained as a sum of the edges incident to that vertex. So, the definition of NEGL yields the degree of every vertex which is nothing but the label value of every vertex. Since the sum of distinct edge labels yields the distinct vertex labels at every vertex, the degree of each vertex is not a constant to its corresponding membership value. Some of the irregular properties of NEGL graphs are as follows:

(i) **Irregular NEGL graphs:** Every vertex of a NEGL graph has distinct degree in its adjacent vertices.

(ii) **Totally irregular NEGL graphs:** Every vertex label of a NEGL graph is nothing but the degrees (distinct) of each vertex, the sum of the degree of each vertex and the corresponding vertex label yields distinct total degree.

(iii) **Neighbourly irregular NEGL graphs:** Every vertex of a NEGL graph ends with distinct degrees. So, the two adjacent vertices of the graph also have distinct degree.

(iv) **Neighbourly total irregular NEGL graphs:** Every vertex of a NEGL graph ends with distinct total degree. So, the same result will be obtained for every two adjacent vertices.

(v) **Highly irregular NEGL graphs:** Every vertex of a NEGL graph ends with distinct degrees. So, the adjacent vertices for every vertex will have distinct degree.

Note 3.
(i) Every NEGL graph is irregular, totally irregular, neighbourly irregular, neighbourly total irregular and highly irregular.

(ii) The NEGL graphs discussed in Section 3 are the examples for the above irregular properties.

5. Applications of Neutrosophic Edge Graceful Labeling Graphs

At present, neutrosophic graphs are very much useful to explicitly deal with the uncertain cases that occur in a system with a separate membership function called indeterminacy. Some new kinds of neutrosophic graphs like single-valued neutrosophic graphs, bipolar neutrosophic graphs, Pythagorean neutrosophic graphs, etc., were developed to solve the problems in different situations and dimensions [2, 5, 6]. But graceful labeling in neutrosophic graphs have not been demonstrated yet to build a model in real-life situations and system. Now, we deal with the current scenario in our day-to-day life with the representation using neutrosophic edge graceful labeling. As we sum up all the edge labels in the neutrosophic edge graceful labeling concept, comparing and analyzing components with highly efficient output can be studied. Here, we have an example to explore the importance of the neutrosophic edge graceful labeling(NEGL) concept in real-life situations.

(i) **Country with best trading practice**: Consider a neutrosophic vertex set that consists of five countries. Let the neutrosophic edge set represent the trade practice of these countries via cargo aircraft. These countries have decided to export their products for a certain amount and import the same valued products from neighboring countries. Here, each country has two neighboring countries. This agreement among the countries will surely benefit each and every country, since they are sharing the products with their neighboring countries, which reduces the transportation cost. After the survey taken on trade practices among these countries for a particular period, it is easy to evaluate the country which is actively involved in trade with their neighboring countries. The corresponding memberships of vertices and edges must have an interlink connection to portray the trade practice among these countries. Here, the truth membership of each vertex represents the efficiency of the country's trade system. The indeterminacy membership of vertices denotes uncertain issues that occur in the system. The false membership of vertices deals with the failure in the system. The truth membership of each edge shows the fine trade via aircraft among these countries. The indeterminacy membership of edges analyses uncertain conditions like weather change, clash/war, delay in shipment of products, etc that disrupt the trade journey among these countries. The false membership of

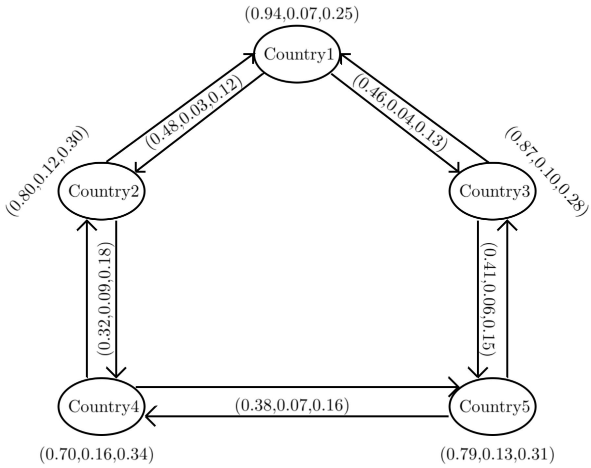

Figure 10. **Trade among neighboring countries**

edges says about the coarse trade due to some defects on aircraft. For example, the edge between country-1 and country-3 shows that the equivalent trade take place between them is 46%, 4% refers to uncertain conditions, and 13% is defective trade practice due to regular usage, technical issues, etc. Finally, summing up the edges (incident on each vertex) gives the total trade that occurred through cargo aircraft in each country (at vertices). On comparing the truth value of each vertex, one can find the country with top neighboring trade. Figure 10 shows that the truth value for country-1 is higher than other countries, which concludes that country-1 is actively involved in neighboring trade practice than other countries. Comparatively, indeterminacy and false membership values for country-1 are lesser than the other countries, which develop the system efficiency (truth value) of country-1 in trade.
The following algorithm explains the general procedure used here for our application.

