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Research Article

Decomposition of complete graphs into disconnected bipartite graphs with seven edges and eight vertices

Dalibor Froncek[1,*], Michael Kubesa[2]

[1]Department of Mathematics and Statistics, University of Minnesota Duluth, Duluth, MN 55812, U.S.A
*dalibor@d.umn.edu

[2]Department of Applied Mathematics, VŠB-Technical University of Ostrava, 17. listopadu 15, 708 33 Ostrava, Czech Republic
michael.kubesa@vsb.cz

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Abstract: In this paper, we continue investigation of decompositions of complete graphs into graphs with seven edges. The spectrum has been completely determined for such graphs with at most six vertices. Connected graphs with seven edges and seven vertices are necessarily unicyclic and the spectrum for bipartite ones was completely determined by the authors. Connected graphs with seven edges and eight vertices are trees and the spectrum was found by Huang and Rosa. As a next step in the quest of completing the spectrum for all graphs with seven edges, we completely solve the case of disconnected bipartite graphs with seven edges and eight vertices.

Keywords: graph decomposition, G-design, ρ-labeling.

AMS Subject classification: 05C51, 05C78

1. Introduction

Graph decompositions have been extensively studied for decades and became one of the classical themes in graph theory. Decomposition of complete graphs into mutually isomorphic subgraphs is probably the most popular topic within this area. We say that a graph G *decomposes* K_n if there exist subgraphs $G_1, G_2, \dots, G_s$ of K_n, all isomorphic to G, such that every edge of K_n appears in exactly one copy G_i of G. One selects a class of graphs $\mathcal{G}$, finite or infinite, and classifies complete graphs that

* *Corresponding Author*

admit a decomposition into all graphs in $\mathcal{G}$. A typical example of this is the Ringel Conjecture [10] stating that every tree on $n+1$ vertices decomposes the complete graph K_{2n+1}.

In this paper we continue the effort to classify all graphs with a given (small) number of vertices and/or edges and determine which complete graphs they decompose. Almost all graphs with up to six edges have been fully classified, as well as almost all graphs with eight edges. For a detailed overview, we refer the reader to [6]. For graphs with seven edges, much less is known. An overview of known results is presented in Section 2.

We continue in this direction by classifying all disconnected bipartite graphs with seven edges and eight vertices decomposing complete graphs. All such graphs are unicyclic with exactly two components. A *unicyclic graph* is a simple finite graph without loops containing exactly one cycle.

Our methods are mostly based on Rosa-type labelings, introduced by Rosa in 1967 [12].

2. Known results

While the graphs with at most six edges, as well as those with eight edges have been almost completely classified except for about a dozen of cases, the class of graphs with seven edges is still wide open.

Graphs with seven edges and five vertices are always connected and were classified by Bermond, Huang, Rosa, and Sotteau [1].

Theorem 1 (Bermond et al. [1]). *There exists a G-decomposition of K_n for a graph G on seven edges and five vertices if and only if*

1. $G = K_5 - K_{1,3}$, $n \equiv 0, 1 \pmod 7, n \geq 14$ *except possibly when* $n \in \{119, 120, 147, 203, 204\}$, *or*
2. $G = K_5 - K_3, n \equiv 1, 7 \pmod{14}$, *except possibly when* $n \in \{119, 120, 147, 203, 204\}$, *or*
3. $G = K_5 - (P_3 \cup P_2), n \equiv 0, 1 \pmod 7, n \neq 8, 14$ *except possibly when* $n \in \{16, 42, 56, 92, 98, 120\}$, *or*
4. $G = K_5 - (P_4), n \equiv 0, 1 \pmod 7, n \neq 8$.

Blinco [2] and Tian, Du, and Kang [13] studied connected graphs with seven edges and six vertices. The only disconnected graph with seven edges and six vertices is $K_4 \cup K_2$ and the spectrum for this graph was also found in [13].

Theorem 2 (Blinco [2], Tian et al. [13]). *There exists a G-decomposition of K_n for a graph G on seven edges and six vertices if and only if* $n \equiv 0, 1 \pmod 7$ *except for eight exceptions when* $n = 7$ *or* $n = 8$.

All graphs with seven edges and seven vertices are either connected and unicyclic or disconnected. A complete solution for connected bipartite (and necessarily unicyclic) graphs was obtained by the authors in [6].

Theorem 3 (Froncek, Kubesa [6]). *There exists a G-decomposition of K_n for a connected bipartite unicyclic graph G on seven edges and seven vertices if and only if $n \equiv 0, 1 \pmod 7$ except for three exceptions when $n = 7$ and two exceptions when $n = 8$.*

The only remaining connected class for seven edges and seven vertices is then unicyclic tripartite graphs. Therefore, we state here our first open problem.

Problem 1. Determine the G-decomposition spectrum for connected tripartite graphs on seven edges and seven vertices (which are necessarily unicyclic).

We do not know any result classifying disconnected graphs with seven edges and seven vertices. Our second open problem is then the following.

Problem 2. Determine the G-decomposition spectrum for disconnected graphs on seven edges and seven vertices.

Connected graphs with seven edges and eight vertices are trees, which were investigated by Huang and Rosa [7].

Theorem 4 (Huang, Rosa [7]). *There exists a G-decomposition of K_n for a connected graph G on seven edges and eight vertices (that is, a tree) if and only if $n \equiv 0, 1 \pmod 7, n \geq 8$ except for nine exceptions when $n = 8$.*

In Section 5 we take first steps towards determining the spectrum for disconnected graphs on seven edges and eight vertices by finding it for all such bipartite graphs. These graphs are necessarily unicyclic. The obvious necessary conditions for K_n to be decomposable into such graphs are $n \geq 8$ and $n \equiv 0, 1 \pmod 7$.
Graphs with seven edges and more than eight vertices are necessarily disconnected. We are not aware of any results in this direction.

3. Definitions and tools

Disclaimer. The whole section is copied almost verbatim from the authors' previous paper [6] as the topic is very similar and tools used here are identical.
The following definition has been used in different variations for years, and we present it just for the sake of completeness.

Definition 1. Let H be a graph. A *decomposition* of the graph H is a collection of pairwise edge-disjoint subgraphs $\mathcal{D} = \{G_1, G_2, \ldots, G_s\}$ such that every edge of H appears in exactly one subgraph $G_i \in \mathcal{D}$.
We say that the collection forms a *G-decomposition* of H (also known as an *(H,G)-design*) if each subgraph G_r is isomorphic to a given graph G. If H is the complete graph K_n, then we can use just the term *G-design*.

Because we focus solely on decompositions of complete graphs, we only use the term G-decomposition or G-design.

Definition 2 (Rosa [12]). A G-decomposition of the complete graph K_n is *cyclic* if there exists an ordering $(x_0, x_1, \ldots, x_{n-1})$ of the vertices of K_n and a permutation φ of the vertices of K_n defined by $\varphi(x_j) = x_{j+1}$ for $j = 0, 1, \ldots, n-1$ inducing an automorphism on $\mathcal{D}$, where the addition is performed modulo n.

Definition 3 (Huang, Rosa [7]). A G-decomposition of the complete graph K_n is *1-rotational* if there exists an ordering $(x_0, x_1, \ldots, x_{n-1})$ of the vertices of K_n and a permutation φ of the vertices of K_n defined by $\varphi(x_j) = x_{j+1}$ for $j = 0, 1, \ldots, n-2$ and $\varphi(x_{n-1}) = x_{n-1}$ inducing an automorphism on $\mathcal{D}$, where the addition is performed modulo $n-1$.

We will use the interval notation $[k, n]$ for the set of consecutive integers $\{k, k+1, k+2, \ldots, n\}$. When $k = 1$, the interval is denoted simply by $[n]$.
One of the basic and most useful tools for finding G-designs is the following labeling.

Definition 4 (Rosa [12]). Let G be a graph with n edges. A *ρ-labeling* (sometimes also called *rosy labeling*) of G is an injective function $f\colon V(G) \to [0, 2n]$ that induces the *length function* $\ell\colon E(G) \to [1, n]$ defined as

$$\ell(uv) = \min\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\}$$

with the property that

$$\{\ell(uv) : uv \in E(G)\} = [1, n].$$

A graph G possessing a ρ-labeling decomposes the complete graph, as proved by Rosa in 1967.

