

## On connected graphs with integer-valued $Q$ -spectral radius

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**Abstract:** The  $Q$ -eigenvalues are the eigenvalues of the signless Laplacian matrix  $Q(G)$  of a graph  $G$ , and the largest  $Q$ -eigenvalue is known as the  $Q$ -spectral radius  $q(G)$  of  $G$ . The edge-degree of an edge is defined as the number of edges adjacent to it. In this article, we characterize the structure of simple connected graphs having integral  $Q$ -spectral radius. We show that the necessary and sufficient condition for such graphs to contain either a double star  $S_r^2$  or its variation  $S_r^{2,1}$  (having exactly one common neighbor between the central vertices) as a subgraph is that the maximum edge-degree is  $2r$ , where  $r = q(G) - 3$ . In particular, we characterize all graphs that contain only double star as a subgraph when  $q(G)$  equals 8 and 9. Further, we characterize all the connected edge-non-regular graphs with a maximum edge-degree equal to 4 whose minimum  $Q$ -eigenvalue does not belong to the open interval  $(0, 1)$  and has an integral  $Q$ -spectral radius.

**Keywords:** edge-degree, integral graph, signless Laplacian matrix,  $Q$ -integral graph,  $Q$ -spectral radius.

**AMS Subject classification:** 05C50, 05C07

### 1. Introduction

All the graphs considered in this article are simple. Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . We call a graph  $G$  as  $H$ -free if  $H$  is not a subgraph of  $G$ . For a vertex  $x \in V(G)$ , the *degree*,  $d_G(x)$ , is the number of vertices adjacent to  $x$  in  $G$ , and  $d_G^{\max}$  is used to denote the *maximum degree of  $G$* . We use  $N(x)$  to denote the *neighborhood of  $x$* . An edge in  $G$  with incident vertices  $x, y$  is denoted by  $xy$ .

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The *cartesian product*  $G_1 \square G_2$  of two graphs  $G_1$  and  $G_2$  is defined by  $V(G_1 \square G_2) = V(G_1) \times V(G_2)$  and  $(x_1, y_1)(x_2, y_2) \in E(G_1 \square G_2)$  if and only if  $x_1 = x_2$  and  $y_1 y_2 \in E(G_2)$  or,  $y_1 = y_2$  and  $x_1 x_2 \in E(G_1)$ . We define a *double star*  $\mathcal{S}_r^2$  obtained by taking two disjoint copies of star graph  $K_{1,r}$  and adding an edge between the vertices of degree  $r$ . Consider  $\mathcal{S}_r^{2,1}$  as a *variation of*  $\mathcal{S}_r^2$ , where the vertices of degree  $r + 1$  in  $\mathcal{S}_r^2$  have exactly one common neighbor, see Figure 3. We call the vertices of degree  $r + 1$  in  $\mathcal{S}_r^2$  and  $\mathcal{S}_r^{2,1}$  as the central vertices. We use  $\lambda_{\min}(B)$  to denote the *minimum eigenvalue* of a real symmetric square matrix  $B$ .

Let the adjacency matrix of  $G$  be  $A(G)$ .  $G$  is called an *integral graph* if the spectrum of  $A(G)$  consists entirely of integers. The question about which graphs are integral dates back to Harary and Schwenk (1974) [7], who remarked that the general problem appeared intractable. For some results on integral graphs, see [2, 5].

Let  $D(G)$  be the diagonal matrix with  $D(G)_{xx} = d_G(x)$ , for any  $x \in V(G)$ . The matrix  $D(G) + A(G)$  is called the *signless Laplacian matrix* of  $G$  and is denoted by  $Q(G)$ . The matrix  $Q(G)$  is positive semidefinite and irreducible. The eigenvalues of  $Q(G)$  are known as the  $Q$ -*eigenvalues* of the graph  $G$ . The  $Q$ -*spectral radius*  $q(G)$  of  $G$  is the largest  $Q$ -eigenvalue. A graph is called  $Q$ -*integral* if the spectrum of  $Q(G)$  consists entirely of integers. Several studies on signless Laplacian matrix of graphs and  $Q$ -integral graphs can be found in [1, 4, 6, 9, 13–18, 20, 21]. The *edge-degree*  $e\text{-deg}_G(e')$  of an edge  $e' = xy \in E(G)$  is  $|N(x)| + |N(y)| - 2$ . We denote the *maximum edge-degree* of a graph  $G$  by  $e\text{-deg}_G^{\max}$ . A graph  $G$  is called *edge-regular* if for all  $e' \in E(G)$ ,  $e\text{-deg}_G(e')$  are equal and is denoted by  $e\text{-deg}_G$ , the *edge-degree* of  $G$ . If a graph is not edge-regular, we call it as *edge-non-regular graph*.

