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Research Article

Weighted topological index of graphs

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Abstract: The definition of the weighted topological index associated with a degree function ϕ is $\Phi(G) = \sum_{uv \in E(G)} \phi(d_u, d_v)$, where d_u denotes the degree of node u and ϕ satisfies symmetric property $\phi(d_u, d_v) = \phi(d_v, d_u)$. In this paper, we characterized extremal graphs and presented several results concerning the function $\Phi(G)$ in terms of various graph invariants. Additionally, we characterize the graphs that achieve these bounds and present multiple bounds for $\Phi(G)$ for the class of cozero divisor graphs defined on commutative rings.

Keywords: topological index, inequalities, extremal graphs, cozero divisor graphs.

AMS Subject classification: 05C50, 05C09, 05C92, 15A18

1. Introduction

All graphs are simple and without any direction. A graph $G = G(V, E)$ consists of a node (vertex) set $V = \{\xi_1, \xi_2, \ldots, \xi_n\}$ and an edge E consisting of unordered pairs of ξ_i's. The cardinality $|V| = n$ is order and the cardinality $|E| = m$ is size of G. The *degree* of $\xi \in V$, denoted by d_ξ, is the number of edges adjacent with v. The *neighbourhood* (open neighbourhood) of $\xi \in V$, denoted by $N(\xi)$, is the set of nodes of G incident to ξ. We note that $d_\xi = |N(\xi)|$. A graph is called *r-regular*, if $d(\xi) = r$

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each node $\xi \in V$. An independent set I of G is a subset of V such that no two nodes in I are adjacent and the cardinality of such a maximal set is known as the independence number of G. A graph is said to be complete if every two distinct nodes are adjacent. A clique is a complete subgraph of a graph. A graph G is complete bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ such that each node of V_1 is adjacent to every node of V_2 and there are no edges among nodes of V_i's. A node of degree one is known as the pendent node and its adjacent node is a quasi pendent node. The join of graphs G_1 and G_2, denoted by $G_1 + G_2$ consists of $G_1 \cup G_2$ and all edges joining a node of G_1 and a node of G_2. For other undefined notations, we follow [8].

A general node-degree-based topological index (or weighted topological index) Φ of a graph G [14] is defined as

$$\Phi(G) = \sum_{uv \in E(G)} \Phi(d_u, d_v),$$

where $\Phi(d_u, d_v)$ is a symmetric function of node degrees, that is, $\Phi(d_u, d_v) = \Phi(d_v, d_u)$. The function $\Phi(d_u, d_v)$ is an edge weight between nodes u and v. For particular values of $\Phi(d_u, d_v)$, we obtain well studied topological indices like the general Randić index [25] $\Phi(d_u, d_v) = (d_u d_v)^{\eta}$, for $\eta = -\frac{1}{2}$, we get Randić index [26] $R = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}$ (also see [22, 27]). For $\Phi(d_u, d_v) = \frac{d_u + d_v}{2\sqrt{d_u d_v}}$, we get arithmetic-geometric index [31], for $\Phi(d_u, d_v) = \left(d_u^2 + d_v^2\right)^{\eta}$, we obtain the general Sombor index, for $\eta = \frac{1}{2}$, we obtain Sombor index $SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$ [13], and for $\eta = -\frac{1}{2}$, we obtain ${}^{m}SO(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u^2 + d_v^2}}$, a modified Sombor index [16]. Similarly, for particular values of function $\Phi(d_u, d_v)$, we obtain several other indices. More about the weighted topological indices can be seen in [11, 14]. The spectral analysis of the weighted function $\Phi(d_u, d_v)$ is very well studied, see [9, 10, 12, 14, 15, 17–21, 28, 33].

In Section 2, we present several results related to the weighted topological index $\Phi(G)$ and characterize the corresponding extremal graphs for such results. Section 3 gives the results related to weighted topological index Φ for the cozero divisor graphs of commutative rings. We give complete formula for $\Phi(\Gamma'(\mathbb{Z}_n))$ for $n = p_1^{n_1} p_2$ and $n = pqr$, where p_i's and p, q, r are primes. Several other inequalities are given for the function $\Phi(\Gamma'(\mathbb{Z}_n))$ along with the classification of graphs attaining them.

2. Some inequalities for the topological index Φ

In this section we will obtain some bounds for the weighted function Φ of a graph G. The first very results establishes bounds for the function Φ of G in terms of independence number.

Theorem 1. *If G is a graph with independence number ℓ such that each node of the independence set share the same neighbourhood. Then*

$$\Phi(G) \geq \ell \sum_{i=1}^{\jmath} \phi(d_0, d_{u_i}),$$

with equality if and only if G is the complete bipartite graph.

Proof. Let $I = \{v_1, v_2, \ldots, v_\ell\}$ such that $N(v_1) = \cdots = N(v_\ell)$. Then it is clear that $d_{v_1} = \cdots = d_{v_\ell}$ and equals some constant say d_0. Let L be the neighbour set of I with cardinality $\jmath$. Thus, $V(G) = I \cup L \cup \Big(V(G) \setminus \big(I \cup L\big)\Big)$. Using the information of the edge weights form a node say v_1 to the nodes of $\jmath$, we have

$$w(v_1) = \phi(d_{v_1}, d_{u_1}) + \phi(d_{v_1}, d_{u_2}), \ldots, \phi(d_{v_1}, d_{u_\jmath}) = \sum_{i=1}^{\jmath} \phi(d_0, d_{u_i}),$$

where $u_i, i = 1, \ldots, \jmath$ are the nodes of L. Repeating the similar with the other nodes of I and summing their weights, we obtain

$$\begin{aligned} \Phi(G) &= \sum_{i=1}^{\ell} w(v_i) + W(L) + W(V(G) \setminus \big(I \cup L\big)) \\ &= \ell \sum_{i=1}^{\jmath} \phi(d_0, d_{u_i}) + W(L) + W(V(G) \setminus \big(I \cup L\big)) \\ &\geq \ell \sum_{i=1}^{\jmath} \phi(d_0, d_{u_i}), \end{aligned} \tag{2.1}$$

where $W(L)$ is edge weight among the nodes of L and $W(V(G) \setminus \big(I \cup L\big))$ is the edge weight from the nodes of L to the nodes of $(V(G) \setminus \big(I \cup L\big))$, which includes the edge weights between nodes of $V(G) \setminus \big(I \cup L\big)$.
Now, equality occurs in (2.1) if and only if $W(L) = W(V(G) \setminus \big(I \cup L\big)) = 0$. Thus, we see that the node sets L and $V(G) \setminus \big(I \cup L\big)$ contributes nothing to $\Phi(G)$. From this, it follows that $V(G) \setminus \big(I \cup L\big) = \emptyset$ and L induces a graph isomorphic to complement of clique. This happens if G is the complete bipartite graph with partite sets I and L and in this case $d_{u_1} = \cdots = d_{u_\jmath} = d_o$. So, $\Phi(G) = \ell \jmath \phi(d_0, d_o)$. Conversely, assume that $G \cong K_{\ell,\jmath}$, then G is biregular graph with nodes of degree ℓ and $\jmath$ with edges weights only between nodes of I and L. Hence, its topological index is given as

$$\Phi(G) = \ell \jmath \phi(d_\ell, d_\jmath).$$

□

The following corollary is a consequence of Theorem 1.

