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Research Article

On the nullity of cycle-spliced $\mathbb{T}$-gain graphs

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Abstract: Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain (or complex unit gain) graph and $A(\Phi)$ be its adjacency matrix. The nullity of Φ, denoted by $\eta(\Phi)$, is the multiplicity of zero as an eigenvalue of $A(\Phi)$, and the cyclomatic number of Φ is defined by $c(\Phi) = e(\Phi) - n(\Phi) + \kappa(\Phi)$, where $n(\Phi)$, $e(\Phi)$ and $\kappa(\Phi)$ are the number of vertices, edges and connected components of Φ, respectively. A connected graph is said to be cycle-spliced if every block in it is a cycle. We consider the nullity of cycle-spliced $\mathbb{T}$-gain graphs. Given a cycle-spliced $\mathbb{T}$-gain graph Φ with $c(\Phi)$ cycles, we prove that $0 \leq \eta(\Phi) \leq c(\Phi) + 1$. Moreover, we show that there is no cycle-spliced $\mathbb{T}$-gain graph Φ of any order with $\eta(\Phi) = c(\Phi)$ whenever there are no odd cycles whose gain has real part 0. We give examples of cycle-spliced $\mathbb{T}$-gain graphs whose nullity equals the cyclomatic number, and we show some properties of those graphs Φ such that $\eta(\Phi) = c(\Phi) - \varepsilon$, $\varepsilon \in \{0, 1\}$. A characterization is given in case $\eta(\Phi) = c(\Phi)$ when Φ is obtained by identifying a unique common vertex of 2 cycle-spliced $\mathbb{T}$-gain graphs Φ_1 and Φ_2. Finally, we compute the nullity of all $\mathbb{T}$-gain graphs Φ with $c(\Phi) = 2$.

Keywords: nullity, cycle-spliced gain graphs, cyclomatic number.

AMS Subject classification: 05C05, 05C50

1. Introduction

All graphs in this paper are finite and simple (without loops and multi-edges). Let $G = (V, E)$ be a simple graph, $n(G)$ and $e(G)$ be the number of vertices and edges in G, respectively. The *cyclomatic number* of G is denoted by $c(G)$ and defined

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as $c(G) = e(G) - n(G) + \kappa(G)$, where $\kappa(G)$ stands for the number of connected components of G. A connected simple graph G ($\kappa(G) = 1$) is a *tree* if $c(G) = 0$; when $c(G) = 1$ or $c(G) = 2$, then G is a *unicyclic* or a *bicyclic* graph, respectively. For any simple graph G a *block* is a maximal connected subgraph with no articulation point or cut vertex. Let $V(G) = \{v_1, \dots v_n\}$ be the vertex set of G. Then the *adjacency matrix* of G is a $n \times n$ symmetric matrix usually denoted by $A(G) = (a_{ij})_{n\times n}$ and defined as: $a_{ij} = 1$ if v_i is adjacent to v_j and 0, otherwise. The nullity (resp. rank) of G stand for the nullity (resp. rank) of $A(G)$, denoted by $\eta(G)$ (resp. $r(G)$). It is obvious that $\eta(G) = n(G) - r(G)$, where $n(G)$ is the order of a simple graph G.

Let $\overrightarrow{E}(G)$ be the set of oriented edges of G. Let $\mathfrak{G}$ be a group. A *gain graph* is a triple $\Phi = (G, \mathfrak{G}, \varphi)$ consisting of an underlying graph $G = (V, E)$, the gain group $\mathfrak{G}$ and a mapping $\varphi : \overrightarrow{E}(G) \to \mathfrak{G}$ such that for every oriented edge of G we have $\varphi(e_{ij}) = \varphi(e_{ji})^{-1}$, which is also called the gain function (see [19]). For simplicity, we denote a $\mathfrak{G}$-gain graph by $\Phi = (G, \varphi)$, when $\mathfrak{G}$ is a gain group. For the particular choice $\mathfrak{G} = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, a gain graph Φ is known as a *complex unit gain graph* or $\mathbb{T}$-*gain graph*. Gain graphs are commonly used in many fields, including electrical engineering, computer science, social sciences, and operations research (see [1, 21, 28] for example). They provide a way to model and analyze complex systems, such as transportation networks, communication networks, and social networks ([7, 12, 17, 20, 23]). The adjacency matrix is a fundamental tool for analyzing the properties and behavior of gain graphs. Considered a gain graph $\Phi = (G, \mathfrak{G}, \varphi)$ with its underlying graph $G = (V, E)$, the *adjacency matrix* of Φ is usually denoted by $A(\Phi) = (a_{ij})_{n\times n}$ and defined by

$$a_{ij} = \begin{cases} \varphi(e_{ij}) & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

When v_i is adjacent to v_j, then $a_{ij} = \varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \overline{\varphi(e_{ji})} = \overline{a}_{ji}$. Therefore, $A(\Phi)$ is Hermitian and its eigenvalues are real.

Most of the concepts defined for simple graphs can be directly extended to complex unit gain graphs. Let Φ be a $\mathbb{T}$-gain graph, the *nullity* $\eta(\Phi)$ and rank $r(\Phi)$ of Φ is the nullity and rank of $A(\Phi)$, respectively. The *cyclomatic number* $c(\Phi)$ of a $\mathbb{T}$-gain graph Φ can be borrowed from its underlying graph G. For any connected $\mathbb{T}$-gain graph Φ ($\kappa(\Phi) = 1$), if $c(\Phi) = 0$, $c(\Phi) = 1$ or $c(\Phi) = 2$, then Φ is a *tree*, a *unicyclic* or a *bicyclic* $\mathbb{T}$-gain graph, respectively. The other concepts i.e., block in a graph, cycle-spliced $\mathbb{T}$-gain graph, and pendant cycles can be inherited directly from simple graphs. Let $W = e_{12}e_{23}\dots e_{(h-1)h}$ be a walk of a $\mathbb{T}-$gain graph; then we denote the gain of W by $\varphi(W) = \varphi(e_{12})\varphi(e_{23})\dots\varphi(e_{(h-1)h})$. A walk is *neutral* if $\varphi(W) = 1$. A cycle is *balanced* if its gain is 1. A $\mathbb{T}-$gain graph is *balanced* if all of its cycles are balanced. A *switching function* is any function $\zeta : V \to \mathbb{T}$. Switching the $\mathbb{T}$-gain graph $\Phi = (G, \varphi)$ means replacing φ by φ^ζ, defined by $\varphi^\zeta(e_{ij}) = \zeta(v_i)^{-1}\varphi(e_{ij})\zeta(v_j)$; this replacement produces the $\mathbb{T}$-gain graph $\Phi^\zeta = (G, \varphi^\zeta)$. We say Φ_1 and Φ_2 are *switching equivalent* , written $\Phi_1 \sim \Phi_2$, when there exists a switching function ζ such that $\Phi_2 = \Phi_1^\zeta$. Switching equivalence is an equivalence relation on gain functions for a fixed underlying graph G. In 1989, Thomas Zaslavsky [30], proved the following result for the balance in gain graphs.

Lemma 1. *[30] Let $\Phi = (\Gamma, \varphi)$ be a gain graph. Then Φ is balanced if and only if $\Phi \sim (G, 1)$.*

The study of $\mathbb{T}-$gain graphs has attracted considerable attention in recent years. In [19], N. Reff studied the spectral characteristics for $\mathbb{T}-$gain graphs and defined their adjacency, Laplacian and incidence matrices. The inertia of $\mathbb{T}-$gain graphs and their related properties were studied by Yu et al., in [29]. Wang et al., in [26], provided a description for the determinant of the Laplacian matrix of $\mathbb{T}-$gain graphs. In [13], Lu et al., considered the $\mathbb{T}-$gain bicyclic graphs and characterized those with rank 2, 3 or 4. Furthermore, Lu et al., in [14] investigated the bounds for the rank of $\mathbb{T}-$gain graphs in terms of the rank of their underlying graph. In [10], He et al., showed that $2m(G) - 2c(G) \leq r(\Phi) \leq 2m(G) + c(G)$, where $m(G)$ is the matching number of G.

A graph G is said to be *singular* (resp. *non-singular*) if the adjacency matrix of G is singular (resp. non-singular). In [24], Collatz and Sinogowitz provided a characterization problem for all singular graphs ($\eta(G) > 0$). This problem motivated researchers to enhance further research in this area. Ma et al., in [16], proved that $\eta(G) \leq 2c(G) + p(G) - 1$ unless G is a cycle of length $0 \; mod \, 4$, where $p(G)$ is the total number of leaves in G. Wang [25] and Chang et al., [5] posed the characterization for all graphs G with $\eta(G) = 2c(G) + p(G) - 1$. Further, some related results for signed graphs were studied by Lu and Wu in [15], where they characterized signed graphs on the basis of their matching number and further proved that there are no signed graphs with $\eta(G, \sigma) = n(G) - 2m(G) + 2c(G) - 1$. The nullity of unicyclic signed graphs was studied by Fan et al. in [9]. In [8], Fan et al. also derived results for the nullity of bicyclic signed graphs. Information on the nullity of signed graphs can be deduced by [2]. In [27], Wong et al. provided bounds for the nullity of cycle-spliced bipartite graphs G in terms of $c(\Gamma)$, i.e., $0 \leq \eta(G) \leq c(G) + 1$. The nullity of cycle-spliced signed graphs has been studied in [6] and [4].

The main results of the paper are collected in three theorems. In Theorem 1 we show that the nullity of a cycle-spliced complex unit gain graph with $c(\Phi)$ cycles is at most $c(\Phi) + 1$ and give a characterization of those graphs whose nullity is exactly $c(\Phi) + 1$. Theorem 2 proves that the nullity of Φ never equals the number of cycles $c(\Phi)$ if Φ does not contain any odd cycle C such that the real part $\mathcal{R}(C)$ of its gain is zero. Examples of cycle-spliced $\mathbb{T}$-gain graphs with $\eta(\Phi) = c(\Phi)$ are given. In Theorem 3 there is a characterization of cycle-spliced $\mathbb{T}$-gain graphs with $\eta(\Phi) = c(\Phi)$ when Φ is obtained by identifying the unique common vertex v of two cycle-spliced $\mathbb{T}$-gain graphs Φ_1 and Φ_2.