In 2008, Simić and Stanić [19] studied the connected  $Q$ -integral graphs with  $e\text{-deg}_G^{\max} \leq 5$ . In 2019, the connected  $Q$ -integral graphs with  $e\text{-deg}_G^{\max} \leq 6$  was studied by Park and Sano [11]. They gave a structural classification for such graphs  $G$  when  $q(G) = 6$ . It is interesting that any connected  $Q$ -integral graph with  $e\text{-deg}_G^{\max} = q(G) = 6$  always contains a double star  $\mathcal{S}_3^2$  as a subgraph. Though, it was proved in [10] that there is no connected  $Q$ -integral bipartite graph having  $\mathcal{S}_3^2$  as an induced subgraph.

Recently, in 2023 [12], the authors studied connected  $Q$ -integral graphs with  $e\text{-deg}_G^{\max} \leq 8$  and gave a structural classification under the restriction  $q(G) = 7$ . They showed that  $\mathcal{S}_4^2$  is a subgraph of the connected  $Q$ -integral graph with  $e\text{-deg}_G^{\max} = 8$ . Besides, they also gave an upper bound and a lower bound for  $e\text{-deg}_G^{\max}$  in terms of  $q(G)$  for  $Q$ -integral graphs and proved that there does not exist any connected edge-non-regular  $Q$ -integral graph with  $q(G) \leq 4$ .

Moreover, it is quite surprising to observe that the double star  $\mathcal{S}_r^2$  is always a subgraph of connected  $Q$ -integral graph with  $e\text{-deg}_G^{\max} = 2r$ , where  $r = q(G) - 3$ ;  $q(G) \in \{5, 6, 7\}$ , see [11, 12, 19]. Eventually, a question arises about the existence of such a double star in a connected  $Q$ -integral graph for any value of  $q(G)$ . Also, it is quite interesting to analyze, whether the condition of integral  $Q$ -spectrum can be relaxed. If so, then what conditions on its  $Q$ -spectrum are required for a graph to have  $\mathcal{S}_r^2$  as a subgraph.

With the quest to answer the above questions, we study connected graphs with integral  $Q$ -spectral radius  $q(G)$  and  $e\text{-deg}_G^{\max} = 2q(G) - 6$ . We give a necessary and sufficient condition for such a connected graph  $G$  to contain  $\mathcal{S}_r^2, \mathcal{S}_r^{2,1}$ , where  $r = q(G) - 3$ , as a subgraph. Using this condition, we also characterize connected graphs having  $q(G) \in \{8, 9\}$  and  $\lambda_{\min}(Q(G)) \notin (0, 1)$  to contain only double star  $\mathcal{S}_5^2, \mathcal{S}_6^2$  as a subgraph, respectively.

In 2008, Simić and Stanić [19] showed that the only connected edge-non-regular  $Q$ -integral graph with  $e\text{-deg}_G^{\max} = 4$  is  $\mathcal{H}^*$  and  $K_{1,2} \square K_2$ , see Figure 1. In this article, we extend this result by characterizing all such edge-non-regular connected graph  $G$ , when it is not  $Q$ -integral and instead have only integral  $q(G)$  and  $\lambda_{\min}(Q(G)) \notin (0, 1)$ .

## 2. Preliminaries

The principal submatrix  $Q_p(H)$  of the signless Laplacian matrix  $Q(G)$ , corresponding to a subset  $H \subseteq V(G)$  is defined by

$$Q_p(H)_{xy} = \begin{cases} d_G(x) & x = y \\ 1 & xy \in E(G) \\ 0 & xy \notin E(G). \end{cases}$$

Let  $M$  be a complex matrix of order  $n$  described in the following block form

$$M = \begin{bmatrix} M_{11} & \dots & M_{1t} \\ \vdots & \ddots & \vdots \\ M_{t1} & \dots & M_{tt} \end{bmatrix}$$

where the blocks  $M_{ij}$  are  $n_i \times n_j$  matrices for any  $1 \leq i, j \leq t$  and  $n = n_1 + \dots + n_t$ . For  $1 \leq i, j \leq t$ , let  $r_{ij}$  denote the average row sum of  $M_{ij}$ , i.e.,  $r_{ij}$  is the sum of all entries in  $M_{ij}$  divided by the number of rows. Then  $\mathcal{E}_M = (r_{ij})$  is called the *quotient matrix* of  $M$ . If, in addition, for each pair  $i, j$ ,  $M_{ij}$  has a constant row sum, then  $\mathcal{E}_M$  is called the *equitable quotient matrix* of  $M$ .