Corollary 1. *With notations and conditions as in above theorem, the topological index Φ satisfies the following inequalities*

(i)

$$\Phi(G) \geq \ell \sum_{i=1}^{\jmath} \phi(d_0, d_{u_i}) + \sum_{u_i u_j \in E(L \setminus V(I \cup (V(G) \setminus (I \cup L))))} \phi(d_{u_i}, d_{u_j}),$$

with equality holding if and only if $V(G) \setminus (I \cup L) = \emptyset$.

(ii)

$$\Phi(G) \geq \ell \sum_{i=1}^{\jmath} \phi(d_0, d_{u_i}) + \frac{\jmath(\jmath - 1)}{2} \phi(d_{u_i}, d_{u_j}),$$

with equality holding if and only if the induced subgraph of L is a clique and $V(G) \setminus (I \cup L) = \emptyset$.

The following results gives another lower bound for $\Phi(G)$ in terms of various parameters of G.

Theorem 2. *If I is an independence set of G of cardinality ℓ, J is a neighbour set of I with cardinality $\jmath$ and Z is the neighbour set of J of cardinality k, such that I share J and J share Z. Then*

$$\Phi(G) \geq \ell \jmath \phi(d_0, d_{00}) + m' \phi(d_{00}, d_{00}) + \jmath \sum_{i=1}^{k} \phi(d_{00}, d_{u_i}),$$

where d_0 is the common degree of nodes of I, d_{00} is the common degree of nodes of W, u_i's are the nodes of Z and $m' = |E(J \setminus (I \cup (V(G) \setminus (I \cup J)))|$. Equality holds if and only if $G \cong H + (\overline{K}_\ell \cup \overline{K}_\jmath)$, where H is induced subgraph of J.

Proof. Let $u_1, u_2, \ldots, u_k$ be the nodes of Z. Clearly $V(G)$ is partitioned into mutually disjoint subsets, that is, $V(G) = I \cup J \cup Z \cup (V(G) \setminus (I \cup J \cup Z))$. Since each node of sets I and J share the same neighborhood, their nodes have common degrees, namely d_0 and d_{00}, respectively. Also, d_{u_i} denotes the degrees of the nodes of Z. One node of I have $\jmath$ neighbours in J, so its edge weight is $j\phi(d_0, d_{00})$. Adding all such edge weights between I and J, we obtain $\ell \jmath \phi(d_0, d_{00})$. The edge weights within the nodes of W, contribute $m'\phi(d_{00}, d_{00})$ to $\Phi(G)$, where m' is the size of the induced subgraph of J. Similarly, calculating the edge weights between the nodes of J and Z, we have

$$\Phi(G) \geq j\phi(d_0, d_{00}) + m'\phi(d_{00}, d_{00}) + j\sum_{i=1}^{k}\phi(d_{00}, d_{u_i}) + W(Z) + W(V(G) \setminus (I \cup J \cup Z))$$
$$\geq j\phi(d_0, d_{00}) + m'\phi(d_{00}, d_{00}) + j\sum_{i=1}^{k}\phi(d_{00}, d_{u_i}),$$

where $W(Z)$ and $W(V(G) \setminus (I \cup J \cup Z))$ are the edge weight of Z and $V(G) \setminus (I \cup J \cup Z)$ nodes. The equality holds if and only if $W(V(G) \setminus (I \cup J \cup Z)) = 0$ and $W(Z) = 0$, which is so if and only if the induced subgraph of Z is totally disconnect graph (without edges) and $V(G) \setminus (I \cup J \cup Z) = \emptyset$. Thus G must be isomorphic to $H + (\overline{K}_\ell \cup \overline{K}_\jmath)$, where H is an induced subgraph of J. Conversely, if $G \cong H + (\overline{K}_\ell \cup \overline{K}_\jmath)$, then it is easy to verify the equality case. □

The following result concerns the lower bound for $\Phi(G)$ in terms of pendent and quasi pendent nodes.

Theorem 3. *Let G be a graph with p pendent nodes. Then*

$$\Phi(G) \geq \sum_{i=1}^{p}\phi(1, d_{u_i})$$

where d_i is the degree of quasi pendent nodes. Equality holding if and only if $G \cong \frac{n}{2}K_2$.

Proof. Let $u_1, \ldots, u_p$ be the nodes adjacent to nodes of degree 1. Then corresponding to each such edge, the edge weight function is $\phi(1, d_{u_i})$, for $i = 1, 2, \ldots, p$. Thus, by definition, we have

$$\Phi(G) = \sum_{i=1}^{p}\phi(1, d_{u_i}) + \Phi(G)' \geq \sum_{i=1}^{p}\phi(1, d_{u_i}),$$

where $\Phi(G)'$ is the edge weight of the remaining edges. Clearly, equality holds if and only if $\Phi(G)' = 0$, which is possible if and only if G is isomorphic to $\frac{n}{2}$ disjoint union of K_2 copies. □

Theorem 3 can be generalized as follows.

Theorem 4. *Let G be a graph with p nodes of degree $r \geq 1$. Then*

$$\Phi(G) \geq \sum_{i=1}^{r}\phi(r, d_{u_{1_i}}) + \sum_{i=1}^{r}\phi(r, d_{u_{2_i}}) + \cdots + \sum_{i=1}^{r}\phi(r, d_{u_{p_i}})$$

where $d_{u_{i_j}}$ is the degree of the nodes u_{i_j} for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r$. Equality holding if and only if $G \cong \frac{n}{r+1}K_{1,r}$.

Next, we have a consequence of the above result.

Corollary 2. *Let G be a graph with p nodes of degree $r \geq 1$. Then*

$$\Phi(G) \geq pr\phi(r,1),$$

with equality holding if and only if $G \cong \frac{n}{r+1} K_{1,r}$.

The following result considers graphs with given number of nodes of some degrees.