Theorem 5 (Rosa [12]). *Let G be a graph with n edges. A cyclic G-decomposition of the complete graph K_{2n+1} exists if and only if G admits a ρ-labeling.*

When a graph G with n edges has a vertex w of degree one and $G - w$ admits a ρ-labeling, a modification of ρ-labeling can be used to find a G-decomposition of K_{2n}. Such labeling is known as *1-rotational ρ-labeling* and was first used by Huang and Rosa in [7], although a formal definition was not stated there.

Definition 5 (Huang, Rosa [7]). Let G be a graph with n edges and edge ww' where $\deg(w) = 1$. A 1-*rotational* ρ-*labeling* of G consists of an injective function $f\colon V(G) \to [0, 2n-2] \cup \{\infty\}$ with $f(w) = \infty$ that induces a *length function* $\ell\colon E(G) \to [1, n-1] \cup \{\infty\}$ which is defined as

$$\ell(uv) = \min\{|f(u) - f(v)|, 2n - 1 - |f(u) - f(v)|\}$$

for $u, v \neq w$ and

$$\ell(ww') = \infty$$

with the property that

$$\{\ell(uv) : uv \in E(G)\} = [1, n-1] \cup \{\infty\}.$$

This technique was used in [7] and proved only for particular graphs studied in that paper. The following theorem is considered folklore.

Theorem 6. *Let G be a graph with n edges. If G admits a 1-rotational ρ-labeling, then there exists a 1-rotational G-decomposition of the complete graph K_{2n}.*

One can observe that a necessary condition for K_n to admit a G-design for a graph G with 7 edges is that the number of edges in K_n must be divisible by 7, implying $n \equiv 0, 1 \pmod 7$. For the graphs we are interested in, the above theorems only allow decompositions of K_{14} and K_{15}. Therefore, we will need additional tools, which are some more restrictive modifications of ρ-labeling.

Definition 6 (Rosa [12]). Let G be a bipartite graph with n edges and a vertex bipartition $U \cup V$. An α-*labeling* of G is a ρ-labeling f with the additional property that there exist λ such that $f(u) \leq \lambda < f(v) \leq n$ for every $u \in U$ and $v \in V$. The length function is then defined as

$$\ell(uv) = f(v) - f(u).$$

There are also labelings that are less restrictive yet also produce G-decompositions of larger complete graphs; that is, K_{2nk+1} for any $k \geq 1$ when G has n edges.

Definition 7 (El-Zanati, Vanden Eynden [4]). Let G be a bipartite graph with n edges and a vertex bipartition $U \cup V$. A σ^+-*labeling* of G is a ρ-labeling f with the additional property that for every $u \in U$ and $v \in V$ if $uv \in E(G)$, then $f(u) < f(v)$ and the length function is defined as

$$\ell(uv) = f(v) - f(u).$$

The σ^+-labeling is a generalization of the α-labeling and can be viewed as "locally α-labeling." Not all labels in set U need to be smaller that all labels in V, but rather only labels of all neighbors of a given vertex $u \in U$ have to be larger than that of u and vice versa, all neighbors of $v \in V$ have to have labels smaller than the label of v. Even the relaxed conditions guarantee decompositions of K_{2nk+1}, as proved by El-Zanati and Vanden Eynden [4].

Theorem 7 (El-Zanati, Vanden Eynden [4]). *Let G be a bipartite graph with n edges. If G admits a σ^+-labeling, then there exists a cyclic G-decomposition of the complete graph K_{2nk+1} for every $k \geq 1$.*

To decompose complete graphs with $2nk$ vertices into graphs with n edges, we will use the 1-rotational σ^+-labeling. Although the technique using such labeling has been used before (see, e.g., [5]), a formal definition has not been introduced yet.

Definition 8. Let G be a bipartite graph with n edges, vertex w of degree one and an edge ww'. A *1-rotational σ^+-labeling* of G is a 1-rotational ρ-labeling with the additional property that for every $u \in U$ and $v \in V$ if $u, v \neq w$ and $uv \in E(G)$, then $f(u) < f(v)$ and the length function is defined as

$$\ell(uv) = f(v) - f(u)$$

for $u, v \neq w$ and

$$\ell(ww') = \infty.$$

It is easy to see that when we have a σ^+-labeling where the longest edge is ww', vertex w is of degree one and all other vertices have labels at most $2n-2$, the labeling can be transformed to a 1-rotational σ^+-labeling.

Observation 8. Let G be a bipartite graph with n edges, an edge ww' where w is of degree one and a σ^+-labeling f. If $f(w) > f(x)$ for every $x \neq w$ and $\ell(ww') = n$, then there exists a 1-rotational σ^+-labeling $g\colon V(G) \to [0, 2n-2] \cup \{\infty\}$ defined as $g(x) = f(x)$ for $x \neq w$ and $g(w) = \infty$.

The following analogue of the above theorems was proved recently. Even more general version of this theorem was proved by Bunge [3] since this paper was originally written.

Theorem 9 (Fahnenstiel, Froncek [5]). *Let G be a bipartite graph with n edges and a vertex of degree one. If G admits a 1-rotational σ^+-labeling, then there exists a 1-rotational G-decomposition of the complete graph K_{2nk} for every $k \geq 1$.*

In our constructions, we will also need to decompose complete bipartite graphs. The tools are similar, based on labelings as well. An equivalent of ρ-labeling for bipartite graphs is called bilabeling and has been used for years by numerous authors. The following definition is adapted from [4].

Definition 9. Let G be a bipartite graph with n edges and a vertex bipartition $U \cup V$. An *α-bilabeling* of G is a function $f : V(G) \to [0, n-1]$ that is injective when restricted to sets U and V, respectively, and the induced length function defined as

$$\ell(uv) = (f(v) - f(u)) (\text{mod } n)$$

has the property that

$$\{\ell(uv) : uv \in E(G)\} = [0, n-1].$$

The following theorem was proved in a simpler form independently by many authors; e.g., in [4].

Theorem 10. *Let G be a bipartite graph with n edges. If G admits an α-bilabeling, then there exists a G-decomposition of the complete bipartite graph $K_{nk,nm}$ for every $k, m \geq 1$.*

4. Catalog

Obviously, disconnected bipartite graphs with seven edges and eight vertices (none of them isolated) are unicyclic with exactly two components.
There are eight such graphs. To catalog them, we use notation defined by Reed and Wilson in [9]. By $XnYm$ we denote the disjoint union of graphs Xn and Ym, where Xn and Ym are catalog names of graphs according to [9]. By $kXnYm$ we denote an edge-disjoint union of k copies of $XnYm$. We will denote the set of these eight graphs by $\mathcal{G}$. The graphs are presented in Figure 1.

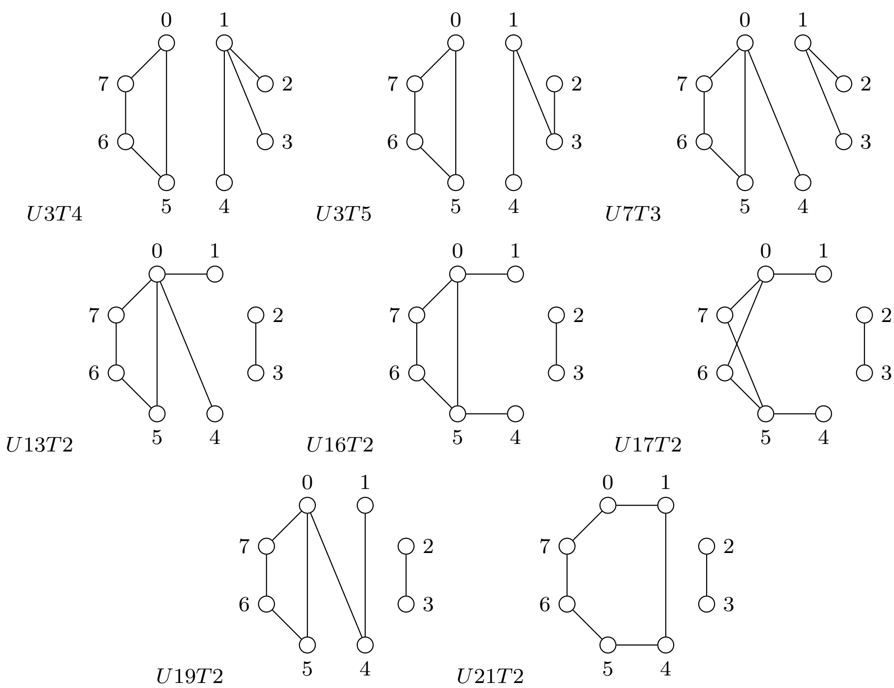

Figure 1. **The unicyclic bipartite graphs with 7 edges and 8 vertices**

5. Decompositions of K_8

The smallest graph satisfying the necessary conditions is K_8. Decompositions of K_8 into graphs $U7T3, U16T2, U17T2, U19T2$ and $U21T2$ are given in Figures 2–6.