We use  $B_{m \times n}$  to denote a matrix  $B$  of order  $m \times n$  and  $B_n$  to denote a square matrix  $B$  of order  $n$ . The spectral radius of a square matrix  $B$  is denoted by  $\rho(B)$  and the spectrum  $\sigma(B)$  is the set of all eigenvalues of  $B$ . For any two non-negative matrices  $B_m = (b_{ij})$  and  $C_m = (c_{ij})$ , we say  $B_m$  dominates  $C_m$  if  $B_m \geq C_m$  (i.e.,  $b_{ij} \geq c_{ij}$  for all  $i, j = 1, \dots, m$ ). Note that, if  $B_m$  dominates  $C_m$ , then  $\rho(B_m) \geq \rho(C_m)$ .

We use  $J$  to mean a matrix with all entries equal to 1 and  $I$  to denote identity matrix.  $K_{1,n}$  is a complete bipartite graph with 1 (resp.  $n$ ) vertex in the first (resp. second) partite set.  $C_n$  is a cycle of order  $n$  and  $P_n$  denotes a path on  $n$  vertices.

We will use the well known theorems, namely Perron-Frobenius Theorem [[8], Theorem 8.4.4] and Interlacing Theorem [[8], Theorem 4.3.17] on eigenvalues to prove

several results in this article. Now we state some important results that we require for our proofs.

**Theorem 1.** ([22], Theorem 2.3). Let  $\mathcal{E}_M$  be the equitable quotient matrix of a complex square matrix  $M$ . Then  $\sigma(\mathcal{E}_M) \subseteq \sigma(M)$ .

**Theorem 2.** ([22], Theorem 2.5). Let  $\mathcal{E}_M$  be the equitable quotient matrix of a non-negative square matrix  $M$ . Then  $\rho(\mathcal{E}_M) = \rho(M)$ .

**Theorem 3.** ([3], Proposition 1.3.9). The number of connected bipartite components of  $G$  is equal to the multiplicity of the  $Q$ -eigenvalue 0 in  $G$ .

**Theorem 4.** ([11], Proposition 2.7). A connected graph  $G$  has  $d_G(v) \leq \lceil q(G) - 1 \rceil$  for any  $v \in V(G)$ , where  $q(G)$  is the  $Q$ -spectral radius of  $G$ . If  $G$  has a vertex  $v$  having  $d_G(v) = q(G) - 1$  and  $q(G) \in \mathbb{Z}^+$ , then  $G = K_{1,q(G)-1}$ .

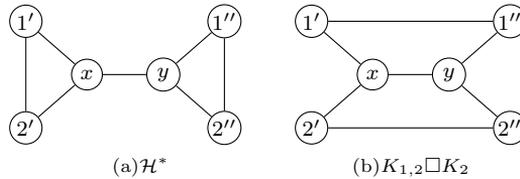


Figure 1. Edge-non-regular connected graphs  $G$  having  $q(G), \lambda_{\min}(Q(G)) \in \mathbb{Z}$  and  $e\text{-deg}_G^{\max} = 4$

**Theorem 5.** ([19], Theorem 3.2). If  $G$  is a connected edge-non-regular  $Q$ -integral graph with maximum edge-degree 4, then  $G$  is one of the two graphs:  $\mathcal{H}^*$  and  $K_{1,2} \square K_2$  (of Figure 1).

The following results give the bounds for the maximum edge-degree of a graph.

**Theorem 6.** ([12], Remark 3.2). For a connected edge-regular graph  $G$ ,  $e\text{-deg}_G = q(G) - 2$ .

**Theorem 7.** ([12], Lemma 3.3, Remark 3.6). Let  $G$  be a connected edge-non-regular graph with  $q(G) \in \mathbb{Z}$ , then  $q(G) - 1 \leq e\text{-deg}_G^{\max} \leq 2q(G) - 6$ .