Theorem 5. *Let G be a graph with p nodes of degree r and $q \geq p$ nodes of degree $s \geq r \geq 1$. Then*

$$\begin{aligned}\Phi(G) \geq & \sum_{i=1}^{r} \phi(r, d_{u_{1_i}}) + \sum_{i=1}^{r} \phi(r, d_{u_{2_i}}) + \cdots + \sum_{i=1}^{r} \phi(r, d_{u_{p_i}}) \\ & + \sum_{i=1}^{s} \phi(s, d_{v_{1_i}}) + \sum_{i=1}^{s} \phi(r, d_{v_{2_i}}) + \cdots + \sum_{i=1}^{s} \phi(s, d_{v_{q_i}}).\end{aligned}$$

where $d_{u_{i_j}}$ is the degree of the nodes u_{i_j} for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r$ and $d_{v_{i_j}}$ is the degree of the nodes v_{i_j} for $i = 1, 2, \ldots, q$ and $j = 1, 2, \ldots, s$. Equality holding if and only if $G \cong \frac{n}{2(r+1)} K_{1,r} \cup \frac{n}{2(s+1)} K_{1,s}$.

The next corollary is a consequence of above result.

Corollary 3. *Let G be a graph with p nodes of degree r and $q \geq p$ nodes of degree $s \geq r \geq 1$. Then*

$$\Phi(G) \geq pr\phi(r,1) + qs\phi(s,1),$$

with equality holding if and only if $G \cong \frac{n}{2(r+1)} K_{1,r} \cup \frac{n}{2(s+1)} K_{1,s}$.

Remark 1. If $q = s = 0$ in Theorem 5, we get Theorem 4 and if $q = s = 0$ and $r = 1$ in Theorem 5 we get Theorem 3. Also, note that for $r = 1$ in Theorem 4 we get Theorem 3. Therefore, Theorem 4 is generalization of Theorem 3 and Theorem 5 is generalization of both Theorems 3 and 4.

We will prove Theorem 5 and Theorem 4 can be similarly proved.

Proof of Theorem 5. As G has p nodes of degree r and q nodes of degree s with $s \geq r$ and $p \leq q$. Labelling these p nodes by $w_1, \ldots, w_p$ and their neighbours by u_{i_j} for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r$. Also, let $x_1, \ldots, x_q$ be the nodes of degree s and let v_{i_j} for $i = 1, 2, \ldots, q$ and $j = 1, 2, \ldots, s$ be their neighbours. Calculating the edge weights for w_1, we get

$$W(w_1) = \phi(r, d_{u_{1_1}}) + \phi(r, d_{u_{1_2}}) + \cdots + \phi(r, d_{u_{1_r}}).$$

Similarly, calculating the weight of other w_i's and summing them, we have

$$\sum_{i=1}^{p} W(w_i) = \sum_{i=1}^{r} \phi(r, d_{u_{1_i}}) + \sum_{i=1}^{r} \phi(r, d_{u_{2_i}}) + \cdots + \sum_{i=1}^{r} \phi(r, d_{u_{p_i}}).$$

Also, calculating the weight of x_i's, we obtain

$$\sum_{i=1}^{p} W(x_i) = \sum_{i=1}^{s} \phi(s, d_{v_{1_i}}) + \sum_{i=1}^{s} \phi(s, d_{v_{2_i}}) + \cdots + \sum_{i=1}^{s} \phi(s, d_{v_{q_i}}).$$

By definition of $\Phi(G)$, we have

$$\begin{aligned}\Phi(G) &= \sum_{i=1}^{p} W(w_i) + \sum_{i=1}^{p} W(x_i) + \Theta(G)\\ &\geq \sum_{i=1}^{r} \phi(r, d_{u_{1_i}}) + \sum_{i=1}^{r} \phi(r, d_{u_{2_i}}) + \cdots + \sum_{i=1}^{r} \phi(r, d_{u_{p_i}}) + \sum_{i=1}^{s} \phi(s, d_{v_{1_i}})\\ &\quad + \sum_{i=1}^{s} \phi(s, d_{v_{2_i}}) + \cdots + \sum_{i=1}^{s} \phi(s, d_{v_{q_i}}),\end{aligned}$$

where $\Theta(G)$ is the edge weight of remaining nodes.
Equality holds if and only if $\Theta(G) = 0$, that is, true if and only if there are no edges among w_i's, among x_i's, between w_i's and x_i's, among u_{i_j}'s, among v_{i_j}'s, between u_{i_j}'s and v_{i_j}'s and no edges between and among remaining nodes of G. Thus, it follows that G union of $\frac{n}{2(r+1)}K_{1,r}$ and $\frac{n}{2(s+1)}K_{1,s}$. □

The following result concerns the topological index of $G_1 + G_2$ in terms of the degree sequences of G_1 and G_2.

Theorem 6. *Let G_1 and G_2 be two graphs of order n_1 and n_2, respectively. Then*

$$\Phi(G_1 + G_2) \geq \sum_{i=1}^{n_1} \sum_{i=1}^{n_2} \phi(d_{u_i+n_2}, d_{v_i+n_1}),$$

where d_{u_i} and d_{v_i} are the degrees of G_1 and G_2 in $G_1 + G_2$, respectively. The equality holds if and only if $G_1 + G_2$ is a complete bipartite graph.

Proof. Let $u_1, \ldots, u_{n_1}$ be the node labelling of G_1 and $v_1, \ldots, v_{n_2}$ be labelling of G_1. Then the degree sequence of $G_1 + G_2$ is $d'_{u_1}, \ldots, d'_{u_{n_1}}, d'_{v_1}, \ldots, d'_{v_{n_2}}$, where $d'_{u_j} =$

$n_2 + d_{u_j}$ and $d'_{v_i} = n_1 + d_{v_i}$, for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. Thus with this labelling of the nodes, we have

$$\begin{aligned}\Phi(G_1 + G_2) &= \phi_1 + \phi_2 + \sum_{i=1}^{n_1}\sum_{i=1}^{n_2} \phi(d'_u, d'_v) \\ &\geq \sum_{i=1}^{n_1}\sum_{i=1}^{n_2} \phi(d'_{u_i}, d'_{v_i}) = \sum_{i=1}^{n_1}\sum_{i=1}^{n_2} \phi(d_{u_i+n_2}, d_{v_i+n_1}),\end{aligned}$$

where $\phi_1 = \sum_{u_iu_j \in E(G_1)} \phi(d_{u_i+n_2}, d_{u_j+n_2})$ and $\phi_2 = \sum_{v_iv_j \in E(G_1)} \phi(d_{v_i+n_1}, d_{v_j+n_1})$. The equality holds if $\phi_1 = \phi_2 = 0$, that is, there are no edges in G_1 and no edges in G_2. Which is possible if $d_{u_1} = \cdots = d_{u_{n_1}} = n_2$ and $d_{v_1} = \cdots = d_{v_{n_2}} = n_1$. This implies that the equality holds for a complete bipartite graph. Conversely it is easy to see that if $G_1 + G_2$ is complete bipartite graph, then equality holds. □

3. Weighted topological indices of cozero divisor graphs of rings

Let R be a commutative ring with unity $1 \neq 0$, the cozero divisor graph associated to R is denoted by $\Gamma'(R)$, and is defined as a simple graph with node set as non-zero non-unit elements of R such that two nodes a and b with $(a \neq b)$ are adjacent if and only if $a \notin Rb$ and $b \notin Ra$, where aR is the ideal generated by a. These graphs are motivated by zero divisor graphs, defined as the graph $\Gamma(R)$ with node set as non-zero zero divisors of R such that two distinct nodes are adjacent if and only if their product is zero. Afkhami and Khashyarmanesh [1–4] in a collection of papers studied the properties of $\Gamma'(R)$ like their graph complements, planarity, identification of commutative rings with forest, star or unicyclic graphs, their relations with comaximal graphs of rings and zero divisor graphs. The cozero divisor graphs of polynomial rings were carried in [5], the spectral analysis of cozero divisor graphs were carried in [23]. Bakhtyiari, Nikandish and Nikmehr [7] obtained results related to coloring of cozero-divisor graphs of commutative von Neumann regular rings. For some other recent progress of cozero divisor, see [6, 24, 29].