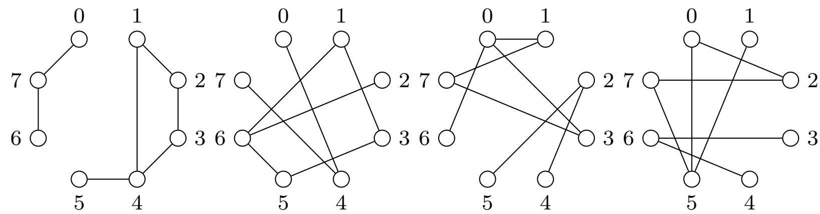

Figure 2. **Decomposition of K_8 into $U7T3$**

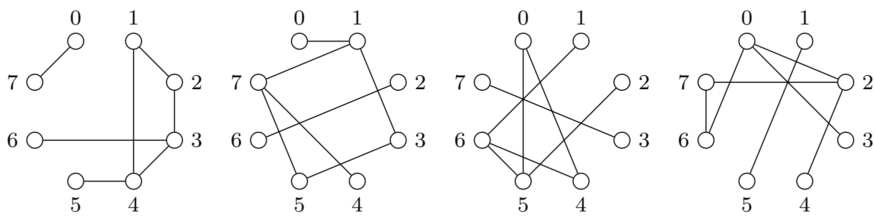

Figure 3. **Decomposition of K_8 into $U16T2$**

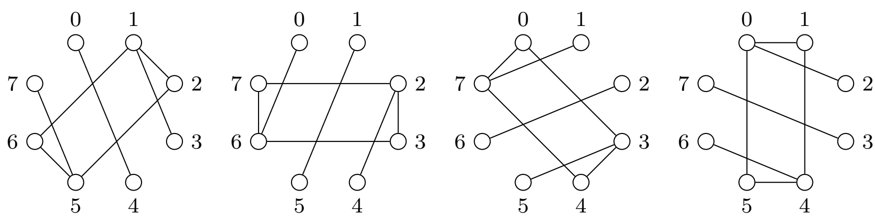

Figure 4. **Decomposition of K_8 into $U17T2$**

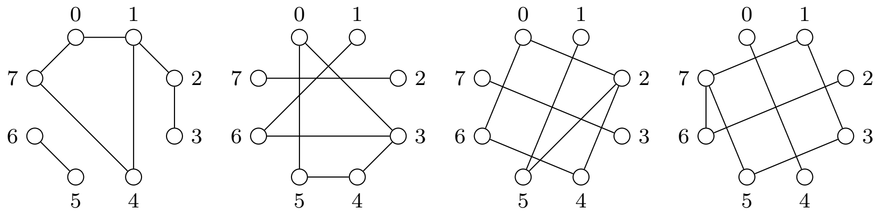

Figure 5. **Decomposition of K_8 into $U19T2$**

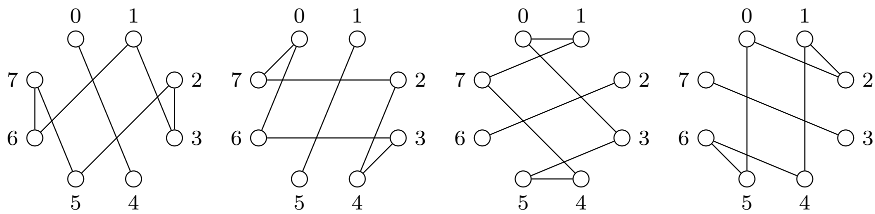

Figure 6. **Decomposition of K_8 into $U21T2$**

For easier reference, we call $\mathcal{G}^+$ the subclass of $\mathcal{G}$ containing the graphs decomposing K_8 and $\mathcal{G}^-$ the subclass of those not decomposing K_8. That is, $\mathcal{G}^+ = \{U7T3, U16T2, U17T2, U19T, U21T2\}$ and $\mathcal{G}^- = \{U3T4, U3T5, U13T2\}$.
From the constructions shown in Figures 2–6, we immediately obtain the following.

Lemma 1. *The graphs in $\mathcal{G}^+ = \{U7T4, U16T2, U17T2, U19T2, U21T2\}$ shown in Figures 2–6 decompose K_8.*

Now we present proofs of non-existence of G-decompositions of K_n for graphs in $\mathcal{G}^- = \{U3T4, U3T5, U13T2\}$ and $n = 8$.
We denote the graphs decomposing K_8 by G_i, where $i = 1, 2, 3, 4$. By $\deg_{G_i}(x)$ we denote the degree of vertex x in G_i.
The *degree set* $\mathrm{DS}(x)$ of a vertex $x \in K_8$ is the unordered multiset $\{\deg G_i(x) | 1 \leq i \leq 4\}$ of degrees of a particular vertex, which is usually listed in non-increasing order.

Lemma 2. *The graph $U13T2$ does not decompose K_8.*

Proof. Let $G_i, i = 1, 2, 3, 4$ be the four copies of $G = U13T2$ (shown in Figure 1) decomposing K_8. We split the vertex set of K_8 into sets $X = \{x_1, \ldots, x_4\}$ and

$Y = \{y_1, \ldots, y_4\}$ and assume that x_i is the vertex of degree four in G_i. By $\langle X \rangle$ and $\langle Y \rangle$ we denote the cliques induced on the vertex sets X and Y, respectively. To simplify our arguments, we color the edges of G_1 blue, of G_2 green, of G_3 red, and of G_4 purple.

All vertices x_i have their degree sets $\mathrm{DS}(x_i) = \{4, 1, 1, 1\}$. If vertex x_1 has all neighbors (called *blue neighbors*) in G_1 in Y, then the fourth vertex of $C_4(x_1)$ must belong to X, say it is x_2. But this is impossible, because x_2 must be in G_1 of degree one.

Now suppose the blue neighbors of x_1 are x_2, y_1, y_2, y_3. Then x_2 cannot belong to $C_4(x_1)$ as it would be of degree two in G_1 and the fourth vertex of $C_4(x_1)$ must be y_4. Therefore, the isolated blue edge must be x_3x_4 and we have two blue edges in $\langle X \rangle$, namely x_1x_2 and x_3x_4.

If x_1 has two or three blue neighbors in X, we have at least two blue edges in $\langle X \rangle$.

This argument can be repeated for all four graphs $G_i, i = 1, 2, 3, 4$ showing that each of them has at least two edges in $\langle X \rangle$, the graph induced by the vertex set X. But this is impossible, because $\langle X \rangle$ is the complete graph K_4 with six edges. This completes the proof. □

Lemma 3. *The graph U3T4 does not decompose K_8.*

Proof. We use the same notation as above, except that x_i is the vertex of degree three in G_i.

First we show that the whole blue star cannot belong to $\langle X \rangle$. Suppose it does. Also suppose that the edges x_2x_3 and x_2x_4 are both green, that is, belong to G_2, and WLOG the third edge of the green star is x_2y_1. Because the blue star is in $\langle X \rangle$, the blue rectangle must be in $\langle Y \rangle$, leaving only two independent edges in $\langle Y \rangle$ uncolored. The green rectangle now must be induced on vertices x_1, y_2, y_3, y_4 with two adjacent green edges in $\langle Y \rangle$. But this is impossible, as the only two non-colored edges in $\langle Y \rangle$ are independent.