The rest of the article is organized in the following way.

In section 2, we provide some preliminary lemmas, notations and some basic results about the rank and nullity of $\mathbb{T}$-gain graphs. In section 3, we present the proof of Theorem 1. Section 4 deals with the proof of Theorem 2 and examples of cycle-spliced

$\mathbb{T}$-gain graphs with nullity $\eta(\Phi) = c(\Phi)$. Section 4 also contains the proof of Theorem 3, which characterizes cycle-spliced $\mathbb{T}$-gain graphs Φ with $\eta(\Phi) = c(\Phi)$ obtained by the coalescence of two graphs Φ_1 and Φ_2 with respect to their unique common vertex v. Further, some properties and auxiliary results on cycles-spliced $\mathbb{T}$-gain graphs of this kind are presented. In Section 5 we compute the nullity of particular configurations of cycle-spliced $\mathbb{T}$-gain graphs, such as bipartite, wedge of cycles and bicyclic graphs.

2. Preliminaries

In this section we give the definition of the five types of $\mathbb{T}$-gain cycles and recall their nullity. Given a cut point x for Φ, some basic results about the nullity of $\mathbb{T}$-gain graphs are provided in terms of the nullity of the components of $\Phi - x$.

A $\mathbb{T}$-gain graph $\Phi = (G, \varphi)$ and any induced gain subgraph (H, φ) of Φ will also be denoted by G^φ and H^φ respectively. For $v \in V(\Phi)$, let $d_\Phi(v)$ be the degree and $N_\Phi(v)$ be the set of neighbors of v. Clearly, $d_\Phi(v) = |N_\Phi(v)|$. If $M \subseteq V(\Phi)$, then the deletion of M together with all incidence edges, is the induced subgraph of Φ, denoted by $\Phi - M$. If $M = \{v_1\}$ or $\{v_1, v_2\}$, then $\Phi - M$ is abbreviated to $\Phi - v_1$ or $\Phi - v_1 - v_2$.

Let C^φ be a weighted cycle and $\{v_1, v_2, \ldots, v_n\}$ be its vertex set such that $v_j v_{j+1} \in E(C^\varphi)$ $(1 \leq j \leq n-1)$, $v_1 v_n \in E(C^\varphi)$. Let $\omega_j = \varphi(v_j v_{j+1})$ and $\omega_n = \varphi(v_n v_1)$.

Definition 1. ([13], Definition 2) Let C^φ be a complex unit gain cycle. Then C^φ is said to be of

$$\begin{cases} \text{A-Type}, & \text{if for even } n, \quad (-1)^{\frac{n}{2}}\overline{\omega}_n = \omega_1\omega_2\omega_3\ldots\omega_{n-2}\omega_{n-1}; \\ \text{B-Type}, & \text{if for even } n, \quad (-1)^{\frac{n}{2}}\overline{\omega}_n \neq \omega_1\omega_2\omega_3\ldots\omega_{n-2}\omega_{n-1}; \\ \text{C-Type}, & \text{if for odd } n, \quad Re((-1)^{\frac{n-1}{2}}\omega_1\omega_2\omega_3\ldots\omega_{n-2}\omega_{n-1}\omega_n) > 0; \\ \text{D-Type}, & \text{if for odd } n, \quad Re((-1)^{\frac{n-1}{2}}\omega_1\omega_2\omega_3\ldots\omega_{n-2}\omega_{n-1}\omega_n) < 0; \\ \text{E-Type}, & \text{if for } n\text{=odd}, \quad Re((-1)^{\frac{n-1}{2}}\omega_1\omega_2\omega_3\ldots\omega_{n-2}\omega_{n-1}\omega_n) = 0. \end{cases}$$

For $Re(\cdot)$ we mean the real part of any complex number.

Let $\varphi(C^\varphi) = \omega_1\omega_2\cdots\omega_{n-1}\omega_n = \varphi(v_1v_2)\varphi(v_2v_3)\cdots\varphi(v_{n-1}v_n)\varphi(v_nv_1)$ be the gain of the cycle C^φ. Then the five Types of cycles of Definition 1 are equivalent to the following ones ([14]):

A-Type, if for even n, $\varphi(C^\varphi) = (-1)^{\frac{n}{2}}$;
B-Type, if for even n, $\varphi(C^\varphi) \neq (-1)^{\frac{n}{2}}$;
C-Type, if for odd n, $Re((-1)^{\frac{n-1}{2}}\varphi(C^\varphi)) > 0$;
D-Type, if for odd n, $Re((-1)^{\frac{n-1}{2}}\varphi(C^\varphi)) < 0$;
E-Type, if for odd n, $Re((-1)^{\frac{n-1}{2}}\varphi(C^\varphi)) = 0$.

Signed graphs are particularly complex unit gain graphs. For $i = 0, 1, 2, 3$, let us denote by $C^i = (C_n, \sigma)$ a signed cycle whose length n is equal to i mod 4. Further, we write C^i_+ for a *balanced* cycle C^i, that is $\sigma(C^i_+) = +1$, and C^i_- for an *unbalanced* cycle C^i, that is $\sigma(C^i_-) = -1$. Then, according to Definition 1,

C^0_+ and C^2_- cycles are gain cycles of A-Type;
C^2_+ and C^0_- cycles are gain cycles of B-Type;
C^1_+ and C^3_- cycles are gain cycles of C-Type;
C^3_+ and C^1_- cycles are gain cycles of D-Type.

Signed cycles of E-Type do not exist.

Lemma 2. *[29] Let C^φ be a $\mathbb{T}-$gain cycle of order n. Then*

$$\eta(C^\varphi) = \begin{cases} 2, & \text{if } C^\varphi \text{ is a cycle of A-Type,} \\ 1, & \text{if } C^\varphi \text{ is a cycle of E-Type,} \\ 0, & \text{otherwise.} \end{cases}$$

The Cauchy-interlacing theorem ([11], Theorem 4.3.8) for Hermitian matrices implies the following result:

Lemma 3. *Let Φ be a complex unit gain graph. Then for any vertex x of Φ, $\eta(\Phi) - 1 \leq \eta(\Phi - x) \leq \eta(\Phi) + 1$.*

Let Φ be a $\mathbb{T}-$gain graph with a cut-vertex x and Φ_1 be a component of $\Phi - x$. Then Φ_1 can be obtained from $\Phi_1 + x$, the subgraph induced by $V(\Phi_1) \cup \{x\}$, by deleting the vertex x. Hence, $\eta(\Phi) - 1 \leq \eta(\Phi - x) \leq \eta(\Phi) + 1$ by Lemma 3. The relationship between $\eta(\Phi)$ and $\eta(\Phi_1 + x)$ summarizes the formulas on the nullity of complex unit gain graphs with cut-vertices as follows:

Lemma 4. *Let $\Phi = (\Gamma, \phi)$ be a complex unit gain graph of order n and u be a cut-point of Φ. Let $\Phi_1, \Phi_2, \ldots, \Phi_q$ be all components of $\Phi - u$. If a component, let say Φ_1, exists among $\Phi_1, \Phi_2, \ldots, \Phi_q$ such that $\eta(\Phi_1) = \eta(\Phi_1 + u) + 1$, then*

$$\eta(\Phi) = \eta(\Phi - u) - 1 = \sum_{j=1}^{q} \eta(\Phi_j) - 1.$$

Proof. Let $A = A(\Phi)$ be the adjacency matrix of Φ. For each j, denoted by $A[\Phi_j]$ the adjacency matrix of the subgraph Φ_j and by $A[u, \Phi_j]$ the sub-vector of $A[u, \Phi]$ corresponding to the vertices of Φ_j. Then the partition of matrix A is given below.

$$A = \begin{bmatrix} 0 & A[u,\Phi_1] & A[u,\Phi_2] & \dots & A[u,\Phi_q] \\ A^*[\Phi_1,u] & A[\Phi_1] & 0 & \dots & 0 \\ A^*[\Phi_2,u] & 0 & A[\Phi_2] & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ A^*[\Phi_q,u] & 0 & 0 & \dots & A[\Phi_q] \end{bmatrix},$$

where, $A[\Phi_j, u] = A^*[u, \Phi_j]^T$, the conjugate transpose of $A[u, \Phi_j]$, for each j. Let us define a row vector $x \in \mathbb{C}^n$ on the vertices of Φ such that $x[\Phi_1] = A[u, \Phi_1]$, and 0 otherwise. Consider a matrix B', obtained from matrix A in term of replacing the vectors $A[u, \Phi]$ and $A^*[\Phi, u]$ by x and $\overline{x^T}$, respectively, i.e.,

$$B' = \begin{bmatrix} 0 & A[u,\Phi_1] & 0 & \dots & 0 \\ A^*[\Phi_1,u] & A[\Phi_1] & 0 & \dots & 0 \\ 0 & 0 & A[\Phi_2] & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A[\Phi_q] \end{bmatrix}.$$

Then the matrix B' is the adjacency matrix of the subgraph Φ' which is obtained from Φ by removing all the edges between u and $\cup_{j=2}^{q} \Phi_j$.