It seems very hard to determine structure of $\Gamma'(R)$, but for some special cases we may find get some information about $\Gamma'(R)$. We consider cozero divisor graphs of the integral modulo ring $\mathbb{Z}_n$. The cozero divisor $\Gamma'(\mathbb{Z}_n)$ can be partitioned into various mutually independent sets (cells). Consider the proper divisors $\tau_i, i \notin \{1, n\}$ of n, and consider cells:

$$C_{\tau_i} = \{x \in \mathbb{Z}_n \ : \ (x, n) = \tau_i\},$$

where (x, n) is the GCD (greatest common divisor) of a and n. Clearly $V(\Gamma'(\mathbb{Z}_n)) = \bigcup_{i=1}^{t} C_{\tau_i}$, where $t = |\{\tau_i \ : \ i \notin \{1, n\}\}|$. Furthermore, for $x, y \in C_{\tau_i}$, we have $\langle x \rangle = \langle y \rangle$. From Young [32], $|C_{\tau_j}| = \phi\left(\frac{n}{\tau_i}\right)$, for $j = 1, \ldots, t$, where $\phi(n)$ is the

number of positive integers less than n and relatively prime to it (Euler's totient function). Also, if $x \in C_{\tau_i}$ and $y \in C_{\tau_j}$ then x and y are adjacent in $\Gamma'(\mathbb{Z}_n)$ if and only $\tau_i \nmid \tau_j$ and $\tau_j \nmid \tau_i$, for $i, j \in \{1, 2, \ldots, \tau(n) - 2\}$, where $\tau(n)$ is divisor function. From [23], the induced subgraph of C_{τ_j} is $\overline{K}_{\phi\left(\frac{n}{\tau_j}\right)}$, for each $i \in \{1, 2, \ldots, \tau(n) - 2\}$. The number of nodes of $\Gamma'(\mathbb{Z}_n)$ is $N = n - \phi(n) - 1$. More about $\Gamma'(\mathbb{Z}_n)$ can be seen in [23].
The very first result of this section gives the weighted topological index of $\Gamma'(\mathbb{Z}_n)$ when n is product of three distinct primes.

Theorem 7. *The general topological index of* $\Gamma'(\mathbb{Z}_n)$ *for* $n = pqr$ *with primes* $p < q < p$*is*

$$\begin{aligned}\Phi(\Gamma'(\mathbb{Z}_n)) = \phi(n)\Big(&\phi(q)\phi(d_{\phi(p)(q+r-1)}, d_{\phi(p)(q+r-1)}) + \phi(r)\phi(d_{\phi(p)(q+r-1)}, d_{\phi(q)(p+r-1)})\\ &+ \phi(d_{\phi(p)(q+r-1)}, d_{qr-1}) + \phi(p)\phi(d_{\phi(r)(p+q-1)}, d_{\phi(q)(p+r-1)})\\ &+ \phi(d_{\phi(q)(p+r-1)}, d_{pr-1}) + \phi(d_{\phi(r)(p+q-1)}, d_{pq-1})\Big) + \phi(qr)\phi(d_{pq-1}, d_{pr-1})\\ &+ \phi(pr)\phi(d_{pq-1}, d_{qr-1}) + \phi(pq)\phi(d_{pr-1}, d_{qr-1}).\end{aligned}$$

Proof. We divide the node set of $\Gamma'(\mathbb{Z}_n)$ for $n = pqr$ as

$$\begin{aligned}&C_1 = \{kp \mid k = 1, \ldots, qr - 1, q \nmid k, r \nmid k\}, C_2 = \{kq \mid k = 1, \ldots, pr - 1, p \nmid k, r \nmid k\},\\ &C_3 = \{kr \mid k = 1, \ldots, pq - 1, p \nmid k, q \nmid k\}, C_4 = \{kpq \mid k = 1, \ldots, r - 1\},\\ &C_5 = \{kpr \mid k = 1, \ldots, q - 1\}, C_6 = \{kqr \mid k = 1, \ldots, p - 1\}.\end{aligned} \tag{3.1}$$

We note that $a \in \langle b\rangle$ if and only if b divides a and in this case a and b are not adjacent in $\Gamma'(\mathbb{Z}_n)$. Consider two arbitrary elements a and b of C_1, we observe that either a divides b or b divides a, since C_1 contains some multiples of p. Thus, we see that no node in C_1 is adjacent to any other node of C_1. Thereby, the induced subgraph of C_1 is isomorphic to $\overline{K}_{|C_1|}$. In a similar manner we can show that the induced subgraphs of other C_i's are isomorphic to the complements of cliques. From Expression (3.1), the cardinalities of other C_i's are $|C_2| = (p-1)(r-1), |C_3| = (p-1)(q-1), |C_4| = r-1, |C_5| = q-1$ and $|C_6| = p-1$. Now, we need to find the adjacency relations among C_i's. Let a be a node of C_1 and b be a node either in C_4 or in C_5, then it is clear that a may be in $\langle b\rangle$ or b may be in $\langle a\rangle$, since either a divides b or vice versa. So, the nodes of C_1 cannot be adjacent to the nodes of C_i, for $i = 4, 5$. Next, consider $a \in C_2$ and $b \in C_4 \cup C_6$, then as before a may divide b and vice versa. So, no node of C_2 is adjacent to any node of C_4 or C_6. Similarly for $a \in C_3$ and $b \in C_5 \cup C_6$, it is clear that $a \in \langle b\rangle$ or $b \in \langle a\rangle$. Again for $a \in C_4$ and $b \in C_3 \cup C_5 \cup C_6$, by (3.1), it follows that a does not divide b, that is $a \notin \langle b\rangle$ and $b \notin \langle a\rangle$. Thereby, we see that each node of C_4 is adjacent to each node of C_3, C_5 and C_6. Thus, with the construction of $\Gamma'(\mathbb{Z}_n)$, the adjacency relations between the nodes of C_i's are completely known. The nodes of C_i's have the common degree $d_{iv_1} = \cdots = d_{iv_{|V|_i}} = d_i$ for $i = 1, \ldots, 6$. More precisely, $d_1 = \phi(pq) + \phi(pr) + \phi(p) = \phi(p)(q + r - 1), d_2 = \phi(q)(p + r - 1), d_3 =$