Therefore, the edges x_2x_3, x_3x_4, x_4x_2 must all belong to different graphs G_i, that is, have different colors. Therefore, each monochromatic rectangle other than blue must have two vertices in X and two in Y. Because all edges in $\langle X \rangle$ have been already used, all edges of say red rectangle must be of type x_jy_k. Then along with the two remaining red star edges, we have six red edges of type x_jy_k. The same is true for red and purple edges fo the same reasons, and we have 18 edges of type x_jy_k. This is nonsense proving that there cannot be any complete monochromatic star in $\langle X \rangle$.

We know that $\mathrm{DS}(x_i) = \{3, 2, 1, 1\}$, so the vertex x_1 must be of degree one in two graphs G_i, say G_2 (green) and G_3 (red). Hence, we have a green x_1x_2 and red x_1x_3. Also, x_3 must be of degree one in two colors other than red. It cannot be blue, because we have the edge x_1x_3 already colored red. So it must be green and purple, and we have x_2x_3 green and x_3x_4 purple. Now x_4 must have two incident edges in $\langle X \rangle$ of other colors than purple. Since the edge x_3x_4 is already purple, it must be a blue x_1x_4 and green x_2x_4. But now we have three green star edges incident with x_2, which was proved impossible above. This contradiction completes the proof. □

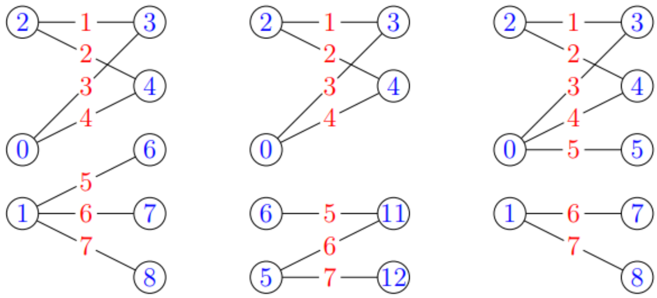

Figure 7. σ^+**-labelings of** $U3T4, U3T5, U7T3$ **(left to right)**

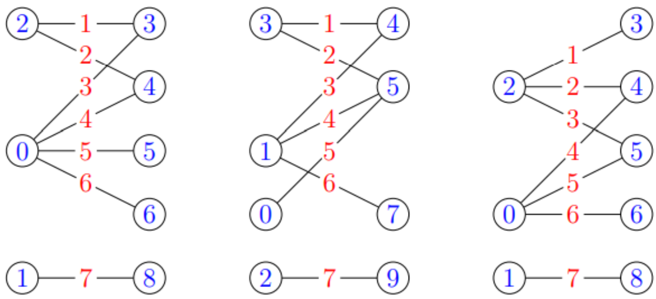

Figure 8. σ^+**-labelings of** $U13T2, U16T2, U17T2$ **(left to right)**

The remaining non-existence result was obtained independently by a computer search by Rosa [11] and Meszka [8].

Lemma 4 (Rosa [11], Meszka [8]). *The graph* $U3T5$ *does not decompose* K_8.

The complete result on G-decompositions of K_8 is a direct consequence of Lemmas 1–4.

Theorem 11. *Let* $G \in \mathcal{G}$. *Then there exists a* G*-decomposition of the complete graph* K_8 *if and only if* $G \in \mathcal{G}^+ = \{U7T4, U16T2, U17T2, U19T2, U21T2\}$.

6. Decompositions of K_n for $n \equiv 0, 1 \pmod{14}$

All decompositions of K_n for $n \equiv 1 \pmod{14}$ are based on σ^+-labelings of the respective graphs.

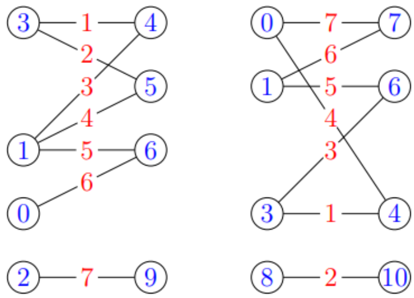

Figure 9. σ^+**-labelings of** $U19T2, U21T2$ **(left to right)**

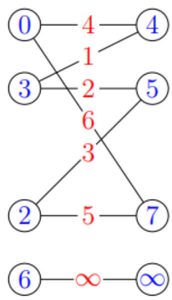

Figure 10. **1-rotational** ρ**-labeling of** $U21T2$

Theorem 12. *There exists a G-decomposition of the complete graph* K_{14k+1} *into each graph* $G \in \mathcal{G}$ *for every* $k \geq 1$.

Proof. Because each graph $G \in \mathcal{G}$ has a σ^+ labeling, a decomposition exists by Theorem 7. □

For decompositions of K_{14k}, the labelings we use can be easily modified to 1-rotational σ^+-labelings by replacing the label 7 with ∞ except for graph $U21T2$, where a 1-rotational σ^+-labeling does not exist. We present the labelings in Figures 7 – 9. Notice that the σ^+-labeling of $U21T2$ does not satisfy requirements of Theorem 9. Therefore, the labeling only guarantees a decomposition of K_{14k+1} but not of K_{14k}. For that decomposition, we need the following construction.

Lemma 5. *A decomposition of the complete graph* K_{14k} *into the graph* $U21T2$ *exists for any* $k \geq 1$.

Proof. For $k = 1$ the result follows from the existence of the 1-rotational ρ-labeling shown in Figure 10. Notice that the edge lengths are calculated in K_{13}, hence the

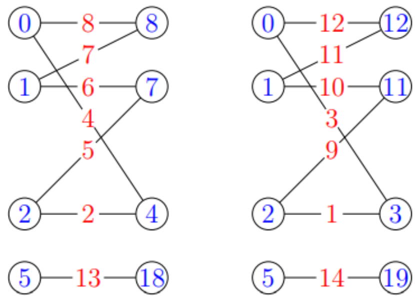

Figure 11. First two copies H_1, H_2 of $U21T2$ in $(2m)U21T2$

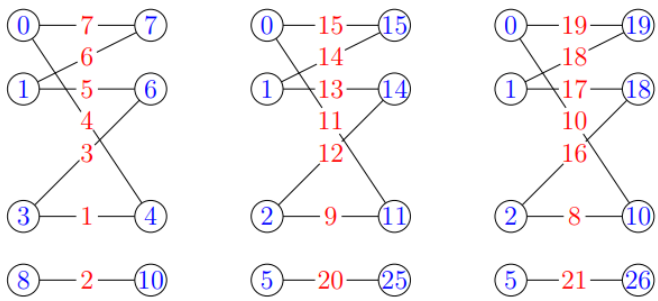

Figure 12. First three copies H_1, H_2, H_3 of $U21T2$ in $(2m+1)U21T2$

edge with the endvertices labeled 0 and 7 has in fact length 6.

Because the labeling in Figure 11 labels an isolated edge with the highest length, it satisfies the requirements of Theorem 9 when we replace the label 19 and the edge length 14 by ∞. Therefore, $U21T2$ decomposes the complete graph K_{14k} for any even $k \geq 2$.

For $k = 3$, we use the labeling used in Figure 12. We call the labeling f and denote the "lower" partite set on the left (consisting of vertices labeled $0, 1, 2, 3, 5, 8$) by X and the "upper" one on the right by Y. Replacing the vertex label 26 and edge length 21 by ∞, we obtain a 1-rotational ρ-labeling satisfying requirements of Theorem 9, which guarantees the decomposition.

For $k = 2m + 1 \geq 5$, we define the labeling recursively. We denote the copies of $U21T2$ in Figure 12 from left to right by H_1, H_2, H_3. For better clarity, we denote the vertices in copy H_i by x^i_j, y^i_j for $i = 1, 2, \ldots, k$ and $j = 1, 2, 3, 4$. The isolated edge is $x^i_4 y^i_4$ and the cycle is labeled in natural order $x^i_1, y^i_1, \ldots, y^i_3$.