**Theorem 8.** ([12], Remark 3.4, Remark 3.6). There does not exist any connected edge-non-regular graph with integral  $q(G) \leq 4$ . Moreover, if  $q(G) = 5$ , then  $e\text{-deg}_G^{\max} = 4$ .

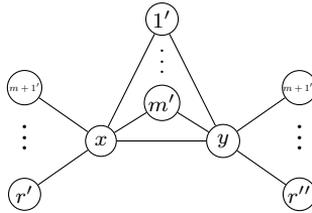
### 3. Main Result

In this section, we study the structure of the graphs  $G$  with integral  $Q$ -spectral radius  $q(G)$  and maximum edge-degree  $2q(G) - 6$ . Thus from now, we consider connected graph  $G$  having  $q(G) \in \mathbb{Z}$ .

For  $q(G) \geq 5$ , it can be observed from Theorem 4, Theorem 6 and Theorem 7 that  $e\text{-deg}_G^{\max} = 2q(G) - 6$  if and only if  $G$  contains at least two adjacent vertices  $x$  and  $y$  with vertex degree  $d_G(x) = d_G(y) = q(G) - 2$ . For any two distinct vertices  $i, j$  of  $G$ , we use  $a_{ij}$  to denote the  $(i, j)$ -th entry of the adjacency matrix  $A(G)$ . We use  $a_{..}$  and  $d_G(\cdot)$  to mean  $a_{xy}$  and  $d_G(z)$  for suitable vertices  $x, y$ , and  $z$ .

**Lemma 1.** *Let  $G$  be a connected graph with integral  $Q$ -spectral radius  $q(G) \geq 5$ . If  $e\text{-deg}_G^{\max} = 2q(G) - 6$ , then the incident vertices on any edge with edge-degree  $2q(G) - 6$  can have at most one common neighbor.*

*Proof.* Let  $xy \in E(G)$  be any arbitrary edge with  $e\text{-deg}(xy) = 2q - 6$ , where  $q = q(G)$ . Thus  $d_G(x) = d_G(y) = q - 2$ . Let  $N(x) = \{y, 1', 2', \dots, r'\}$  and  $N(y) = \{x, 1'', 2'', \dots, r''\}$ , where  $r = q - 3$ , be the neighborhood sets of  $x$  and  $y$ , respectively.



**Figure 2.**  $x$  and  $y$  with  $m$  common neighbors,  $r = q - 3$

Suppose  $x$  and  $y$  have exactly  $m$  common neighbors say,  $1' = 1'', \dots, m' = m''$ , where  $2 \leq m \leq r$ . The principal submatrix  $Q_p(H)$  of the signless Laplacian matrix  $Q(G)$  corresponding to the vertex set  $H = N(x) \cup N(y) = \{x, y, 1', 2', \dots, m', m + 1', \dots, r', m + 1'', \dots, r''\}$  is given by

$$Q_p(H) = \begin{bmatrix} q-2 & 1 & 1 & \dots & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 1 & q-2 & 1 & \dots & 1 & 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & 1 & d_G(1') & \dots & a_{1'm'} & a_{1'm+1'} & \dots & a_{1'r'} & a_{1'm+1''} & \dots & a_{1'r''} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & a_{1'm'} & \dots & d_G(m') & a_{m'm+1'} & \dots & a_{m'r'} & a_{m'm+1''} & \dots & a_{m'r''} \\ 1 & 0 & a_{1'm+1'} & \dots & a_{m'm+1'} & d_G(m+1') & \dots & a_{m+1'r'} & a_{m+1'm+1''} & \dots & a_{m+1'r''} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & a_{1'r'} & \dots & a_{m'r'} & a_{m+1'r'} & \dots & d_G(r') & a_{r'm+1''} & \dots & a_{r'r''} \\ 0 & 1 & a_{1'm+1''} & \dots & a_{m'm+1''} & a_{m+1'm+1''} & \dots & a_{r'm+1''} & d_G(m+1'') & \dots & a_{m+1'r''} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & a_{1'r''} & \dots & a_{m'r''} & a_{m+1'r''} & \dots & a_{r'r''} & a_{m+1'r''} & \dots & d_G(r'') \end{bmatrix}$$

where  $1 \leq d_G(\cdot) \leq r + 1$ ,  $d_G(i') \geq 2$  for  $i = 1, \dots, m$ , and  $a_{..} \in \{0, 1\}$ . By Interlacing Theorem [[8], Theorem 4.3.17] and Perron-Frobenius Theorem [[8], Theorem 8.4.4],  $\rho(Q_p(H)) \leq \rho(Q(G)) = q$ . Then for any possible choices of  $d_G(\cdot)$  and  $a_{..}$ , the matrix  $Q_p(H)$  dominates the following matrix

$$M = \begin{bmatrix} q-2 & 1 & 1 & \dots & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 1 & q-2 & 1 & \dots & 1 & 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & 1 & 2 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \dots & 2 & 0 & \dots & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}.$$