$\phi(r)(p+q-1), d_4 = pq-1, d_5 = pr-1$ and $d_6 = qr-1$. Therefore, with the information and the definition of $\Phi(\Gamma'(\mathbb{Z}_n))$, we have

$$\begin{aligned}\Phi(\Gamma'(\mathbb{Z}_n)) = &\sum_{uv\in E(C_1\cup C_2)} \phi(d_1,d_2) + \sum_{uv\in E(C_1\cup C_3)} \phi(d_1,d_3) + \sum_{uv\in E(C_1\cup C_6)} \phi(d_1,d_6)\\ &+ \sum_{uv\in E(C_2\cup C_3)} \phi(d_2,d_3) + \sum_{uv\in E(C_2\cup C_5)} \phi(d_2,d_5) + \sum_{uv\in E(C_3\cup C_4)} \phi(d_3,d_4)\\ &+ \sum_{uv\in E(C_4\cup C_5)} \phi(d_4,d_5) + \sum_{uv\in E(C_4\cup C_6)} \phi(d_4,d_6) + \sum_{uv\in E(C_5\cup C_6)} \phi(d_5,d_6).\end{aligned}$$

Now making the necessary calculations, we get the required result. □

Lemma 1 ([30]). *Let $n = p_1^{n_1}p_2$ (or $n = p_1p_2^{n_2}$, n_2 is a positive integer) where p_1, p_2 are primes and n_1 is a positive integer. Then $\Gamma'(\mathbb{Z}_n)$ is a bipartite graph.*

Theorem 8. *If $n = p_1^{n_1}p_2$, then*

$$\Phi(\Gamma'(\mathbb{Z}_n)) = \sum_{i=1}^{n_1} \phi(p_1^{i-1}p_2) \sum_{j=1}^{n_1+1-i} \phi(p_1^{n_1+1-j})\phi(d_{u_i}, d_{v_j}),$$

where $d_{u_i} = p_1^{n_1} - p_1^{i-1}$ and $d_{v_j} = \phi(p_2)p_1^{n_1-j}$, for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_1$.

Proof. By Lemma 1, the cozero divisor graph $\Gamma'(\mathbb{Z}_n)$ is bipartite for $n = p_1^{n_1}p_2$. So its node set can be divided into two subsets. Let C_i's be the small subsets in one set and B_j's subsets in other set. So there will be edges between C_i's and B_j's for some i and j. Also, C_i's correspond to the divisors p_1^i, for $i = 1, \ldots, n_1$ and B_j's correspond to the divisors $p_1^{n_1-j}p_2$, for $j = 1, \ldots, n_1$. The divisor $p_1^{n_1}$ is not multiple of any $p_1^{n_1-i}p_2$, for $i = 1, 2, \ldots, n_1$. By definition each node in cells C_i (or B_i) share the same neighbourhood, so the degree sequence of $\Gamma'(\mathbb{Z}_n)$ can be determined. Let u_i be common degree of C_i and v_j be the common degree of B_j. Thus, we get

$$d_{u_1} = \phi(p_1) + \cdots + \phi(p_1^{n_1}) = p_1^{n_1} - 1,$$

since each node of C_1 is adjacent to every node of B_j and we note the number theoretic fact $\sum_{i=1}^{t} \phi(p^i) = p^t - 1$, for prime p and positive integer $t \geq 2$. Also degree of each node of C_2 is $\phi(p_1^2) + \cdots + \phi(p_1^{n_1}) = p_1^{n_1} - p_1$ as each node of C_2 is adjacent to each node of $B_1 \cup B_{n_1-1}$. Similarly, degrees of other u_i's is given by

$$d_{u_3} = \phi(p_1^3) + \cdots + \phi(p_1^{n_1}) = p_1^{n_1} - p_1^2, \ldots, d_{u_{n_1-2}} = +\phi(p_1^{n_1-2}) + \phi(p_1^{n_1}) = p_1^{n_1} - p_1^{n_1-3},$$
$$d_{u_{n_1-1}} = \phi(p_1^{n_1}) + \phi(p_1^{n_1-1}) = p_1^{n_1} - p_1^{n_1-2}, d_{u_{n_1}} = \phi(p_1^{n_1}) = p_1^{n_1} - p_1^{n_1-1}.$$

Also, each node of B_1 is adjacent to each node of C_i for all i. So, the degree of each node in B_1 is

$$d_{v_1} = \phi(p_2)+\phi(p_1p_2)+\cdots+\phi(p_1^{n_1-1}p_2) = \phi(p_2)\Big(1+\phi(p_1)+\cdots+\phi(p_1^{n_1-1})\Big) = \phi(p_2)p_1^{n_1}-1.$$

Likewise, the common degree of other nodes in B_j is given by

$$\begin{aligned} d_{v_2} &= \phi(p_2) + \phi(p_1p_2) + \cdots + \phi(p_1^{n_1-2}p_2) = \phi(p_2)p_1^{n_1-2}, d_{v_3} = \phi(p_2) + \phi(p_1p_2) \\ &+ \cdots + \phi(p_1^{n_1-3}p_2) = \phi(p_2)(p_1^{n_1-3}), \ldots, d_{v_{n_1-2}} = \phi(p_2) + \phi(p_1p_2) + \phi(p_1^2p_2) \\ &= \phi(p_1)p_1^2 \\ d_{v_{n_1-1}} &= \phi(p_2) + \phi(p_1p_2) = \phi(p_2)p_1, d_{v_1} = \phi(p_2). \end{aligned}$$