Now we define the labeling f' of $(2m + 1)U21T2$ as follows.

For $x^i_j, y^i_j \in H_i, j = 1, 2, 3$ we have $f'(x^i_j) = f(x^i_j)$ and $f'(y^i_j) = f(y^i_j)$. For $x^i_j \in H_i, i > 3$ we set $f'(x^i_j) = f(x^{i-2}_j)$ and $f'(y^i_j) = f(y^{i-2}_j) + 14$. This way, the copies H_{2s} and H_{2s+1} for $s = 1, 2, \ldots, m$ contain edges of lengths $7(2s - 1) + 1, 7(2s - 1) +$

$2, \ldots, 7(2s-1)+14$ and the longest edge of length $7k$ is always the isolated edge $x_4^k y_4^k = x_4^{2m+1} y_4^{2m+1}$ in copy $H_k = H_{2m+1}$. Finally, we replace the label $f'(y_4^k) = 26 + 7(k-3)$ by $f'(y_4^k) = \infty$ and the length of edge $x_4^k y_4^k$ is now ∞.
This way, we obtained a 1-rotational ρ-labeling of the graph $H = (2m+1)U21T2$ which is an edge-disjoint union of $2m+1$ copies of the graph $U21T2$. Because the longest edge of length $7(2m+1) = 7k$ is incident with a vertex of degree one, the labeling satisfies conditions of Theorem 6 and H decomposes K_{14k}. Because H can be decomposed into $2m+1$ copies of $U21T2$, the proof is complete. □

Now we can prove the result for $n \equiv 0 \pmod{14}$.

Theorem 13. *There exists a G-decomposition of the complete graph K_{14k} into each graph $G \in \mathcal{G}$ for every $k \geq 1$.*

Proof. Except for $U21T2$, all other graphs in $\mathcal{G}$ satisfy assumptions of Theorem 9 and therefore decompose K_{14k} for every $k \geq 1$. The graph $U21T2$ decomposes K_{14k} for every $k \geq 1$ by Lemma 5. This completes the proof. □

7. Decompositions of K_n for $n \equiv 7 \pmod{14}$

In this case, we let $n = 14k+7$ and first decompose K_{14k+7} into graphs $K_{14k}, K_{14} - K_7$ and $2k-1$ copies of $K_{7,7}$ and then in turn show decompositions of these graphs into each $G \in \mathcal{G}$.
The decomposition of K_{14k+7} into the above mentioned graphs should be obvious. We first decompose K_{14k+7} into K_{14k}, K_7 and $K_{14k,7}$ and then split $K_{14k,7}$ into $2k$ copies of $K_{7,7}$. Finally, we add K_7 back to one of the copies of $K_{7,7}$ to obtain $K_{14} - K_7$.
It should not be difficult to observe that this forms a G-decomposition of K_{14k+7} whenever $K_{14} - K_7$ and $K_{7,7}$ are decomposable into G, because K_{14k} is G-decomposable by Theorem 13.
To show that $K_{7,7}$ is G-decomposable, it is enough to find an α-bilabeling of G. They are shown in Figures 13, 14 and 15.
A lemma follows immediately.

Lemma 6. *The complete bipartite graph $K_{7,7}$ is G-decomposable for every $G \in \mathcal{G}$.*

For decompositions of $K_{14} - K_7$, we combine decompositions of $K_{7,7}$ and packings of $G - e$ into K_7, where e is a pendant edge of G. Because K_7 has 21 edges and the packing has 18 edges, we obtain a leave (that is, a set of edges that do not appear in any copy of G in K_7) with three edges. We denote the copies of $G - e$ as H^1, H^2, H^3 and the copies of G in the decomposition of $K_{7,7}$ as G^j for $j = 1, 2, \ldots, 7$.
Now we "swap" edges between the leave and graphs G^j. We take a suitable copy G^j, remove and edge e^j (denoted in Figure 16 by a blue dotted line) corresponding to e and add it to H^i (as a blue solid line), obtaining a graph $\widetilde{H^i}$ isomorphic to G. Then

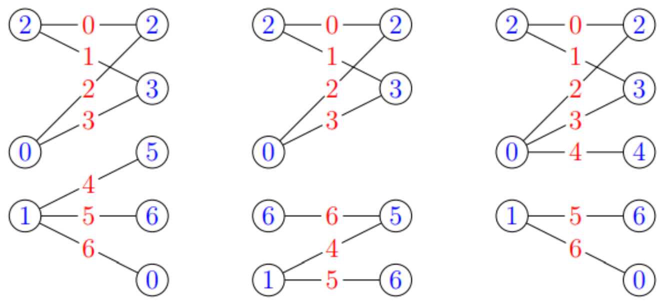

Figure 13. **α-bilabelings of $U3T4, U3T5, U7T3$ (left to right)**

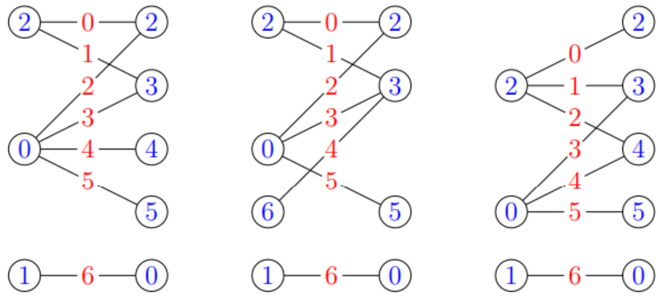

Figure 14. **α-bilabelings of $U13T2, U16T2, U17T2$ (left to right)**

we pick one leave edge (drawn as a red dotted line) and place it to $G^j - e^j$ (as a solid red line) to obtain a graph $\widetilde{G^j}$ isomorphic to G as well.
In Figures 16–23 we show the graphs $\widetilde{H^i}$ arising from packings and the three corresponding copies $\widetilde{G^j}$. The remaining graphs in the decompositions are the copies of G^j that were not modified, and are not shown in the figures. The leave edges are shown

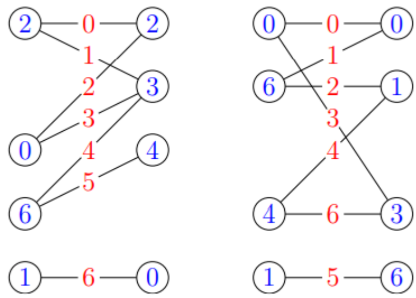

Figure 15. **α-bilabelings of $U19T2, U21T2$ (left to right)**

in red, the edges moved from G^j to $\widetilde{H^i}$ in blue; the original position is dotted, the new placement is solid. For graphs $U16T2$ and $U19T2$ we move two edges from the same copy G^j to two different copies H^i. In this case one edge is blue and the other one green for better readability.

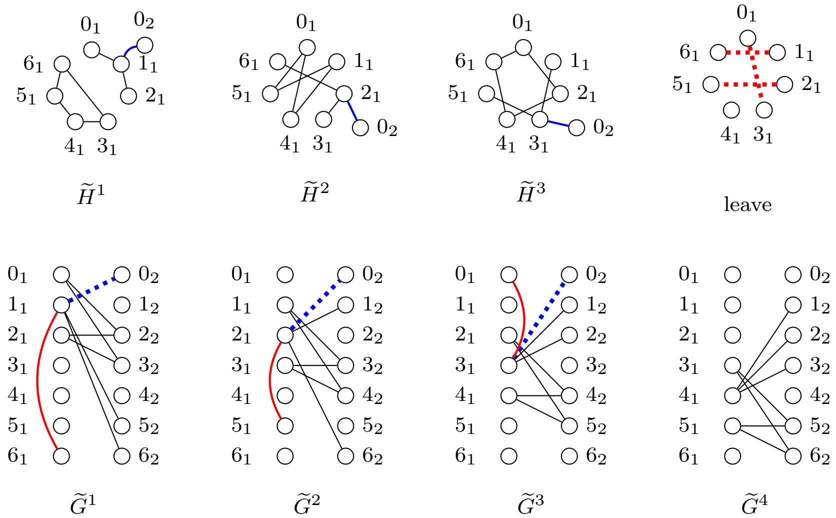

Figure 16. **Copies $\widetilde{H}^i$ and $\widetilde{G}^j$ of $U3T4$ in $K_{14} - K_7$**