Note that since $\eta(\Phi_1) = \eta(\Phi_1 + u) + 1$, then $r(A[\Phi_1 + u]) = r(A[\Phi_1]) + 2$, and hence the row vector $A[u, \Phi_1]$ is linearly independent of all the row vectors in $A[\Phi_1]$. Consequently, the row vector x, and the vector $A[u, \Phi]$ are linearly independent with respect to all other row vectors in matrix A. Also, the column vectors $A^*[\Phi, u]$ and $\overline{x^T}$ are linearly independent with respect to all other vectors in matrix A and B', respectively. Then $r(A) = r(A[\Phi - u]) + 2$ and $r(B') = r(B'[\Phi' - u]) + 2$, which implies that

$$r(A) = r(B').$$

Consequently,

$$\begin{aligned} \eta(\Phi) = n - r(A) = n - r(B') = (n-1) - r(\Phi - u) - 1 = \eta(\Phi - u) - 1 \\ = \sum_{j=1}^{q} \eta(\Phi_j) - 1. \end{aligned}$$

This completes the proof. □

Lemma 5. *Let $\Phi = (\Gamma, \phi)$ be a complex unit gain graph of order n and u be a cut-point of Φ. Let Φ_1 be a component of $\Phi - u$. If $\eta(\Phi_1) = \eta(\Phi_1 + u) - 1$. Then*

$$\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi - \Phi_1).$$

Proof. Let $A = A(\Phi)$ be the adjacency matrix of Φ. We define a row vector $x \in \mathbb{C}^n$ on the vertices of Φ such that $x[\Phi_1] = 0$ and $x[\Phi - \Phi_1] = A[u, \Phi - \Phi_1]$, where $A[u, \Phi - \Phi_1]$ represents the subvector of $A[u, \Phi]$ corresponding to the vertices in $\Phi - \Phi_1$. Consider a matrix B', obtained from matrix A in term of replacing the vectors $A[u, \Phi]$ and $A^*[\Phi, u]^T$ by x and $\overline{x^T}$, respectively, i.e.,

$$A = \begin{bmatrix} A[\Phi_1] & A[\Phi_1, u] & 0 \\ A^*[u, \Phi_1] & 0 & A[u, \Phi - \Phi_1 - u] \\ 0 & A^*[\Phi - \Phi_1 - u, u] & A[\Phi - \Phi_1 - u] \end{bmatrix},$$

and

$$B' = \begin{bmatrix} A[\Phi_1] & 0 & 0 \\ 0 & 0 & A[u, \Phi - \Phi_1 - u] \\ 0 & A^*[\Phi - \Phi_1 - u, u] & A[\Phi - \Phi_1 - u] \end{bmatrix}.$$

Then matrix B' is the adjacency matrix of the $\Phi_1 \cup (\Phi - \Phi_1)$. Since $\eta(\Phi_1) = \eta(\Phi_1 + u) - 1$, $r(A[\Phi_1]) = r(A[\Phi_1 + u])$ and hence the row vector $A[u, \Phi_1 + u]$ can be written in a linear combination of the row vectors of $A[\Phi_1, \Phi_1 + u]$. In a similar way, the column vector $A[\Phi_1, U]$ can be written in a linear combination of the column vectors of $A[\Phi_1]$, since $A[u] = 0$. Thus, we have

$$r(A) = r(B').$$

Consequently,

$$\eta(\Phi) = n - r(A) = n - r(B') = \eta(\Phi_1) + \eta(\Phi - \Phi_1).$$

This completes the proof. □

The nullity of a $\mathbb{T}$-gain tree is independent of the gain (see [29], Theorem 4.), so the following result holds.

Lemma 6. *Let P_n^{φ} be a gain path. Then $\eta(P_n^{\varphi}) = 1$ if n is odd, and $\eta(P_n^{\varphi}) = 0$ if n is even.*

We say that a connected induced subgraph H of a graph G is a *pendant* subgraph of G with root x if there exists exacly one vertex $x \in V(H)$ such that $N_G(x)$ contains at least one vertex $y \in V(G) - V(H)$. If H is a pendant subgraph of a complex unit gain graph Φ with root x, then x is a cut vertex of Φ. Thus, Lemma 4 and Lemma 5 can be stated respectively as part (i) and part (ii) of the following result:

Lemma 7. *Let Φ_1 be a pendant subgraph of Φ with root x.*
(i) If $\eta(\Phi_1 - x) = \eta(\Phi_1) + 1$, then $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi - \Phi_1)$.
(ii) If $\eta(\Phi_1 - x) = \eta(\Phi_1) - 1$, then $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi - \Phi_1 + x) - 1$.

When we apply this result to the particular case of a cycle C^φ as pendant subgraph Φ_1 of Φ, we get what follows.

Lemma 8. *Let C^φ be a pendant cycle of Φ with root x.*
(i) If C^φ is a cycle of A-Type, then $\eta(\Phi) = \eta(\Phi - C^\varphi + x) + 1$.
(ii) If C^φ is a cycle of B-Type, then $\eta(\Phi) = \eta(\Phi - C^\varphi)$.
(iii) If C^φ is a cycle of E-Type, then $\eta(\Phi) = \eta(\Phi - C^\varphi + x)$.

Proof. (i) $\Phi_1 = C^\varphi$, an A-Type cycle, hence, by Lemma 6 and Lemma 2, $1 = \eta(\Phi_1 - x) = 2 - 1 = \eta(\Phi_1) - 1$. By Lemma 7 (ii), $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi - \Phi_1 + x) - 1 = 2 + \eta(\Phi - C^\varphi + x) - 1 = \eta(\Phi - C^\varphi + x) + 1$.

(ii) $\Phi_1 = C^\varphi$, a B-Type cycle, hence, by Lemma 6 and Lemma 2, $1 = \eta(\Phi_1 - x) = 0 + 1 = \eta(\Phi_1) + 1$. According to Lemma 7 (i), $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi - \Phi_1) = 0 + \eta(\Phi - C^\varphi) = \eta(\Phi - C^\varphi)$.

(iii) $\Phi_1 = C^\varphi$, an E-Type cycle, hence, by Lemma 6 and Lemma 2, $0 = \eta(\Phi_1 - x) = 1 - 1 = \eta(\Phi_1) - 1$. According to Lemma 7 (ii), $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi - \Phi_1 + x) - 1 = 1 + \eta(\Phi - C^\varphi + x) - 1 = \eta(\Phi - C^\varphi + x)$. □

We deduce that a pendant E-type cycle C^φ does not change the nullity of a graph. More generally, if (C_1^φ, v_1), (C_2^φ, v_2), $\ldots$ (C_s^φ, v_s) are s pendant E-type cycles of a $\mathbb{T}$-gain graph Φ, with corresponding cut vertices v_i, $i = 1, \ldots, s$, then

$$\eta(\Phi) = \eta(\Phi - C_1^\varphi - C_2^\varphi - \cdots - C_s^\varphi + v_1 + v_2 + \cdots + v_s).$$

Following [29], if y is a pendant vertex of a complex unit gain graph Φ and z is its unique neighbour, the process of obtaining $\Phi - y - z$ from Φ is known as *pendant K_2 deletion*. The following lemma says that pendant K_2 deletion does not change the nullity of a complex unit gain graph.

Lemma 9. *([29] Theorem 1) Let Φ be a complex unit gain graph with a pendant vertex y such that $yz \in E(\Phi)$. If $H^\varphi = \Phi - y - z$, then $\eta(\Phi) = \eta(H^\varphi)$.*

3. Cycle-spliced $\mathbb{T}$-gain graphs Φ with $\eta(\Phi) = c(\Phi) + 1$

We show that the nullity of a cycle-spliced $\mathbb{T}$-gain graph Φ does not exceed $c(\Phi)+1$, then we give a characterization of extremal graphs Φ such that $\eta(\Phi) = c(\Phi) + 1$.
In order to prove Theorem 1, we need the following lemma on cycle-spliced complex unit gain graphs with a pendant cycle C^φ and a cut vertex x of Φ.

Lemma 10. *Let Φ be a cycle-spliced complex unit gain graph and C^φ be a pendant cycle of Φ with a cut vertex x. Let $H = \Phi - C^\varphi + x$.*
(i) If C^φ is a cycle of A-Type, then $\eta(\Phi) = \eta(H^\varphi) + 1$.
(ii) If C^φ is a cycle of the remaining types, then $\eta(\Phi) \leq \eta(H^\varphi) + 1$.

Proof. (i) Let C^φ be a cycle of A-Type and x be the cut vertex of Φ. By Lemma 5, we have

$$\eta(\Phi) = \eta(H^\varphi) + \eta(C^\varphi - x) = \eta(H^\varphi) + 1.$$

(ii) Let y be a vertex in C^φ adjacent with x. By Lemma 3, we have $\eta(\Phi) \leq \eta(\Phi - y) + 1$. If C^φ is a B-Type cycle, applying pendant K_2 deletions on $\Phi - y$, we have $\eta(\Phi - y) = \eta(H^\varphi)$. Hence, $\eta(\Phi) \leq \eta(H^\varphi) + 1$. Now we assume that C^φ is an odd cycle, that is $\eta(C^\varphi)$ is 0 or 1. It follows from Lemma 3 that $\eta(H^\varphi) - 1 \leq \eta(H^\varphi - x) \leq \eta(H^\varphi) + 1$. We now consider the following three cases.

Case 1. $\eta(H^\varphi - x) = \eta(H^\varphi) + 1$.
Clearly, H^φ is a pendant subgraph of Φ with root x. By Lemma 6 and Lemma 7, we have $\eta(\Phi) = \eta(H^\varphi) + \eta(\Phi - H^\varphi) = \eta(H^\varphi) + \eta(C^\varphi - x) = \eta(H^\varphi) < \eta(H^\varphi) + 1$.

Case 2. $\eta(H^\varphi - x) = \eta(H^\varphi)$.
Lemmas 3 and 6 imply that $\eta(\Phi) \leq \eta(\Phi - x) + 1 = \eta(C^\varphi - x) + \eta(H^\varphi - x) + 1 = \eta(H^\varphi) + 1$.

Case 3. $\eta(H^\varphi - x) = \eta(H^\varphi) - 1$.
Note that H^φ is a pendant subgraph of Φ with root x. Then Lemmas 2, 7 and 8 imply that $\eta(\Phi) = \eta(H^\varphi) + \eta(C^\varphi) - 1 \leq \eta(H^\varphi) < \eta(H^\varphi) + 1$, as desired. □

Theorem 1. *Let $\Phi = (\Gamma, \varphi)$ be a cycle-spliced complex unit gain graph with $c(\Phi)$ cycles. Then*
(i) $0 \leq \eta(\Phi) \leq c(\Phi) + 1$.
(ii) $\eta(\Phi) = c(\Phi) + 1$, if and only if every cycle C^φ in Φ is of A-Type.