Thus, by the definition of the topological index ϕ, we have

$$\begin{aligned} \Phi(\Gamma'(\mathbb{Z}_n)) &= \phi(p_2)\Big(\phi(p_1^{n_1})\phi(d_{u_1}, d_{v_1}) + \phi(p_1^{n_1-1})\phi(d_{u_1}, d_{v_2}) + \phi(p_1^{n_1-2})\phi(d_{u_1}, d_{v_3}) \\ &+ \cdots + \phi(p_1^3)\phi(d_{u_1}, d_{v_{n_1-2}}) + \phi(p_1^2)\phi(d_{u_1}, d_{v_{n_1-1}}) + \phi(p_1)\phi(d_{u_1}, d_{v_{n_1}})\Big) \\ &+ \phi(p_1p_2)\Big(\phi(p_1^{n_1})\phi(d_{u_2}, d_{v_1}) + \phi(p_1^{n_1-1})\phi(d_{u_2}, d_{v_2}) + \phi(p_1^{n_1-2})\phi(d_{u_2}, d_{v_3}) + \ldots \\ &+ \phi(p_1^3)\phi(d_{u_2}, d_{v_{n_1-2}}) + \phi(p_1^2)\phi(d_{u_2}, d_{v_{n_1-1}})\Big) + \phi(p_2p_1)\Big(\phi(p_1^{n_1})\phi(d_{u_3}, d_{v_1}) \\ &+ \phi(p_1^{n_1-1})\phi(d_{u_3}, d_{v_2}) + \phi(p_1^{n_1-2})\phi(d_{u_3}, d_{v_3}) + \phi(p_1^3)\phi(d_{u_3}, d_{v_{n_1-2}})\Big) \\ &\vdots \\ &+ \phi(p_2p_1^{n_1-3})\Big(\phi(p_1^{n_1})\phi(d_{u_{n_1-2}}, d_{v_1}) + \phi(p_1^{n_1-1})\phi(d_{u_{n_1-2}}, d_{v_2}) + \phi(p_1^{n_1-2})\phi(d_{u_{n_1-2}}, d_{v_3})\Big) \\ &+ \phi(p_2p_1^{n_1-2})\Big(\phi(p_1^{n_1})\phi(d_{u_{n_1-1}}, d_{v_1}) + \phi(p_1^{n_1-1})\phi(d_{u_{n_1-1}}, d_{v_2})\Big) \\ &+ \phi(p_2p_1^{n_1-1})\phi(p_1^{n_1})\phi(d_{u_{n_1}}, d_{v_1}) \\ &= \phi(p_2)\sum_{j=1}^{n_1}\phi(p_1^{n_1+1-j})\phi(d_{u_1}, d_{v_j}) + \phi(p_1p_2)\sum_{j=1}^{n_1-1}\phi(p_1^{n_1+1-j})\phi(d_{u_2}, d_{v_j}) + \phi(p_2p_1^2) \\ &\sum_{j=1}^{n_1-2}\phi(p_1^{n_1+1-j})\phi(d_{u_3}, d_{v_j}) + \cdots + \phi(p_2p_{n_1-3})\sum_{j=1}^{n_1-(n_1-3)}\phi(p_1^{n_1+1-j})\phi(d_{u_{n_1-2}}, d_{v_j}) \\ &+ \phi(p_2p_{n_1-2})\sum_{j=1}^{n_1-(n_1-2)}\phi(p_1^{n_1+1-j})\phi(d_{u_{n_1-1}}, d_{v_j}) + \phi(p_2p_{n_1-1})\phi(p_1^{n_1})\phi(d_{u_{n_1}}, d_{v_1}) \\ &= \sum_{i=1}^{n_1}\phi(p_2p_1^{i-1})\sum_{j=1}^{n_1+1-i}\phi(p_1^{n_1+1-j})\phi(d_{u_i}, d_{v_j}), \end{aligned}$$

$d_{u_i} = p_1^{n_1} - p_1^{i-1}$ and $d_{v_j} = \phi(p_2)p_1^{n_1-j}$, for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_1$. This completes the proof. □

Theorem 9. *Let $n = \prod_{i=1}^{t} p_i$ be the product of $t \geq 3$ primes. Then*

$$\Phi(\Gamma'(\mathbb{Z}_n)) \geq \sum_{j=1}^{t-1} D_j \sum_{i=j+1}^{t} D_i\phi(d_{u_j}, d_{u_i}) + \sum_{i=1}^{t} D_iE_i\phi(d_{u_i}, d_{v_i}) + \sum_{j=1}^{t-1} E_j \sum_{i=j+1}^{t} E_i\phi(d_{v_j}, d_{v_i}),$$

where $D_i = \phi\Big(\prod_{\substack{j=1\\i\neq j}}^{t} p_j\Big), E_i = \phi(p_i), d_{u_i} = \sum_{\substack{j=1\\i\neq j}}^{t} D_i + E_i + \alpha$ and $d_{v_i} = \sum_{\substack{j=1\\i\neq j}}^{t} E_i + D_i + \beta$, for $i = 1, \ldots, t$ with $\alpha, \beta \geq 0$. Equality holds if and only if $t = 3$.

Proof. Let $n = \prod_{i=1}^{t} p_i$ be the product of $t \geq$ primes with $p_1 < \cdots < p_t$ and let $G \cong \Gamma'(\mathbb{Z}_n)$ be the corresponding cozero divisor graphs. The structure of $\Gamma'(\mathbb{Z}_n)$ depends upon the proper divisors of n. We investigate the cells C_{p_i} and C_{e_i} with $e_i = \prod_{\substack{j=1\\ i\neq j}}^{t} p_j$, for $i = 1, \ldots, t$. The cardinalities of C_{p_i} are $D_i = \phi\Big(\prod_{\substack{j=1\\ i\neq j}}^{t} p_j\Big)$ and that of C_{e_i}'s are $E_i = \phi(p_i)$, for $i = 1, \ldots, t$. Now, we see that each $a \in C_{p_i}$ is adjacent to every $b \in C_{b_j}$, since p_i doest not divide p_j with $i < j$. Also, each $a \in C_{e_i}$ is adjacent to every $b \in C_{e_j}$, since e_i doest not divide e_j with $i < j$. Furthermore, each node in C_{p_i} is adjacent to each node in C_{e_i} as p_i does not divide $e_i = \prod_{\substack{j=1\\ i\neq j}}^{t} p_j$. Thus we know adjacency relation among C_{p_i}'s, C_{e_i}'s and between them. Certainly there are more cells C_{d_i} for some proper divisors of n other than p_i's and e_i's along with new adjacency relation, since G is connected. As cells C_i's share neighbourhood in terms of other cells, so degree of each node in each cell is common. Let d_{v_i} be the common degree of C_{p_i} and d_{u_i} be the common degree of each node in $C_{d_{e_i}}$ for $i = 1, \ldots, t$. Thus, we have

$$d_{u_i} = \sum_{\substack{j=1\\ i\neq j}}^{t} D_i + E_i + \alpha \quad d_{v_i} = \sum_{\substack{j=1\\ i\neq j}}^{t} E_i + D_i + \beta,$$

for $i = 1, \ldots, t$ with $\alpha, \beta \geq 0$. Therefore, by the definition of ϕ, we have