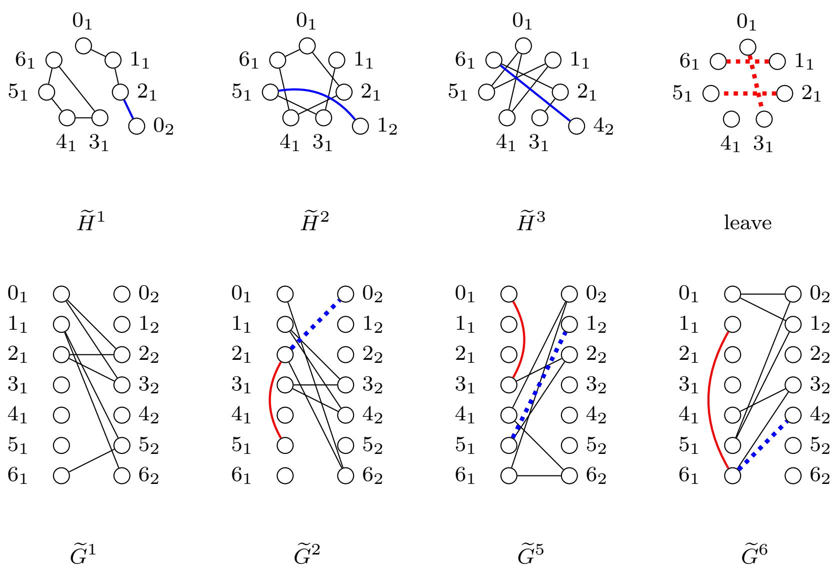

Figure 17. **Copies $\widetilde{H}^i$ and $\widetilde{G}^j$ of $U3T5$ in $K_{14} - K_7$**

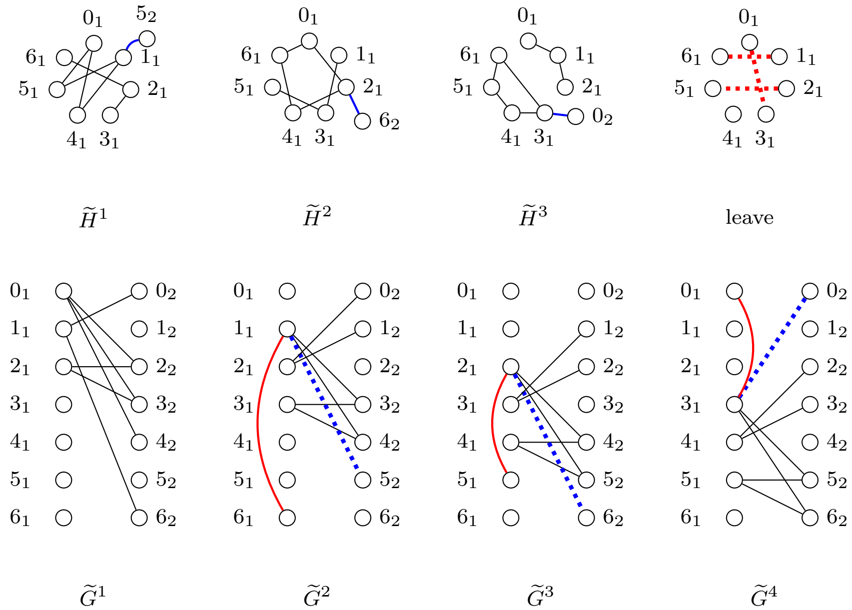

Figure 18. **Copies $\widetilde{H}^i$ and $\widetilde{G}^j$ of $U7T3$ in $K_{14} - K_7$**

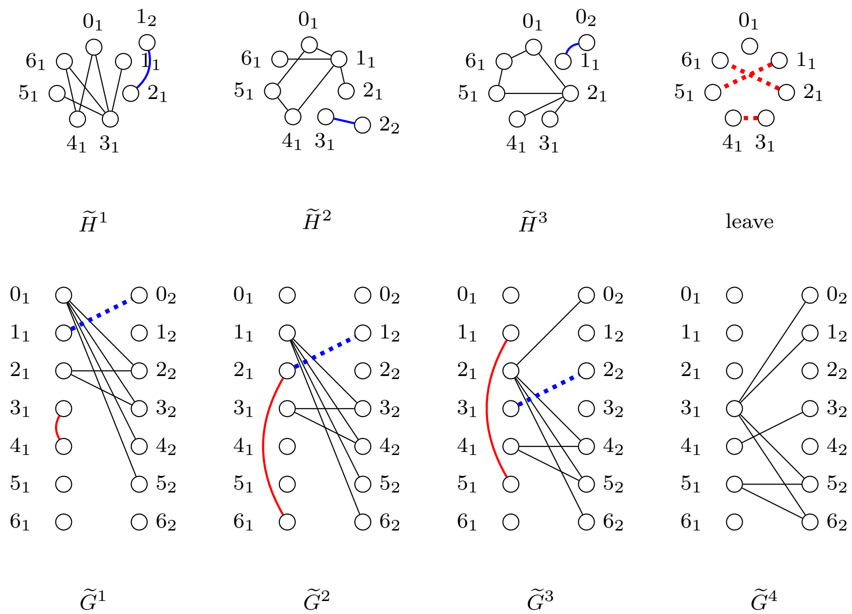

Figure 19. **Copies $\widetilde{H}^i$ and $\widetilde{G}^j$ of $U13T2$ in $K_{14} - K_7$**

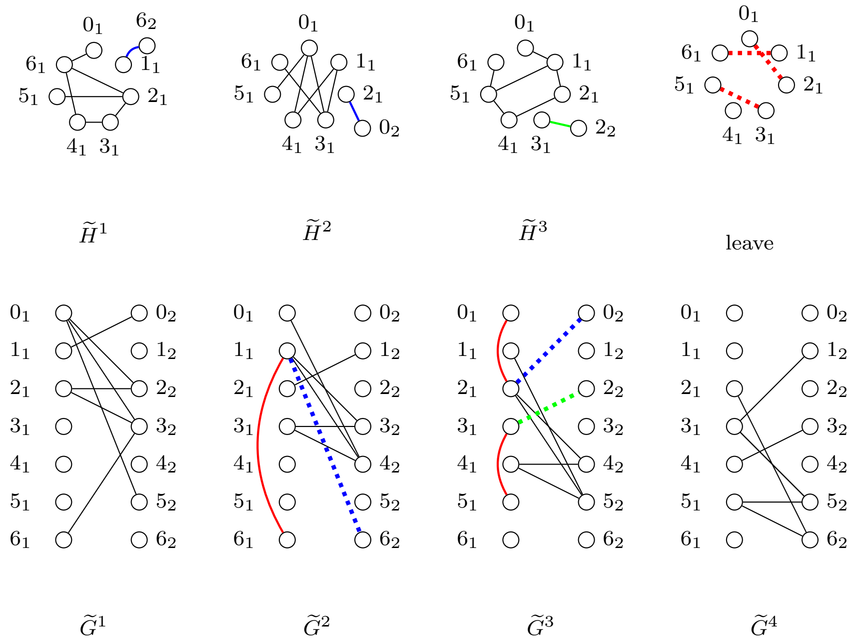

Figure 20. **Copies $\widetilde{H}^i$ and $\widetilde{G}^j$ of $U16T2$ in $K_{14} - K_7$**