Proof. (i) We proceed by induction on the number of cycles $c(\Phi)$ to prove that $\eta(\Phi) \leq c(\Phi) + 1$. If $c(\Phi) = 1$ then $\Phi = C^\varphi$ consists of a gain cycle C^φ. If C^φ is an $A-$type cycle, then

$$\eta(\Phi) = \eta(C^\varphi) = 2 \leq 1 + 1 = c(\Phi) + 1.$$

If C^φ is an E-type cycle, then

$$\eta(\Phi) = \eta(C^\varphi) = 1 < 1 + 1 = c(\Phi) + 1.$$

If C^φ is a cycle of Type B, C or D, then

$$\eta(\Phi) = \eta(C^\varphi) = 0 < 1 + 1 = c(\Phi) + 1.$$

Now suppose that $\eta(\Phi') \leq c(\Phi') + 1$ for any cycle-spliced gain graph Φ' with a number of cycles $c(\Phi') \leq t$ (induction hypothesis). Let us consider a cycle-spliced bipartite gain graph Φ with a number of cycles $c(\Phi) = t + 1$. Let C_1^φ be a pendant cycle of Φ with cut vertex y and let $H_1^\varphi = \Phi - C_1^\varphi + y$. By Lemma 10,

$$\eta(\Phi) \leq \eta(H_1^\varphi) + 1.$$

H_1^φ has got t cycles, so by the inductive hypothesis,

$$\eta(H_1^\varphi) \leq c(H_1^\varphi) + 1,$$

hence

$$\eta(\Phi) \leq \eta(H_1^\varphi) + 1 \leq c(H_1^\varphi) + 2 = c(\Phi) + 1.$$

Part (i) is then proved.

Now we prove part (ii): $\eta(\Phi) = c(\Phi) + 1$ if and only if all cycles appearing in Φ are of A-Type. We use induction again. For $\Phi = C^\varphi$ an A-Type cycle, we just showed that $\eta(\Phi) = \eta(C^\varphi) = 2 = c(\Phi) + 1 = c(C^\varphi) + 1$. Now suppose that $\eta(\Phi') = c(\Phi') + 1$ for any cycle-spliced complex unit gain graph Φ' with a number of cycles $c(\Phi') \leq t$, where all cycles appearing in the graph are of A-Type (induction hypothesis). Let us consider a cycle-spliced bipartite complex unit gain graph Φ with a number of cycles $c(\Phi) = t + 1$, each of them of A-Type. Let C_1^φ be a pendant cycle of Φ with cut vertex y and let $H_1^\varphi = \Phi - C_1^\varphi + y$. The cycle C_1^φ is of A-Type. H_1^φ has got t cycles of A-Type. By Lemma 10 and the inductive hypothesis,

$$\eta(\Phi) = \eta(H_1^\varphi) + 1 = c(H_1^\varphi) + 2 = c(\Phi) + 1.$$

The proof for sufficiency of (ii) is complete.

Now, we proceed by induction on $c(\Phi)$ to prove $\eta(\Phi) < c(\Phi) + 1$ if Φ has at least one cycle of Type B, C, D or E. If Φ has only one cycle, it follows that $\eta(\Phi) \leq 1 < c(\Phi)+1$. Now we assume that $c(\Phi) \geq 2$. Then Φ has at least two pendant cycles. Let C^φ be one of such cycles. If C^φ is of $A-$type, then by similar discussion as above, we have $\eta(\Phi) = \eta(H^\varphi)+1$, where $H^\varphi = \Phi - C^\varphi + x$ and x is the cut-vertex of Φ on C^φ. Noting that H^φ has at least one cycle of B-Type and it has one cycle less than those of Φ, we have $\eta(H^\varphi) < c(H^\varphi) + 1$. Hence, $\eta(\Phi) = \eta(H^\varphi) + 1 < c(\Phi) + 1$, as required. Now suppose all pendant cycles of Φ are not of A-Type. Then $H^\varphi = \Phi - C^\varphi + x$ has at least one cycle not of A-Type. The induction hypothesis implies that $\eta(H^\varphi) < c(H^\varphi) + 1$. By Lemma 10 (ii) and the previous inequality, $\eta(\Phi) \leq \eta(H^\varphi) + 1 < (c(H^\varphi) + 1) + 1 = c(\Phi) + 1$ which proves the necessity of (ii).

This completes the proof of the theorem. □

4. Cycle-splice $\mathbb{T}$-gain graphs with $\eta(\Phi) = c(\Phi)$ or $\eta(\Phi) = c(\Phi)-1$

In this section, we consider the properties of cycle-spliced complex unit gain graphs Φ with $\eta(\Phi) = c(\Phi)$ or $\eta(\Phi) = c(\Phi) - 1$. We prove that $\eta(\Phi)$ never equals $c(\Phi)$ if Φ does not contain any E-Type cycle. Examples of cycle-spliced $\mathbb{T}$-gain graphs with $\eta(\Phi) = c(\Phi)$ are given. We also consider cycle-spliced $\mathbb{T}$-gain graphs obtained by identifying the unique common vertex v of two cycle-spliced $\mathbb{T}$-gain graphs Φ_1 and Φ_2. For this kind of graphs, a characterization in case $\eta(\Phi) = c(\Phi)$ is given and some properties and auxiliary results are presented.

In order to prove Theorem 2, first we need the following lemma on the property of cycle-spliced complex unit gain graphs with nullity $\eta(\Phi) = c(\Phi) + 1$, hence bipartite graphs.

Lemma 11. *Let Φ be a cycle-spliced complex unit gain graph with $c(\Phi)$ cycles. If $\eta(\Phi) = c(\Phi) + 1$, then $\eta(\Phi - x) = \eta(\Phi) - 1$ for any $x \in V(\Phi)$.*

Proof. We proceed with the induction on $c(\Phi)$ to prove $\eta(\Phi - x) = \eta(\Phi) - 1$ for any $x \in V(\Phi)$. If $c(\Phi) = 1$, then Φ is an A-Type cycle, since $\eta(\Phi) = c(\Phi) + 1 = 2$. It follows from Lemma 2 that $1 = \eta(\Phi - x) = \eta(\Phi) - 1$, as required. Assume that the result holds for a cycle-spliced complex unit gain graphs with $c(H^\varphi) = p$ cycles and assume that Φ has $p+1$ such cycles. Theorem 1 (ii) implies that all cycles in Φ are of A- Type. Let C^φ be a pendant cycle of Φ with a cut vertex x and $H^\varphi = \Phi - C^\varphi + x$. Then Lemma 10 (i) implies that $\eta(\Phi) = \eta(H^\varphi) + 1$ since C^φ is a cycle of A-Type. Since H^φ is a cycle-spliced complex unit gain graph with one cycles less than Φ, by Theorem 1 (ii), $\eta(H^\varphi) = c(H^\varphi) + 1$. Note that $c(H^\varphi) = p$. Then by induction hypothesis, we have $\eta(H^\varphi - x) = \eta(H^\varphi - 1)$ for any $x \in V(H)$. Let y be any arbitrary vertex in Φ. We consider the following two cases for y according to its position in Φ.
Case i. y does not lie on C^φ.
In this case, C^φ is also a pendant cycle of $\Phi - y$ with root x. Let $H^\varphi - y = (\Phi - y) - C^\varphi + x$. Then Lemma 8 (i) implies that $\eta(\Phi - y) = \eta(H^\varphi - y) + 1$. Hence, $\eta(\Phi - y) = (\eta(H^\varphi) - 1) + 1 = \eta(\Phi) - 1$, as desired.
Case ii. y lies on C^φ.
If $d_\Phi(y, x)$ is even or possibly zero, then by applying pendant K_2 deletions on $\Phi - y$, we get $\eta(\Phi - y) = \eta(H^\varphi - x) + 1 = (\eta(H^\varphi - 1) + 1 = \eta(\Phi) - 1$. If $d_\Phi(y, x)$ is odd, then pendant K_2 deletions on $\Phi - y$ produce $\eta(\Phi - y) = \eta(H^\varphi) = \eta(\Phi) - 1$, and the proof is completed. □

Theorem 2. *For any cycle-spliced complex unit gain graph $\Phi = (G, \varphi)$ of order n with $c(\Phi)$ cycles, none of E-Type, $\eta(\Phi) \neq c(\Phi)$.*

Proof. We proceed with the induction on $c(\Phi)$ to prove $\eta(\Phi) \neq c(\Phi)$ in case no cycle in Φ is of E-Type. If $c(\Phi) = 1$, then Φ is a cycle. Thus $\eta(\Phi) \neq c(\Phi)$, by Lemma 2. Assume the result true for cycle-spliced complex unit gain graphs having p cycles

and consider Φ with $p+1$ cycles. Let C^{φ} be a pendant cycle of Φ with a cut vertex x. Let $H^{\varphi} = \Phi - C^{\varphi} + x$. Then $c(H^{\varphi}) = p$. The induction hypothesis says that $\eta(H^{\varphi}) \neq c(H^{\varphi})$. By Theorem 1, we have $\eta(H^{\varphi}) = c(H^{\varphi}) + 1$ or $\eta(H^{\varphi} \leq c(H^{\varphi}) - 1$. For this, we consider the following two cases.
Case i. $\eta(H^{\varphi}) = c(H^{\varphi}) + 1$.
By Lemma 11, $\eta(H^{\varphi} - x) = \eta(H^{\varphi}) - 1$ since $\eta(H^{\varphi}) = c(H^{\varphi}) + 1$. It follows from Lemma 7 (ii), $\eta(\Phi) = \eta(H^{\varphi}) + \eta(C^{\varphi}) - 1$. If C^{φ} is of A-Type, we have $\eta(\Phi) = \eta(H^{\varphi}) + 1 = (c(H^{\varphi}) + 1) + 1 = c(\Phi) + 1$. If C^{φ} has nullity 0, that is C^{φ} is of Type B,C or D, we have $\eta(\Phi) = \eta(H^{\varphi}) - 1 = (c(H^{\varphi}) + 1) - 1 = c(\Phi) - 1$.
Case ii. $\eta(H^{\varphi}) \leq c(H^{\varphi}) - 1$.
By Lemma 10, we have $\eta(\Phi) \leq \eta(H^{\varphi}) + 1$, so we have $\eta(\Phi) \leq \eta(H^{\varphi}) + 1 \leq (c(H^{\varphi}) - 1) + 1 = c(\Phi) - 1$.
In any case, $\eta(\Phi) \neq c(\Phi)$.

□

As we pointed out, Theorem 2 does not hold if the complex unit gain graph Φ contains E-Type cycles. The smallest counterexample is $\Phi = C^{\varphi}_{E}$, a graph made by one cycle of E-Type. In this case, $\eta(\Phi) = \eta(C^{\varphi}_{E}) = 1 = c(\Phi)$, by Lemma 2. The following Proposition gives infinite examples of complex unit gain graphs Φ such that $\eta(\Phi) = c(\Phi)$.