$$\begin{aligned}
\Phi(G) &= D_1\Big(D_2\phi(d_{u_1}, d_{u_2}) + \cdots + D_t\phi(d_{u_1}, d_{u_t})\Big) + D_2\Big(D_3\phi(d_{u_2}, d_{u_3}) + \cdots + D_t\phi(d_{u_2}, d_{u_t})\Big) \\
&\quad + \cdots + D_{t-2}\Big(D_{t-1}\phi(d_{u_{t-2}}, d_{u_{t-1}}) + D_t\phi(d_{u_{t-2}}, d_{u_t})\Big) + D_t D_{t-1}\phi(d_{u_{t-1}}, d_{u_t}) \\
&\quad + D_1E_1\phi(d_{u_1}, d_{v_1}) + \cdots + D_tE_t\phi(d_{u_t}, d_{v_t}) + E_1\Big(E_2\phi(d_{v_1}, d_{v_2}) + \cdots + E_t\phi(d_{v_1}, d_{v_t})\Big) \\
&\quad + E_2\Big(E_3\phi(d_{v_2}, d_{v_3}) + \cdots + E_t\phi(d_{v_2}, d_{v_t})\Big) + \cdots + E_{t-2}\Big(E_{t-1}\phi(d_{v_{t-2}}, d_{v_{t-1}}) \\
&\quad + E_t\phi(d_{v_{t-2}}, d_{v_t})\Big) + E_tE_{t-1}\phi(d_{v_{t-1}}, d_{v_t}) + \Theta \\
&= D_1\sum_{i=2}^{t} D_i\phi(d_{u_1}, d_{u_i}) + D_2\sum_{i=3}^{t} D_i\phi(d_{u_2}, d_{u_i}) + \cdots + D_{t-2}\sum_{i=t-1}^{t} D_i\phi(d_{u_{t-2}}, d_{u_i}) \\
&\quad + D_tD_{t-1}\phi(d_{u_{t-1}}, d_{u_t}) + \sum_{i=1}^{t} D_iE_i\phi(d_{u_i}, d_{v_i}) + E_1\sum_{i=2}^{t} E_i\phi(d_{v_1}, d_{v_i}) + E_2\sum_{i=3}^{t} E_i\phi(d_{v_2}, d_{v_i}) \\
&\quad + \cdots + E_{t-2}\sum_{i=t-1}^{t} E_i\phi(d_{v_{t-2}}, d_{v_i}) + E_tE_{t-1}\phi(d_{v_{t-1}}, d_{v_t}) + \Theta \\
&= \sum_{j=1}^{t-1} D_j \sum_{i=j+1}^{t} D_i\phi(d_{u_j}, d_{u_i}) + \sum_{i=1}^{t} D_iE_i\phi(d_{u_i}, d_{v_i}) + \sum_{j=1}^{t-1} E_j \sum_{i=j+1}^{t} E_i\phi(d_{v_j}, d_{v_i}) + \Theta \\
&\geq \sum_{j=1}^{t-1} D_j \sum_{i=j+1}^{t} D_i\phi(d_{u_j}, d_{u_i}) + \sum_{i=1}^{t} D_iE_i\phi(d_{u_i}, d_{v_i}) + \sum_{j=1}^{t-1} E_j \sum_{i=j+1}^{t} E_i\phi(d_{v_j}, d_{v_i}),
\end{aligned}$$

where $D_i = \phi\Big(\prod_{\substack{j=1\\ i\neq j}}^{t} p_j\Big)$, $E_i = \phi(p_i)$, $d_{u_i} = \sum_{\substack{j=1\\ i\neq j}}^{t} D_i + E_i + \alpha$ and $d_{v_i} = \sum_{\substack{j=1\\ i\neq j}}^{t} E_i +$

$D_i+\beta$, for $i=1,\ldots,t$ with $\Theta \geq 0$, (a weight quantity corresponding to the remaining nodes of G.) Next, we consider the equality case. Suppose equality holds, then we must have $\Theta=0$. We need to verify that there are not edges present in G other than the edges among C_{p_i}, C_{e_i} and between C_{p_i} to C_{e_i} for each i. If we take $t=3$, then we have edges among $C_{p_i}, i=1,2,3$, edges among C_{e_i} where $e_1=p_1p_2, e_2=p_1p_3$ and $e_3=p_2p_3$ and edges between each node of C_{p_i} to C_{e_i} and equality holds in this case, since $\Theta=0$ (see Theorem 7). For $t\geq 4$, there are cells from $C_{p_1p_2p_3}$ (for some i,j,k) to $C_{p_1\ldots p_{t-2}}$ which contribute the non zero quantity to $\Phi(G)$, since there are edges with non-trivial edge weights from each node of $C_{p_ip_jp_k}$ to each node of $C_{p_rp_sp_l}$ and there are also edges from each node of C_{p_z} to every node of $C_{p_ip_jp_k}$ for $z\notin\{i,j,k\}$. In general there are $\binom{n}{2}+\cdots+\binom{n}{n-2}$ cells and each node of each cell in $\binom{n}{i}$ is adjacent to each node of remaining $\binom{n}{i}-1$ cells, for $i=2,\ldots,n-2$. Besides, there are more edge weights among all cells. Thus equality cannot hold when t is at least 4.

Conversely assume that $n=p_1p_2p_3$, then from Theorem 7, there are edges among nodes of C_{p_i}, among nodes of C_{e_i} and between each node of C_{p_i} to every edges of C_{e_i} for each i. Thus, equality holds with $D_1=\phi(p_2p_3), D_2=\phi(p_1p_3), D_3=\phi(p_1p_2), E_1=\phi(p_1), E_2=\phi(p_2)$ and $E_3=\phi(P_3)$. Therefore, we have

$$\begin{aligned}\Phi(G) &= D_1D_2\phi(d_{u_1,d_{u_2}})+D_1D_3\phi(d_{u_1,d_{u_3}})+D_2D_3\phi(d_{u_2,d_{u_3}})+D_1E_1\phi(d_{u_1},d_{v_1})+D_2E_2\phi(d_{u_2},d_{v_2})\\ &\quad + D_3E_3\phi(d_{u_3},d_{v_3})D_1D_2\phi(d_{u_1,d_{u_2}})+D_1D_3\phi(d_{u_1,d_{u_3}})+D_2D_3\phi(d_{u_2,d_{u_3}}).\end{aligned}$$

Now, the other steps are same as in Theorem 7. □

Theorem 10. *Let $n=p_1^{n_1}p_2^{n_2}$. Then*

$$\begin{aligned}\Phi(\Gamma'(\mathbb{Z}_n)) &\geq \phi(p_1^{n_1-1}p_2^{n_2})\phi(p_1^{n_1}p_2^{n_2-1})\Phi(p_2^{n_2-1}\phi(p_1^{n_1}),p_1^{n_1-1}\phi(p_2^{n_2}))+\phi(p_1^{n_1-1}p_2^{n_2})\phi(p_1^{n_1}p_2^{n_2-2})\\ &\quad \Phi(p_2^{n_2-1}\phi(p_1^{n_1}),p_1^{n_1-1}(p_2^{n_2}-p_2^{n_2-2}))+\phi(p_1^{n_1}p_2^{n_2-1})\phi(p_1^{n_1-2}p_2^{n_2})\Phi(p_1^{n_1-1}\phi(p_2^{n_2}),\\ &\quad p_2^{n_2-1}(p_1^{n_1}-p_1^{n_1-2}))+\phi(p_1^{n_1-2}p_2^{n_2})\phi(p_1^{n_1}p_2^{n_2-2})\Phi(p_2^{n_2-1}(p_1^{n_1}-p_1^{n_1-2}),p_1^{n_1-1}(p_2^{n_2}-p_2^{n_2-2})).\end{aligned}$$