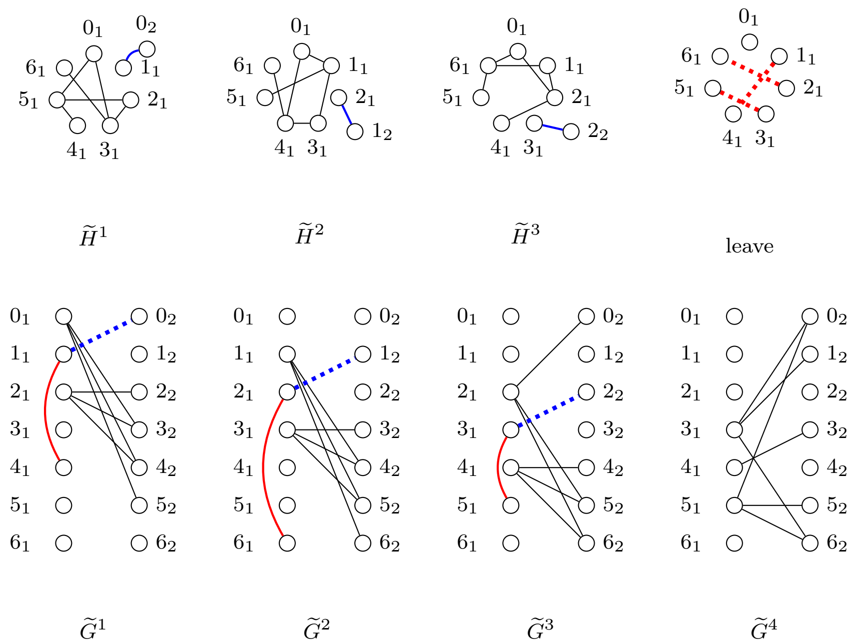

Figure 21. **Copies $\widetilde{H}^i$ and $\widetilde{G}^j$ of $U17T2$ in $K_{14} - K_7$**

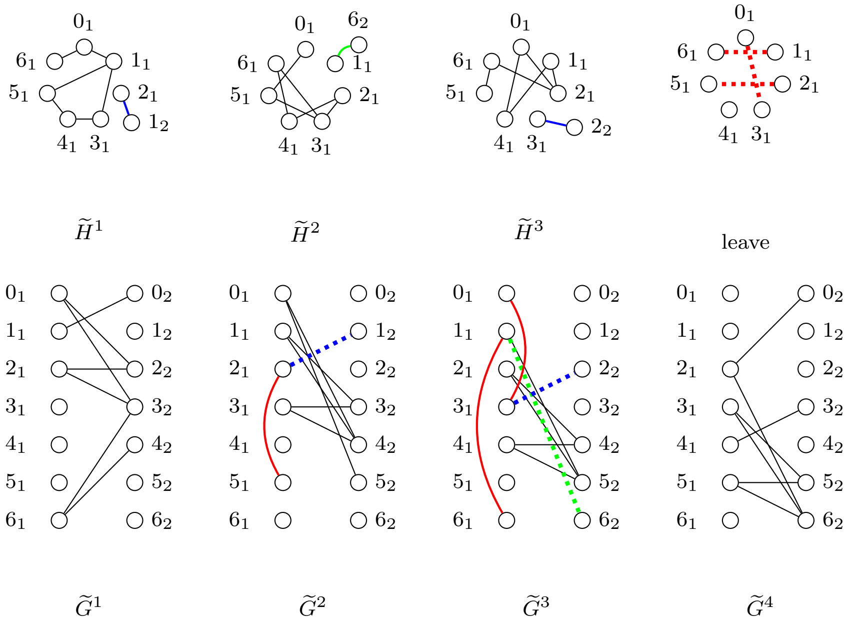

Figure 22. **Copies $\widetilde{H}^i$ and $\widetilde{G}^j$ of $U19T2$ in $K_{14} - K_7$**

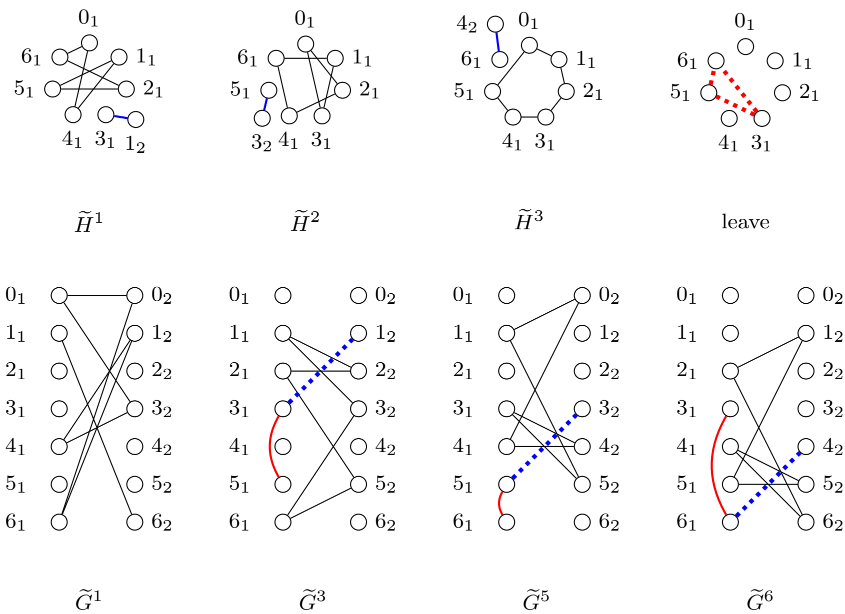

Figure 23. **Copies $\widetilde{H}^i$ and $\widetilde{G}^j$ of $U21T2$ in $K_{14} - K_7$**

Lemma 7. *The graph $K_{14} - K_7$ is G-decomposable for every $G \in \mathcal{G}$.*

Proof. Each decomposition consists of graphs $\widetilde{H^i}$ and $\widetilde{G^j}$ shown in Figures 16–23 and additional copies G^s arising from the α-bilabeling shown in Figures 13, 14 and 15. □

The fact that K_n for $n \equiv 7 \pmod{14}$ and $n > 7$ is decomposable into graphs $K_{n-7}, K_{14} - K_7$ and $K_{7,7}$ along with Lemmas 6 and 7 immediately yield the following.

Theorem 14. *The graph K_n is G-decomposable for every $G \in \mathcal{G}$ when $n \equiv 7 \pmod{14}$ and $n > 7$.*

8. Decompositions of K_n for $n \equiv 8 \pmod{14}$

In this case, we use a similar approach as in Section 7 but we will need one more ingredient. This time we let $n = 14k + 8$ and first decompose K_{14k+8} into graphs $K_{14k+1}, K_{14} - K_7, K_{8,7}$ and $2k - 2$ copies of $K_{7,7}$.
The above decomposition of K_{14k+8} is similar to the one in the previous section. We first decompose K_{14k+8} into K_{14k+1}, K_7 and $K_{14k+1,7}$ and then split $K_{14k+1,7}$ into $2k-$ copies of $K_{7,7}$ and one copy of $K_{8,7}$. Then we add K_7 back to one copy of $K_{7,7}$ to get $K_{14} - K_7$.
This indeed forms a G-decomposition of K_{14k+8} whenever K_8 is decomposable into G, because K_{14k+1} is G-decomposable by Theorem 12, and the graphs $K_{14} - K_7$ and $K_{7,7}$ are G-decomposable by Lemmas 7 and 6, respectively.
Because $K_{8,7}$ can be decomposed into two graphs $K_{4,7}$, it is enough to show G-decompositions of $K_{4,7}$ into each $G \in \mathcal{G}$. They are shown in Figures 24–31.