Proposition 1. *Let Φ be a cycle-spliced complex unit gain graph.*
(i) If $c(\Phi) - 1$ cycles are of A-Type and one pendant cycle C^{φ}_{E} is of E-Type, with root x, then $\eta(\Phi) = c(\Phi)$.
(ii) If all cycles of Φ are of E-Type, then $\eta(\Phi) = 1$.

Proof. (i) All cycles in the subgraph $\Phi' = \Phi - C^{\varphi}_{E} + x$ are of A-Type, so, by Theorem 1 (ii), $\eta(\Phi') = c(\Phi') + 1$. According to Lemma 8 (iii), $\eta(\Phi) = \eta(\Phi') = c(\Phi') + 1 = c(\Phi)$.
(ii) It follows from Lemma 8 (iii). □

More generally, starting from a complex unit gain graph Φ' such that $\eta(\Phi') = c(\Phi') - p$, with p any integer $p \geq -1$, we can attach $s \geq 1$ E-Type cycles at any vertex x of Φ', to build a new graph Φ such that $\eta(\Phi) = \eta(\phi') = c(\Phi') - p = c(\Phi) - s - p$.

Lemma 12. *Let Φ be a complex unit gain graph obtained from two cycle-spliced complex unit gain graphs Φ_1 and Φ_2 by identifying the unique common vertex x. If $\eta(\Phi_1) \leq c(\Phi_1) - p$, then $\eta(\Phi) \leq c(\Phi) - p + 1$. If Φ_2 does not contain E-Type cycles, then $\eta(\Phi) \leq c(\Phi) - p$.*

Proof. Let us introduce the variable ε such that $\varepsilon = 0$ if Φ contains E-Type cycles, and $\varepsilon = 1$ if there is no E-Type cycle in Φ. By Theorem 1, $\eta(\Phi_2) \leq c(\Phi_2) + 1$. If $\eta(\Phi_2) = c(\Phi_2) + 1$, then by Lemma 11, $\eta(\Phi_2 - x) = \eta(\Phi_2) - 1$, so Lemma 7 (ii) implies $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi_2) - 1 \leq (c(\Phi_1) - p) + (c(\Phi_2) + 1) - 1 = c(\Phi) - p$, as required. If $\eta(\Phi_2) \neq c(\Phi_2) + 1$, we have $\eta(\Phi_2) \leq c(\Phi_2) - \varepsilon$ by Theorem 2.
Lemma 3 implies that $\eta(\Phi_2) - 1 \leq \eta(\Phi_2 - x) \leq \eta(\Phi_2) + 1$. Now we consider the following three cases.

Case i. $\eta(\Phi_2 - x) = \eta(\Phi_2) + 1$.
Clearly, Φ_2 is a pendant subgraph of Φ with root x. By Lemma 7 (i), we have

$$\begin{aligned}\eta(\Phi) &= \eta(\Phi_2) + \eta(\Phi - \Phi_2)\\ &= \eta(\Phi_2) + \eta(\Phi_1 - x)\\ &\leq \eta(\Phi_2) - \varepsilon + (\eta(\Phi_1) + 1)\\ &\leq c(\Phi_2) - \varepsilon + (c(\Phi_1) - p) + 1 = c(\Phi) - p + 1 - \varepsilon.\end{aligned}$$

If Φ_2 does not contain E-Type cycles, then $\eta(\Phi_2) \leq c(\Phi_2) - 1$ and $\eta(\Phi) \leq c(\Phi) - p$.
Case ii. $\eta(\Phi_2 - x) = \eta(\Phi_2)$.
Clearly, Φ_1 is a pendant subgraph of Φ with root x. If $\eta(\Phi_1 - x) = \eta(\Phi_1) + 1$, then by Lemma 7 (i), we have

$$\begin{aligned}\eta(\Phi) &= \eta(\Phi_1) + \eta(\Phi - \Phi_1)\\ &= \eta(\Phi_1) + \eta(\Phi_2 - x)\\ &= \eta(\Phi_1) + (\eta(\Phi_2))\\ &\leq (c(\Phi_1) - p) + c(\Phi_2) - \varepsilon = c(\Phi) - p - \varepsilon.\end{aligned}$$

If $\eta(\Phi_1 - x) \leq \eta(\Phi_1)$, then Lemma 3 implies that

$$\begin{aligned}\eta(\Phi) &\leq \eta(\Phi - x) + 1\\ &= \eta(\Phi_1 - x) + \eta(\Phi_2 - x) + 1\\ &\leq \eta(\Phi_1) + \eta(\Phi_2) + 1\\ &\leq (c(\Phi_1) - p) + (c(\Phi_2)) - \varepsilon + 1 = c(\Phi) - p + 1 - \varepsilon.\end{aligned}$$

If Φ_2 does not contain E-Type cycles, then $\eta(\Phi_2) \leq c(\Phi_2) - 1$ and $\eta(\Phi) \leq c(\Phi) - p$.
Case iii. $\eta(\Phi_2 - x) = \eta(\Phi_2) - 1$.
Since Φ_2 is a pendant subgraph of Φ with root x, by Lemma 7 (ii), we have

$$\begin{aligned}\eta(\Phi) &= \eta(\Phi_1) + \eta(\Phi_2) - 1\\ &\leq (c(\Phi_1) - p) + c(\Phi_2) - \varepsilon - 1 = c(\Phi) - p - 1 - \varepsilon < c(\Phi) - p.\end{aligned}$$

□

Lemma 13. *Let Φ be a complex unit gain graph obtained from two cycle-spliced complex unit gain graphs Φ_1 and Φ_2 by identifying the unique common vertex x. Suppose that $\eta(\Phi) = c(\Phi) - p$, $p \geq 0$.*
(i) If $\eta(\Phi_1) = c(\Phi_1) + 1$ then $\eta(\Phi_2) = c(\Phi_2) - p$;
(ii) If $\eta(\Phi_j) \leq c(\Phi_j) - p$ for $j = 1, 2$, then $\eta(\Phi_j) = c(\Phi_j) - p$ for $j = 1, 2$ if neither Φ_1 nor Φ_2 contain E-Type cycles. If only Φ_1 contains E-Type cycles (not Φ_2), then $\eta(\Phi_2) = c(\Phi_2) - p - \varepsilon_2$ and $\eta(\Phi_1) = c(\Phi_1) - p$.
If both Φ_1 and Φ_2 contain E-Type cycles, then $\eta(\Phi_i) = c(\Phi_i) - p - \varepsilon_i$, for $i = 1, 2$ and $\varepsilon_i = 0, 1$.

Proof. (i) Since $\eta(\Phi_1) = c(\Phi) + 1$, by Lemma 11 we have, $\eta(\Phi_1 - x) = \eta(\Phi_1) - 1$. Moreover, Lemma 7 (ii) implies that $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi_2) - 1$. It follows that $\eta(\Phi_2) = \eta(\Phi) - \eta(\Phi_1) + 1 = c(\Phi) - p - \eta(\Phi_1) + 1 = (c(\Phi_1) + c(\Phi_2) - p) - \eta(\Phi_1) + 1 = (c(\Phi_1) - \eta(\Phi_1) + 1) + c(\Phi_2) - p = 0 + c(\Phi_2) - p$.

(ii) If Φ_2 does not contain E-Type cycles and $\eta(\Phi_1) \leq c(\Phi_1) - p - 1$, then by Lemma 12 it would be $\eta(\Phi) \leq c(\Phi) - p - 1$ which is a contradiction. Hence $\eta(\Phi_1) = c(\Phi_1) - p$. If Φ_1 does not contain E-Type cycles too, then $\eta(\Phi_2) = c(\Phi_2) - p$ too. We deduce that $\eta(\Phi_j) = c(\Phi_j) - p$ for $j = 1, 2$ if Φ does not contain E-Type cycles. For $p = 0$, it is not possible that neither Φ_1 nor Φ_2 contain E-Type cycles by Theorem 2.
If Φ_2 contains at least one E-Type cycle and $\eta(\Phi_1) \leq c(\Phi_1) - p - 2$, then by Lemma 12 it would be $\eta(\Phi) \leq c(\Phi) - p - 1$, which is a contradiction. Hence $\eta(\Phi_1) = c(\Phi_1) - p - 1$ or $\eta(\Phi_1) = c(\Phi_1) - p$. Analogous considerations hold in case Φ_1 contains at least one E-Type cycle. □

The case $p = 0$ in the previous result implies that Φ has at least one E-Type cycle by Theorem 2. If $\eta(\Phi_1) = c(\Phi_1) + 1$, then $\eta(\Phi_2) = c(\Phi_2)$, hence all cycles in Φ_1 are of A-Type and there exists at least one E-Type cycle in Φ_2. In case $\eta(\Phi_i) \leq c(\Phi_i)$ for $i = 1, 2$, if only Φ_1 contains E-Type cycles, then $\eta(\Phi_2) = c(\Phi_) - 1$ and $\eta(\Phi_1) = c(\Phi_1)$; if Φ_1 and Φ_2 both contain E-Type cycles, then $\eta(\Phi_i) = c(\Phi_i) - \varepsilon_i$ for ε_i equal to 0 or 1 and $i =, 0, 1$.

The following result ([22], Theorem 2.4) has been proved for signed graphs, but its proof can be adapted to the $\mathbb{T}$-gain case. Here it is formulated in terms of nullity of $\mathbb{T}$-gain graphs, instead of rank, as in the original version for signed graphs.

Proposition 2. *Let x be a cut-vertex of a connected complex unit gain graph Φ and Φ' a disjoint union of some connected components of $\Phi - x$. The following statements hold true.*
(i) If $\eta(\Phi' + x) = \eta(\Phi') - 2$, then $\eta(\Phi') + \eta(\Phi - \Phi') - 2 \leq \eta(\Phi) \leq \eta(\Phi') + \eta(\Phi - \Phi')$.
(ii) If $\eta(\Phi' + x) = \eta(\Phi')$, then $\eta(\Phi') + \eta(\Phi - \Phi') - 1 \leq \eta(\Phi) \leq \eta(\Phi') + \eta(\Phi - \Phi') + 1$.