Algorithm:
The existing algorithms in fuzzy and intuitionistic fuzzy labeling graphs lack effectiveness in case of explicit uncertainties. Neutrosophic graph algorithms provide a versatile framework to handle uncertainties explicitly and independently in complex environments. The feasibility of neutrosophic graph algorithms is promising but contingent on overcoming several theoretical, computational, and practical challenges. With advances in neutrosophic theory, computational methods, and real-world applications, neutrosophic graph algorithms could become a valuable tool for modeling and solving complex problems involving uncertainty and indeterminacy. However, they are still in an exploratory stage, requiring further research and development to reach their full potential. In general, the neutrosophic algorithms

are feasible for neutrosophic graphs because of their scalability and flexibility in representing uncertainty and vagueness, even though there comes computational complexity. Here, an algorithm for NEGL (a specialized case) is considered that can't be compared with the general existing algorithms of neutrosophic graph theory. An algorithm for this specific model is sequenced as below:

Step-1: Input the truth, indeterminacy, and false membership values for all edges(trade practice via cargo aircraft).
Step-2: Calculate the truth, indeterminacy and false membership values of all vertices using the expression $\sigma(v_i) = \sum \mu(e_j)=(\sum T_2(e_j), \sum I_2(e_j), \sum F_2(e_j))$, where $i = 1, 2, \cdots, n$.
Step-3: Find the neighboring countries of a country using directed edges between them.
Step-4: Find the active country in trade by the maximum truth membership, minimum indeterminacy membership, and minimum false membership of a vertex(trade system of a country).

(ii) **Decision-Making Application**: In general, there will be a relationship among every vertex and edge of a neutrosophic graph, which is involved in the decision-making analysis. But the edge graceful labeling of a neutrosophic graph pertains to the condition for some edges incident to a particular vertex. In this case, the overall relationship can't be stated for this analysis. A neutrosophic edge graceful labeling(NEGL) can be widely applied to illustrate decision-making analysis like social networks, supply chain management, telecommunication networks, etc. Figure 10 can be modified for the below assumptions:

(a) The structure of a NEGL social network graph is constructed by taking the edges as relationships and the vertices as persons. Each edge represents the relationship between persons with the memberships (honesty, unpredictable variations, dishonesty), and the vertices represent the persons who value their neighbors in total. Through this model, one can decide which person has got most trustworthy neighbors.

(b) A NEGL supply chain network is considered by assuming vertices to be the good suppliers and the edges denote the goods flow between them. Here, edge membership is taken as (delivery on time, uncertain cases like delay or shortages, failure to deliver). The vertices are the suppliers who value the neighboring delivery in total. This network model exemplifies the supplier with a good neighboring supply.

(c) The quality in the telecommunication network is studied here through the NEGL telecommunication network. Here, the edges represent the communication channels and the vertices represent the communication. The edge membership is taken as (quality in the channel, uncertain degradation, failure in the channel) and the vertices are the communications of the neighboring channels in total. This model marks the communication with good neighboring channels.

6. Conclusion

Graph labeling facilitates an efficient network design and also much helpful in data organization. It is widely used in other fields for problem-solving, pattern recognition, allocation of resources, etc. But there comes a complexity in finding optimal labeling and also in the case of graph structure and scalability limitation. A brief analysis and interpretation is done to rectify these issues and it results to the present fuzzy extensions. In this paper, edge graceful labeling is taken specifically on neutrosophic graphs to analyze its optimality and structural limitation. The intuitionistic fuzzy edge graceful labeling and neutrosophic edge graceful labeling have been explored as a consequence of the comprehensive discussion and analysis of the edge graceful labeling approach via intuitionistic fuzzy and neutrosophic graphs. In neutrosophic edge graceful labeling(NEGL) graphs, certain irregular aspects related to the degree of each vertex membership have been described and briefly explored. Finally, a real-world application and algorithm for NEGL graph is shown, which aid us to grasp the existing need and significance of such graphs. Our future work is to enrich the content and application of (i) graceful labeling on neutrosophic graph, (ii) vertex and edge-vertex graceful labeling on neutrosophic graph, (iii) Triangular graceful labeling on neutrosophic graph, etc.

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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