Proof. Let $n=p_1^{n_1}p_2^{n_2}$, where p_1,p_2 are primes and n_1,n_2 are positive integers and $G\cong\Gamma'(\mathbb{Z}_n)$ be its cozero divisor graph. From the structural properties of G, each node of A_{p_1} is adjacent to each node of $A_{p_2^i}$, since p_1 does not divide any p_2^i for $i=1,\ldots,n_2$. Similarly, each node of A_{p_2} is adjacent to every node of cells $A_{p_1^i}$, for $i=1,\ldots,n_1$. Also cardinalities of $A_{p_1^i}$ is $\phi(p_1^{n_1-i}p_2^{n_2})$ and that of $A_{p_2^j}$ is $\phi(p_1^{n_1}p_2^{n_2-j})$, for $i=1,\ldots,n_1$ and $j=1,\ldots,n_2$. Thus the common degree of each node of A_{p_1} is

$$d_1=\phi(p_1^{n_1}p_2^{n_2-1})+\phi(p_1^{n_1}p_2^{n_2-2})+\cdots+\phi(p_1^{n_1})=\phi(p_1^{n_1})p_2^{n_2-1}.$$

The common degree of each node of A_{p_2} is

$$d_1'=\phi(p_2^{n_2}p_1^{n_1-1})+\phi(p_2^{n_2}p_1^{n_1-2})+\cdots+\phi(p_2^{n_2})=\phi(p_2^{n_2})p_1^{n_1-1}.$$

Also, each node of $A_{p_1^2}$ is adjacent to every node of $A_{p_2^i}$ and $A_{p_1 p_2^i}$ for $i = 1, \ldots, n_2$. With this information, the degree of each node in $A_{p_1^2}$ is

$$\begin{aligned} d_2 = {} & \phi(p_1^{n_1} p_2^{n_2-1}) + \phi(p_1^{n_1} p_2^{n_2-2}) + \cdots + \phi(p_1^{n_1}) + \phi(p_1^{n_1-1} p_2^{n_2-1}) + \phi(p_1^{n_1-1} p_2^{n_2-2}) \\ & + \cdots + \quad \phi(p_1^{n_1-1}) = p_2^{n_2-1}\phi(p_1^{n_1}) + \phi(p_1^{n_1-1})p_2^{n_2-1} = p_2^{n_2-1}(p_1^{n_1} - p_1^{n_1-2}). \end{aligned}$$

In a similar way $A_{p_2^2}$ is adjacent to every node of $A_{p_1^i}$ and $A_{p_1^i p_2}$ for $i = 1, \ldots, n_1$. So, the degree of each node in $A_{p_2^2}$ is

$$\begin{aligned} d_2' = {} & \phi(p_2^{n_2} p_1^{n_1-1}) + \phi(p_2^{n_2} p_1^{n_1-2}) + \cdots + \phi(p_2^{n_2}) + \phi(p_1^{n_1-1} p_2^{n_2-1}) + \phi(p_1^{n_1-2} p_2^{n_2-1}) \\ & + \cdots + \quad \phi(p_2^{n_2-1}) = \phi(p_2^{n_2})p_1^{n_1-1} + \phi(p_2^{n_2-1})p_1^{n_1-1} = p_1^{n_1-1}(p_2^{n_2} - p_2^{n_2-2}). \end{aligned}$$

There are other adjacency relations between cells between $A_{p_1^i}$ and $A^j_{p_2}$ and between $A_{p_1^i}$ and $A^j_{p_2}$ with $A_{p_1^i p_2^j}$ for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. Thus by the definition of function Φ, we have

$$\begin{aligned} \Phi(G) = {} & \sum_{uv \in E(A_{p_1} \cup A_{p_2})} \phi(d_u, d_v) + \sum_{uv \in E(A_{p_1} \cup A_{p_2^2})} \phi(d_u, d_v) + \sum_{uv \in E(A_{p_1^2} \cup A_{p_2})} \phi(d_u, d_v) \\ & + \sum_{uv \in E(A_{p_1^2} \cup A_{p_2^2})} \phi(d_u, d_v) + \Theta \end{aligned}$$

where $\theta \geq 0$ is the contribution of other edges weights among the nodes of A_{d_i}'s for the function Φ of G. Thus, with the above information, we have

$$\begin{aligned} \Phi(G) \geq {} & \phi(p_1^{n_1-1} p_2^{n_2})\phi(p_1^{n_1} p_2^{n_2-1})\Phi(d_1, d_1') + \phi(p_1^{n_1-1} p_2^{n_2})\phi(p_1^{n_1} p_2^{n_2-2})\Phi(d_1, d_2') \\ & + \phi(p_1^{n_1} p_2^{n_2-1})\phi(p_1^{n_1-2} p_2^{n_2})\Phi(d_1', d_2) + \phi(p_1^{n_1-2} p_2^{n_2})\phi(p_1^{n_1} p_2^{n_2-2})\Phi(d_2, d_2') \\ = {} & \phi(p_1^{n_1-1} p_2^{n_2})\phi(p_1^{n_1} p_2^{n_2-1})\Phi(p_2^{n_2-1}\phi(p_1^{n_1}), p_1^{n_1-1}\phi(p_2^{n_2})) + \phi(p_1^{n_1-1} p_2^{n_2})\phi(p_1^{n_1} p_2^{n_2-2}) \\ & \Phi(p_2^{n_2-1}\phi(p_1^{n_1}), p_1^{n_1-1}(p_2^{n_2} - p_2^{n_2-2})) + \phi(p_1^{n_1} p_2^{n_2-1})\phi(p_1^{n_1-2} p_2^{n_2})\Phi(p_1^{n_1-1}\phi(p_2^{n_2}), \\ & p_2^{n_2-1}(p_1^{n_1} - p_1^{n_1-2})) + \phi(p_1^{n_1-2} p_2^{n_2})\phi(p_1^{n_1} p_2^{n_2-2})\Phi(p_2^{n_2-1}(p_1^{n_1} - p_1^{n_1-2}), p_1^{n_1-1}(p_2^{n_2} - p_2^{n_2-2})). \end{aligned}$$

That proves the result. $\square$

4. Conclusion

Several results for the general topological index of graphs, and specifically for algebraic graphs of comaximal graphs of commutative rings, are presented in the manuscript. For weighted topological index, it is generally very difficult to characterize extremal graphs because, aside from symmetric property, not enough information is available. Nonetheless, the findings hold true for every kind of topological index currently in use. It appears that it is preferable to take specific values of the function

$\Phi(d_u, d_v)$ into account and perform a more thorough analysis for such a function for applications to specific classes of graphs as well as for general graphs.

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