Proposition 3. *[3] Let Φ be a complex unit gain graph obtained from two complex unit gain graphs Φ_1 and Φ_2 by identifying the unique common vertex x. Then the characteristic polynomial $P_\Phi(\lambda) = |\lambda I - A(\Phi)|$ is given by $P_{\Phi_1}(\lambda)P_{\Phi_2 - x}(\lambda) + P_{\Phi_1 - x}(\lambda)P_{\Phi_2}(\lambda) - \lambda \cdot P_{\Phi_1 - x}(\lambda)P_{\Phi_2 - x}(\lambda)$.*

Corollary 1. *Let $\Phi = \Phi_1 \bigvee_x \Phi_2$. If $\eta(\Phi_i - x) = \eta(\Phi_i)$ for $i = 1, 2$, then $\eta(\Phi) \geq \eta(\Phi_1) + \eta(\Phi_2)$.*

Proof. Let $m_i = \eta(\Phi_i) = \eta(\Phi_i - x)$. Then $P_{\Phi_i}(\lambda) = \lambda^{m_i} \cdot q_i(\lambda)$ and $P_{\Phi_i - x}(\lambda) = \lambda^{m_i} \cdot q_i'(\lambda)$ where the polynomials $q_i(\lambda)$ and $q_i'(\lambda)$ are not divisible by λ. According to Proposition 3, $P_\Phi(\lambda) = \lambda^{m_1} q_1(\lambda) \lambda^{m_2} q_2'(\lambda) + \lambda^{m_1} q_1'(\lambda) \lambda^{m_2} q_2(\lambda) - \lambda \cdot \lambda^{m_1} \lambda^{m_2} q_1'(\lambda) q_2'(\lambda) = \lambda^{m_1 + m_2}(q_1(\lambda) q_2'(\lambda) + q_1'(\lambda) q_2(\lambda) - \lambda q_1'(\lambda) q_2'(\lambda))$. This proves that $\eta(\Phi) \geq \eta(\Phi_1) + \eta(\Phi_2)$. □

Theorem 3. *Let Φ be a complex unit gain graph obtained from two complex unit gain cycle-spliced graphs Φ_1 and Φ_2 by identifying the unique common vertex x. Then $\eta(\Phi) = c(\Phi)$ if and only if one of the following conditions is satisfied:*
(i) There is one of Φ_i ($i = 1, 2$), say Φ_1 , such that $\eta(\Phi_1) = c(\Phi_1) + 1$ and $\eta(\Phi_2) = c(\Phi_2)$;
(ii) $\eta(\Phi_i) = c(\Phi_i)$ for $i = 1, 2$, $\eta(\Phi_1 - x) = \eta(\Phi_1) + \varepsilon$, $\eta(\Phi_2 - x) = \eta(\Phi_2)$ (hence $\eta(\Phi - x) = c(\Phi) + \varepsilon$);
(iii) $\eta(\Phi_1) = c(\Phi_1) = \eta(\Phi_1 - x) - \varepsilon$, $\eta(\Phi_2) = c(\Phi_2) - 1 = \eta(\Phi_2 - x) - \varepsilon$ (hence $\eta(\Phi - x) = c(\Phi) + 2\varepsilon - 1$) and $\eta(\Phi - x) = \eta(\Phi) - 1$ when $\varepsilon = 0$.

Proof. *Only if* part. If $\eta(\Phi_1) = c(\Phi_1) + 1$, then Lemma 13 (i) for $p = 0$ implies that $\eta(\Phi_2) = c(\Phi_2)$, so (i) holds. If $\eta(\Phi_i) \le c(\Phi_i)$ for $i = 1, 2$, then by Lemma 13 (ii), we have $\eta(\Phi_i) = c(\Phi_i) - \varepsilon_i$ for $i = 1, 2$.
Case $\varepsilon_i = 0$ for $i = 1, 2$. Let $\eta(\Phi_i) = c(\Phi_i)$ for $i = 1, 2$. Lemma 3 implies that $\eta(\Phi_i) - 1 \le \eta(\Phi_i - x) \le \eta(\Phi_i) + 1$. Now we consider the following three cases.

Case 1. $\eta(\Phi_1 - x) = \eta(\Phi_1) + 1$.
Φ_1 is a pendant subgraph of Φ with root x. By Lemma 7 (i), we have $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi_2 - x)$. If $\eta(\Phi_2 - x) = \eta(\Phi_1) - 1$, then $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi_2 - x) = c(\Phi_1) + c(\Phi_2) - 1 = c(\Phi) - 1$, which is a contradiction. Then $\eta(\Phi_2 - x) = \eta(\Phi_2) + \varepsilon$ ($\varepsilon = 0$ or 1). But $\eta(\Phi - x) = \eta(\Phi_1 - x) + \eta(\Phi_2 - x) = c(\Phi_1) + 1 + c(\Phi_2) + \varepsilon = c(\Phi) + 1 + \varepsilon$, which holds if $\varepsilon = 0$. Then $\eta(\Phi_1 - x) = \eta(\Phi_1) + 1$ and $\eta(\Phi_2 - x) = \eta(\Phi_2)$. It follows that $\eta(\Phi - x) = c(\Phi) + 1$.

Case 2. $\eta(\Phi_1 - x) = \eta(\Phi_1)$.
Again Φ_2 is also a pendant subgraph of Φ with root x. If $\eta(\Phi_2 - x) = \eta(\Phi_2) - 1$, then by Lemma 7 (i), we have $\eta(\Phi) = \eta(\Phi_2) + \eta(\Phi_1) - 1 = c(\Phi_2) + c(\Phi_1) - 1 = c(\Phi) - 1$, which is a contradiction. Then we have $\eta(\Phi_2 - x) = \eta(\Phi_2) + \varepsilon$. Moreover, note that $\eta(\Phi - x) = \eta(\Phi_1 - x) + \eta(\Phi_2 - x) = \eta(\Phi_1) + \eta(\Phi_2) + \varepsilon = c(\Phi) + \varepsilon$. Hence $\eta(\Phi_1 - x) = \eta(\Phi_1)$, $\eta(\Phi_2 - x) = \eta(\Phi_2) + \varepsilon$ and $\eta(\Phi - x) = c(\Phi) + \varepsilon$.

Case 3. $\eta(\Phi_1 - x) = \eta(\Phi_1) - 1$.
Φ_1 is a pendant subgraph of Φ with root x. Then Lemma 7 (ii) implies that $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi_2) - 1 = c(\Phi_1) + c(\Phi_2) - 1 = c(\Phi) - 1$, which is a contradiction.
Then (ii) holds.

Case $\varepsilon_1 = 0$ and $\varepsilon_2 = 1$. Let $\eta(\Phi_1) = c(\Phi_1)$ and $\eta(\Phi_2) = c(\Phi_2) - 1$. Lemma 3 implies that $\eta(\Phi_i) - 1 \le \eta(\Phi_i - x) \le \eta(\Phi_i) + 1$. Now we consider the following three cases.

Case 1. $\eta(\Phi_1 - x) = \eta(\Phi_1) + 1$.
Φ_1 is a pendant subgraph of Φ with root x. By Lemma 7 (i), we have $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi_2 - x) = c(\Phi_1) + \eta(\Phi_2 - x)$. If $\eta(\Phi_2 - x) = \eta(\Phi_2) + 1$, then $\eta(\Phi - x) = \eta(\Phi_1 - x) + \eta(\Phi_2 - x) = (\eta(\Phi_1) + 1) + (\eta(\Phi_2) + 1) = c(\Phi_1) + 1 + c(\Phi_2) = c(\Phi) + 1$. If $\eta(\Phi_2 - x) = \eta(\Phi_2)$ or $\eta(\Phi_2 - x) = \eta(\Phi_2) - 1$, then $\eta(\Phi) \le c(\Phi_1) + c(\Phi_2) - 1$ which is a contradiction.

Case 2. $\eta(\Phi_1 - x) = \eta(\Phi_1)$.
Again Φ_2 is also a pendant subgraph of Φ with root x. If $\eta(\Phi_2 - x) = \eta(\Phi_2) - 1$, then by Lemma 7 (i), we have $\eta(\Phi) = \eta(\Phi_2) + \eta(\Phi_1) - 1 = c(\Phi_2) - 1 + c(\Phi_1) - 1 = c(\Phi) - 2$,

which is a contradiction. If $\eta(\Phi_2 - x) = \eta(\Phi_2) + 1$, then by Lemma 7 (i), we have $\eta(\Phi) = \eta(\Phi_2) + \eta(\Phi_1 - x) = c(\Phi_2) - 1 + c(\Phi_1) = c(\Phi) - 1$, which is a contradiction. Then we have $\eta(\Phi_2 - x) = \eta(\Phi_2)$. Moreover, note that $\eta(\Phi - x) = \eta(\Phi_1 - x) + \eta(\Phi_2 - x) = \eta(\Phi_1) + \eta(\Phi_2) = c(\Phi_1) + c(\Phi_2) - 1 = c(\Phi) - 1$.

Case 3. $\eta(\Phi_1 - x) = \eta(\Phi_1) - 1$.
Φ_1 is a pendant subgraph of Φ with root x. Then Lemma 7 (ii) implies that $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi_2) - 1 = c(\Phi_1) + c(\Phi_2) - 1 = c(\Phi) - 1$, which is a contradiction. Hence (iii) holds.

Case $\varepsilon_i = 1$ for $i = 1, 2$.
Let $\eta(\Phi_1) = c(\Phi_1) - 1$ and $\eta(\Phi_2) = c(\Phi_2) - 1$. Lemma 3 implies that $\eta(\Phi_i) - 1 \leq \eta(\Phi_i - x) \leq \eta(\Phi_i) + 1$. Now we consider the following three cases.

Case 1. $\eta(\Phi_1 - x) = \eta(\Phi_1) + 1$.
Φ_1 is a pendant subgraph of Φ with root x. By Lemma 7 (i), we have $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi_2 - x) \leq c(\Phi_1) - 1 + c(\Phi_2) - 1 + 1 = c(\Phi) - 1$ which is a contradiction.