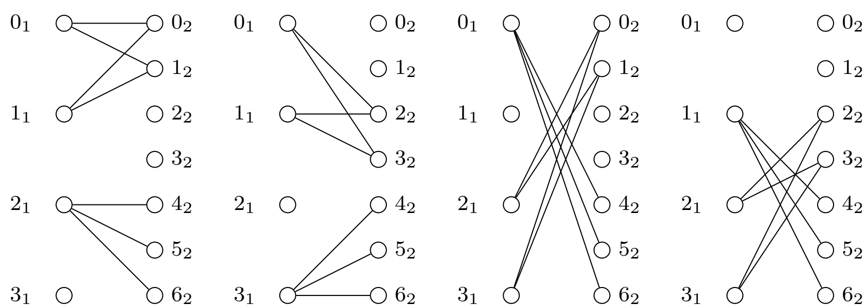

Figure 24. **Decomposition of $K_{4,7}$ into $U3T4$**

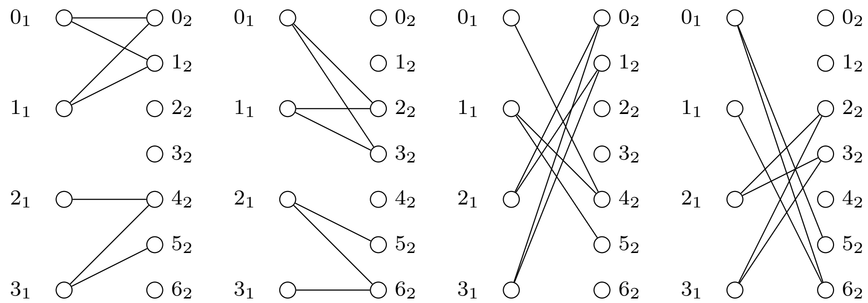

Figure 25. **Decomposition of $K_{4,3}$ into $U3T5$**

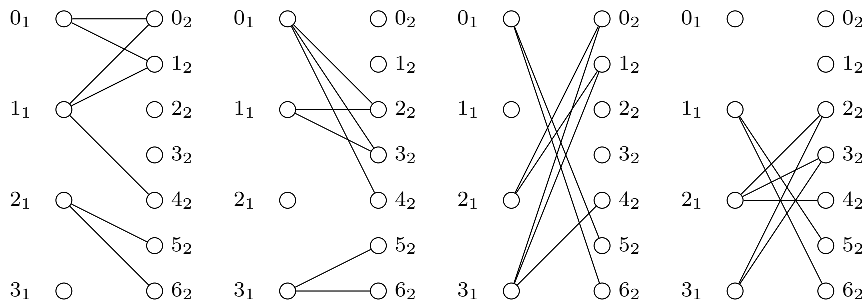

Figure 26. **Decomposition of $K_{4,3}$ into $U7T3$**

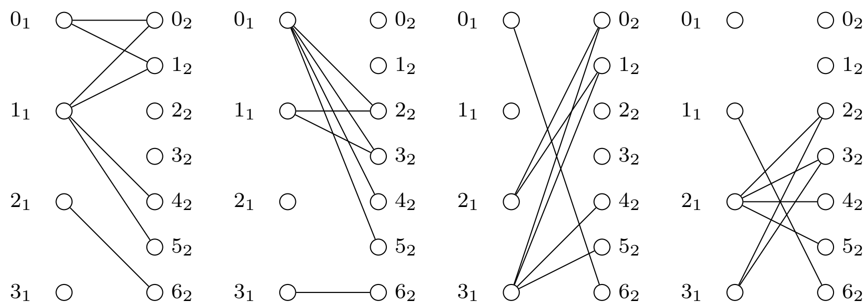

Figure 27. **Decomposition of $K_{4,3}$ into $U13T2$**

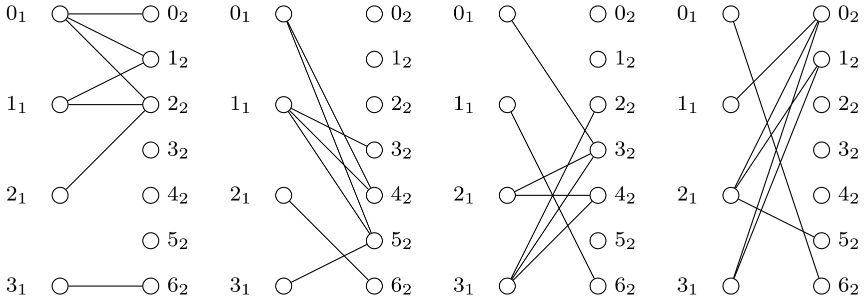

Figure 28. **Decomposition of $K_{4,3}$ into $U16T2$**

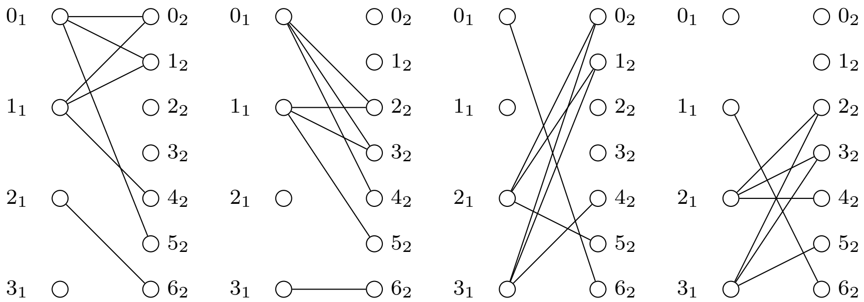

Figure 29. **Decomposition of $K_{4,3}$ into $U17T2$**

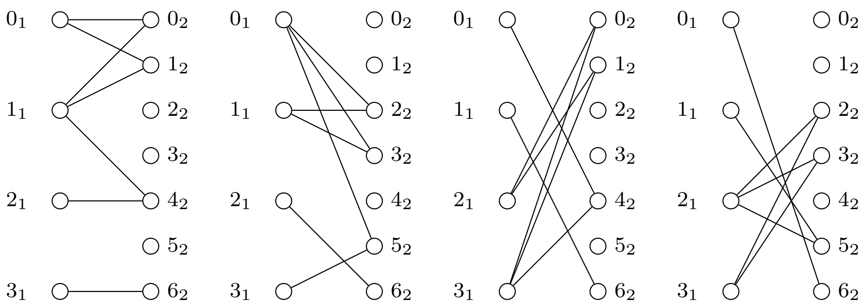

Figure 30. **Decomposition of $K_{4,3}$ into $U19T2$**

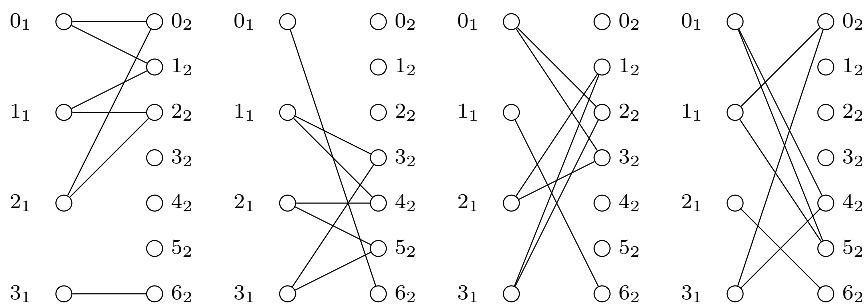

Figure 31. **Decomposition of $K_{4,3}$ into $U21T2$**

Lemma 8. *The complete bipartite graph $K_{8,7}$ is G-decomposable for every $G \in \mathcal{G}$.*

Proof. Because $K_{8,7}$ can be decomposed into two copies of $K_{4,7}$ and there exists a G-decomposition of $K_{4,7}$ for every $G \in \mathcal{G}$, the Lemma follows. □

We now again have all ingredients needed for the complete result on this subclass for $n \equiv 8 \pmod{14}$.

Theorem 15. *The complete graph K_n for $n \equiv 8 \pmod{14}$ is G-decomposable for a graph $G \in \mathcal{G}$ if and only if $G \in \mathcal{G}^+$ and $n \geq 8$ or $G \in \mathcal{G}^-$ and $n > 8$.*

Proof. Follows directly from the fact that K_{14k+8} is decomposable into K_{14k+1}, $K_{14} - K_7, K_{8,7}$ and $2k-2$ copies of $K_{7,7}$, Lemmas 6, 7, 8 and Theorem 12. □

9. Conclusion

Our main result now follows.

Theorem 16. *The complete graph K_n has a G-decomposition for any $G \in \mathcal{G}$ if and only if $n \equiv 0, 1 \pmod{7}, n > 7$, except when $n = 8$ and $G \in \mathcal{G}^- = \{U3T4, U3T5, U13T2\}$.*

Proof. Decompositions of K_8 are characterized in Theorem 11. The case of $n \equiv 0, 1 \pmod{14}$ is covered by Theorems 12 and 13. The case of $n \equiv 7 \pmod{14}$ is proved by Theorem 14 and for $n \equiv 8 \pmod{14}$ by Theorem 15. □

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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