Case 2. $\eta(\Phi_1 - x) = \eta(\Phi_1)$.
Again Φ_2 is also a pendant subgraph of Φ with root x. If $\eta(\Phi_2 - x) = \eta(\Phi_2) - 1$, then by Lemma 7 (ii), we have $\eta(\Phi) = \eta(\Phi_2) + \eta(\Phi_1) - 1 = c(\Phi_2) - 1 + c(\Phi_1) - 1 - 1 = c(\Phi) - 3$, which is a contradiction. If $\eta(\Phi_2 - x) = \eta(\Phi_2) + 1$, then by Lemma 7 (i), we have $\eta(\Phi) = \eta(\Phi_2) + \eta(\Phi_1 - x) = c(\Phi_2) - 1 + c(\Phi_1) - 1 = c(\Phi) - 2$, which is a contradiction. Then we have $\eta(\Phi_2 - x) = \eta(\Phi_2)$. Now we note that $\eta(\Phi - x) = \eta(\Phi_1 - x) + \eta(\Phi_2 - x) = \eta(\Phi_1) + \eta(\Phi_2) = (c(\Phi_1) - 1) + (c(\Phi_2) - 1) = c(\Phi) - 2$, which is a contradiction since $c(\Phi) - 1 = \eta(\Phi) - 1 \leq \eta(\Phi - x) \leq \eta(\Phi) + 1 = c(\Phi) + 1$.

Case 3. $\eta(\Phi_1 - x) = \eta(\Phi_1) - 1$.
Φ_1 is a pendant subgraph of Φ with root x. Then Lemma 7 (ii) implies that $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi_2) - 1 = c(\Phi_1) - 1 + c(\Phi_2) - 1 - 1 = c(\Phi) - 3$, which is a contradiction. Hence, only conditions (i), (ii) and (iii) occur if $\eta(\Phi) = c(\Phi)$ ($\eta(\Phi_1) = c(\Phi_1) - 1$ and $\eta(\Phi_2) = c(\Phi_2) - 1$ never occur).

If part. (i) By Lemma 11, $\eta(\Phi_1 - x) = \eta(\Phi_1) - 1$ since $\eta(\Phi_1) = c(\Phi_1) + 1$. Applying Lemma 7 (ii), we have $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi_2) - 1 = (c(\Phi_1) + 1) + c(\Phi_2) - 1 = c(\Phi)$.
(ii) Case $\varepsilon = 0$.
$\eta(\Phi_i) = c(\Phi_i) = \eta(\Phi_i - x)$, $i = 1, 2$, hence $\eta(\Phi - x) = c(\Phi)$. By Proposition 2 (ii), we get $c(\Phi) - 1 \leq \eta(\Phi) \leq c(\Phi) + 1$. $\eta(\Phi) = c(\Phi) + 1$ is not possible because Φ contains at least one E-Type cycle, being $\eta(\Phi_i) = c(\Phi_i)$. So, $\eta(\Phi) = c(\Phi)$ or $\eta(\Phi) = c(\Phi) - 1$. By Corollary 1, $\eta(\Phi) \geq \eta(\Phi_1) + \eta(\Phi_2) = c(\Phi_1) + c(\Phi_2) = c(\Phi)$, hence $\eta(\Phi) = c(\Phi)$.

Case $\varepsilon = 1$.
$\eta(\Phi_i) = c(\Phi_i)$, $i = 1, 2$, $\eta(\Phi_1 - x) = \eta(\Phi_1) + 1$, $\eta(\Phi_2 - x) = \eta(\Phi_2)$, hence $\eta(\Phi - x) = c(\Phi) + 1$. By Lemma 7, $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi_2 - x) = c(\Phi_1) + c(\Phi_2) = c(\Phi)$.
(iii) Case $\varepsilon = 0$:
$\eta(\Phi_1) = c(\Phi_1) = \eta(\Phi_1 - x)$, $\eta(\Phi_2 - x) = \eta(\Phi_2) = c(\Phi_2) - 1$, hence $\eta(\Phi - x) = c(\Phi) - 1$. By Proposition 2 (ii), we get $c(\Phi) - 2 \leq \eta(\Phi) \leq c(\Phi)$. According to Corollary 1, $\eta(\Phi) \geq \eta(\Phi_1) + \eta(\Phi_2) = c(\Phi_1) + c(\Phi_2) - 1 = c(\Phi) - 1$, hence $\eta(\Phi) = c(\Phi) - 1$ or

$\eta(\Phi) = c(\Phi)$ but, by hypothesis, $\eta(\Phi) = \eta(\Phi - x) + 1 = (c(\Phi) - 1) + 1 = c(\Phi)$.
Case $\varepsilon = 1$:
$\eta(\Phi_1) = c(\Phi_1) = \eta(\Phi_1 - x) - 1$, $\eta(\Phi_2 - x) = \eta(\Phi_2) + 1 = c(\Phi_2)$, hence $\eta(\Phi - x) = c(\Phi) + 1$. By Lemma 7 (i), $\eta(\Phi) = \eta(\Phi_1) + \eta(\Phi_2 - x) = c(\Phi_1) + c(\Phi_2) = c(\Phi)$. □

If $c(\Phi_i) = 1$ for $i = 1, 2$, then $\eta(\Phi) = \eta(\Phi_1 \bigvee_x \Phi_2) = 2 = c(\Phi)$ iff Φ_1 is an A-Type cycle and Φ_2 is an E-Type cycle. For, $\eta(\Phi_i) = c(\Phi_i) - \varepsilon_i = \eta(\Phi_i - x)$ holds only if $\varepsilon = 1$ and this occurs if Φ_i is a C- or D-Type cycle. In this case, $\eta(\Phi) \neq 2 = c(\Phi)$.

If $c(\Phi_1) = 1$ and $c(\Phi_2) = 2$ and one of the conditions (ii), (iii) and (iv) holds, then Φ_1 is a C- or D-Type cycle and Φ_2 is as described above (an A-Type and an E-Type cycle joined in a common vertex). One can verify that in this case $\eta(\Phi_2 - x) < \eta(\Phi_2)$.

Lemma 14. *Let Φ be a complex unit gain graph obtained from two complex unit gain cycle-spliced graphs Φ_1 and Φ_2 by identifying the unique common vertex x. Suppose that Φ does not contain E-Type cycles. Then $\eta(\Phi) = c(\Phi) - 1$ if and only if one of the following conditions is satisfied:*
(i) There is one of Φ_i ($i = 1, 2$), say Φ_1 , such that $\eta(\Phi_1) = c(\Phi_1)+1$ and $\eta(\Phi_2) = c(\Phi_2)-1$;
(ii) $\eta(\Phi_i) = c(\Phi_i) - 1$ and $\eta(\Phi_i - x) = \eta(\Phi_i) - 1$ for $i = 1, 2$;
(iii) $\eta(\Phi_i) = c(\Phi_i) - 1$ and $\eta(\Phi_i - x) = \eta(\Phi_i)$ for $i = 1, 2$; moreover $\eta(\Phi) = \eta(\Phi - x) + 1$.

This result has been proved in [4] for signed graphs and it also holds for $\mathbb{T}$-gain graphs. The absence of E-Type cycles implies that there are no induced subgraphs whose nullity equals the number of cycles, so there are fewer cases to be taken into account. The complete characterization in the general case remains an open problem.

5. Nullity of special configurations

The results presented in the previous sections can be applied to know the nullity of particular configurations of cycle-spliced $\mathbb{T}$-gain graphs, such as bipartite, wedge of cycles and bicyclic graphs.

5.1. Bipartite cycle-spliced complex unit gain graphs Φ with $\eta(\Phi) = c(\Phi) - 1$

We focus on bipartite cycle-spliced complex unit gain graphs. All cycles in these graphs are of A-Type or of B-Type.

Lemma 15. *Let Φ be a cycle-spliced bipartite complex unit gain graph with $c(\Phi)$ cycles. If $\eta(\Phi) = c(\Phi) - 1$, then $\eta(\Phi - v) \neq \eta(\Phi)$ for any $v \in V(\Phi)$.*

The corresponding result for signed graphs has been proved in [4]. With a similar technique, one can prove the analogous result for $\mathbb{T}$-gain graphs.

Lemma 16. *Let Φ be a cycle-spliced bipartite complex unit gain graph with $c(\Phi) \geq 2$ and all the pendant cycles of Φ are B-Type. If $\eta(\Phi) = c(\Phi) - 1$, then*
(i) $\eta(\Phi - y) = \eta(\Phi) + 1$ for any cut vertex y of Φ.
(ii) $d_\Phi(y, x)$ is even for any two cut vertices y and x in Φ.
(iii) $\eta(\Phi - y) = \eta(\Phi) + 1$ for $y \in V(\Phi)$ such that the distance between y and any cut vertex is even.
(iv) $\eta(\Phi - y) = \eta(\Phi) - 1$ for $y \in V(\Phi)$ such that the distance between y and any cut vertex of Φ is odd.

Lemma 17. *Let Φ be a cycle-spliced bipartite complex unit gain graph in which every non-pendant cycle has exactly two cut-vertices as shown in Figure 1. If exactly one pendant cycle is of B-Type and all the other cycles in Φ are of A-Type, and the distance between any two cut vertices of Φ is even then*
(i) $\eta(\Phi) = c(\Phi) - 1$.
(ii) $\eta(\Phi - x) = \eta(\Phi) + 1$ for any cut vertex x of Φ.
(iii) $\eta(\Phi - y) = \eta(\Phi) + 1$ for $y \in V(\Phi)$ such that the distance between y and any cut vertex of Φ is even.
(iv) $\eta(\Phi - y) = \eta(\Phi) - 1$ for $y \in V(\Phi)$ such that the distance between y and any cut vertex of Φ is odd.

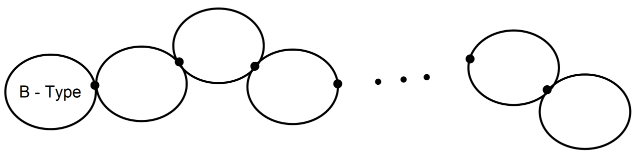

Figure 1. A cycle-spliced bipartite complex unit gain graph Φ in which every non-pendant cycle has exactly two cut vertices.

The following two results give a structural characterization of $\mathbb{T}$-gain cycle-spliced bipartite graphs Φ satisfying $\eta(\Phi) = c(\Phi) - 1$. We give a reference for the corresponding results in the signed case. The proof techniques equally apply to the gain case.

Theorem 4. *[4] Let Φ be a complex unit cycle-spliced bipartite gain graph with $c(\Phi) \geq 2$ and all pendant cycles are of B-Type. Then $\eta(\Phi) = c(\Phi) - 1$ if and only if the distance between any two cut vertices of Φ is even.*

Theorem 5. *[4] For any complex unit cycle-spliced bipartite gain graph Φ with $c(\Phi)$ cycles, $\eta(\Phi) = c(\Phi) - 1$ if and only if Φ is obtained from a complex unit cycle-spliced bipartite gain graph Φ' with $\eta(\Phi') = c(\Phi') - 1$ in which every pendant cycle (if any) is of B-Type by attaching $c(\Phi) - c(\Phi')$ cycles of A-Type on arbitrary vertex of Φ'.*

5.2. Wedge of cycles

For any natural number $t \geq 2$, let $\Phi_1, \ldots, \Phi_t$ be complex unit gain rooted graphs with root v_i, respectively. We denote by $\bigvee_{i=1}^{t} \Phi_i$ or $\bigvee_v \Phi_i$ the *wedge* of the Φ_i's, that is, the

graph obtained by identifying their roots at a unique vertex v. If the rooted graphs are $\mathbb{T}$-gain cycles, then their wedge is equivalent to a cycle-spliced $\mathbb{T}$-gain graph with exactly one cut-vertex v.
The following proposition is about the nullity of a wedge of $\mathbb{T}$-gain cycles of any Type. See Figure 2.

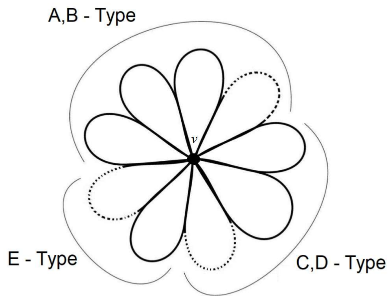

Figure 2. A wedge Φ of m cycles of A- or B-Type, h cycles of C- or D-Type, p cycles of E-Type, all having a common vertex v.

Proposition 4. *Let Φ be a $\mathbb{T}$-gain graph with $c(\Phi) \geq 2$ cycles, obtained from the wedge of m cycles C_j of even length, $j = 1, \dots, m$, p cycles of E-Type, $p \geq 0$, and $h \geq 0$ cycles of C-or D-Type, all having a common vertex v. Then*
(i) $\eta(\Phi) = m + 1$ if all cycles in Φ are of A-Type or of E-Type.
(ii) $\eta(\Phi) = m - 1$ if at least one of the cycles C_j is of B-Type.
(iii) $\eta(\Phi) \in \{0, 1\}$ if its cycles are of C-, D- or E-Type.
(iv) $\eta(\Phi) = \eta(W_0) + m_A$ if $h \geq 1$, W_0 is the wedge at the vertex v of all the h cycles of C- or D-Type, m_A is the number of A-Type cycles and there is no B-Type cycle in Φ.

Proof. According to Lemma 8 (iii), E-Type cycles do not give any contribution to $\eta(\Phi)$ and a wedge Φ of p cycles, all of E-Type, is 1. (i) This comes from Theorem 1 (ii) and (iii) of Lemma 8. If $p = 0$, then $\eta(\Phi) = m + 1 = c(\Phi) + 1$ and this is the only case of wedge Φ whose nullity equals $c(\Phi) + 1$.
(ii) By Lemma 8 (iii), $\eta(\Phi) = \eta(\Phi')$, where Φ' is the subgraph made by the m even cycles $C_1, \dots, C_m$ and the C- or D-Type cycles C'_l, $l = 1, \dots, h$. Assume that a cycle, say C_1, is of B-Type and let v be the unique cut vertex of Φ'. By Lemma 8 (ii), $\eta(\Phi') = \eta(\Phi' - C_1) = \eta(\cup_{j=2}^{m}(C_j - v)) + \eta(\cup_{l=1}^{h}(C'_l - v)) = m - 1$.
(iii) It follows from Lemma 3 and Lemma 6.
(iv) This follows from Lemma 8 (i). □

5.3. Bicyclic cycle-spliced $\mathbb{T}$-gain graphs

In this section we compute the nullity of a cycle-spliced $\mathbb{T}$-gain graph Φ whose cyclomatic number is 2. Then $\Phi = C \bigvee_v C'$, and C and C' are $\mathbb{T}$-gain cycles with a vertex v in common.

Let Φ be a complex unit gain graph. The characteristic polynomial of Φ is

$$P_\Phi(\lambda) = |\lambda I - A(\Phi)| = \lambda^n + a_1(\Phi)\lambda^{n-1} + \cdots + a_{n-1}(\Phi)\lambda + a_n(\Phi).$$

From [18] we get the following two useful results on $P_\Phi(\lambda)$:

$$a_i(\Phi) = \sum_{H\in\mathcal{H}_i} (-1)^{p(H)} 2^{c(H)} \prod_{C\in\mathcal{C}(H)} \mathcal{R}(C), \quad i = 1, \ldots, n$$

where H is a $\mathbb{T}$-gain subgraph of Φ, spanned over i vertices, whose components are edges or cycles (of length at least 3), $p(H)$ is the number of components of H, $c(H)$ is the number of cycles in H, $\mathcal{R}(C)$ is the real part of the gain $\varphi(C)$ of the cycle C and $\mathcal{H}_i$ is the set of all subgraphs as H of order i.

In particular,

$$det\, A(\Phi) = \sum_{H\in\mathcal{H}_n} (-1)^{n-p(H)} 2^{c(H)} \prod_{C\in\mathcal{C}(H)} \mathcal{R}(C).$$

If n is even, then a_n in $|\lambda I - A(\Phi)|$ is equal to $|A(\Phi)|$; if n is odd, then a_n in $|\lambda I - A(\Phi)|$ is equal to $-|A(\Phi)|$.

According to Proposition 4, we deduce that
$\eta(\Phi) = 3$ if and only if C and C' are both of A-Type;
$\eta(\Phi) = 2$ if and only if C is of of A-Type and C' is of E-Type;
$\eta(\Phi) = 1$ if C is of of A-Type and C' is of B-, C- or D-Type, or C and C' are both of B-Type.

As a consequence of Lemma 8, we obtain the following information:
$\eta(\Phi) = 0$ if C is of of B-Type and C' is of C-, D- or E-Type, or C is of E-Type and C' is of C- or D-Type;
$\eta(\Phi) = 1$ if C and C' are both of E-Type.

The remaining cases (a) both C and C' are of C-Type, (b) both C and C' are of D-Type, (c) C is of C-Type and C' is of D-Type need to be investigated. According to Lemma 10 (or Lemma 12), the nullity of Φ in these cases is 0 or 1. It can be precisely established by looking at the determinant of $A(\Phi)$ or, equivalently, the coefficient $a_n(\Phi)$ of the characteristic polynomial $P_\Phi(\lambda)$. A cycle C is of C-Type if its length is 1 mod 4 and the real part $\mathcal{R}(C) = \mathcal{R}(\varphi(C))$ of its gain $\varphi(C)$ is positive (we denote by $\mathcal{C}^1_+$ such a cycle), or if its length is 3 mod 4 and the real part $\mathcal{R}(C)$ of its gain is negative (we denote by $\mathcal{C}^3_-$ such a cycle).

A cycle C is of D-Type if its length is 3 mod 4 and the real part $\mathcal{R}(C)$ of its gain is positive (we denote by $\mathcal{C}^3_+$ such a cycle), or if its length is 1 mod 4 and the real part $\mathcal{R}(C)$ of its gain is negative (we denote by $\mathcal{C}^1_-$ such a cycle).

Proposition 5. *Let $\Phi = C \bigvee_v C'$, where C and C' are of C- or D-Type. Then $\eta(\Phi) = 1$ if and only if C and C' are of different Types (one is of C-Type and the other is of D-Type) and one of the following conditions holds:*
(i) $\mathcal{R}(C) = \mathcal{R}(C')$ when $C = \mathcal{C}^1_+$ and $C' = \mathcal{C}^3_+$, or $C = \mathcal{C}^3_-$ and $C' = \mathcal{C}^1_-$;
(ii) $\mathcal{R}(C) = -\mathcal{R}(C')$ when $C = \mathcal{C}^1_+$ and $C' = \mathcal{C}^1_-$, or $C = \mathcal{C}^3_-$ and $C' = \mathcal{C}^3_+$.

Proof. The coefficient $a_n(\Phi) = \sum_{H\in\mathcal{H}_n}(-1)^{p(H)}2^{c(H)}\prod_{C\in\mathcal{C}(H)}\mathcal{R}(C)$ consists of 2 terms coming from 2 subgraphs: H made by the cycle C together with the disjoint edges (K_2 components) of $C'-v$, and H' made by the cycle C' together with the disjoint edges (K_2 components) of $C-v$. Let us denote by $4h+1$ and $4h+3$ the length of C depending on whether C is a $\mathcal{C}^1$ or a $\mathcal{C}^3$ cycle. Similarly, let $4k+1$ and $4k+3$ be the length of C' depending on whether C' is a $\mathcal{C}^1$ or a $\mathcal{C}^3$ cycle. Then $p(H') = 1+2h$ ($p(H) = 1+2k$) if we take the K_2 components in a $\mathcal{C}^1$ cycle, and $p(H') = 2+2h$ ($p(H) = 2+2k$) if we take the K_2 components in a $\mathcal{C}^3$ cycle. Then

$$a_n(\Phi) = 2((-1)^{p(H)}\mathcal{R}(C) + (-1)^{p(H')}\mathcal{R}(C')).$$

The coefficient $a_n(\Phi)$ will be positive, negative or zero according to the parity of $p(H)$ and $p(H')$ and the sign of $\mathcal{R}(C)$ and $\mathcal{R}(C')$. Our result follows by analyzing all the possibilities for C and C'. □

In [6] the authors computed the nullity of all possible cycle-spliced signed graphs with three cycles. This study required a distinction among many different cases, depending on the sign of the cycles and the arrangement of the cycles in the graph. If this was the case for a graph with only three cycles, we can imagine what the result would be for a graph with a number of cycles greater than 3 and the gain of cycles varying in $\mathbb{T}$. Therefore the idea is to continue investigating particular configurations with conditions on the gain of cycles.

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