The equitable quotient matrix of  $M$  is

$$\mathcal{E}_M = \begin{bmatrix} q - 1 & m & q - m - 3 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The characteristics polynomial of  $\mathcal{E}_M$  is

$$\mathcal{P}_{\mathcal{E}_M}(x) = x^3 - (q + 2)x^2 + (2q - m + 2)x - 4.$$

Note that when  $x = q$ , we get

$$\mathcal{P}_{\mathcal{E}_M}(q) = -(m - 2)q - 4 < 0, \quad \text{since } m \geq 2.$$

Also when  $x = q + 1$ , we have for  $2 \leq m \leq q - 3$ ,

$$\mathcal{P}_{\mathcal{E}_M}(q + 1) = (q^2 + 2q - 3) - m(q + 1) \geq 4q > 0.$$

Thus we observe that  $\mathcal{P}_{\mathcal{E}_M}(x)$  has a root in  $(q, q + 1)$  and hence  $\rho(\mathcal{E}_M) > q$ . Since  $M$  is a non-negative matrix, by Theorem 2, we have  $\rho(M) = \rho(\mathcal{E}_M) > q$ . Further,  $\rho(Q_p(H)) \geq \rho(M) > q$ , which is a contradiction to  $\rho(Q_p(H)) \leq \rho(Q(G)) = q(G) = q$ . Therefore,  $x$  and  $y$  can not have  $m(2 \leq m \leq q - 3)$  common neighbors in  $G$ . Thus,  $x$  and  $y$  can have at most 1 common neighbor in  $G$ . Hence the lemma holds.  $\square$

Let  $\mathcal{S}_r^2$  be the double star, as shown in Figure 3(a), with  $V(\mathcal{S}_r^2) = \{x, y, 1', \dots, r', 1'', \dots, r''\}$  and  $E(\mathcal{S}_r^2) = \{xy, x1', \dots, xr', y1'', \dots, yr''\}$ . Also, consider  $\mathcal{S}_r^{2,1}$ , as shown in Figure 3(b), with  $V(\mathcal{S}_r^{2,1}) = \{x, y, 1', 2', \dots, r', 2'', \dots, r''\}$  and  $E(\mathcal{S}_r^{2,1}) = \{xy, x1', x2', \dots, xr', y1', y2'', \dots, yr''\}$ .

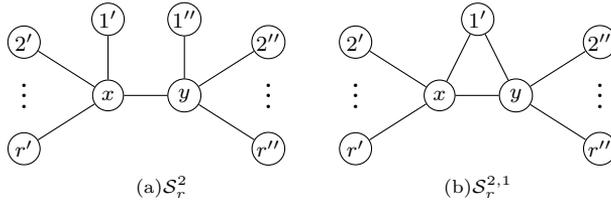


Figure 3. Possible subgraphs of connected graphs  $G$  having  $q(G) \in \mathbb{Z}$  and  $e\text{-deg}_G^{\max} = 2r$ , where  $r = q(G) - 3$

**Remark 1.** Now consider the connected graph  $G$  with maximum edge-degree  $e\text{-deg}_G^{\max}$  equal to  $2q(G) - 6$ . It can be verified from Theorem 4, Theorem 6 and Theorem 8 that there is no such graphs having  $q(G) = 1, 2, 3$ . Also,  $q(G) = 4$  if and only if  $G = K_{1,3}$  or  $G = C_n$ ,  $n \geq 3$ .

**Theorem 9.** Let  $G(\neq K_{1,3})$  be a connected graph with integral  $Q$ -spectral radius  $q(G) \geq 4$ . Then  $e\text{-deg}_G^{\max} = 2r$  if and only if  $G$  contains  $S_r^2$  or  $S_r^{2,1}$  as a subgraph, where  $r = q(G) - 3$ .

*Proof.* By Remark 1, the theorem holds when  $q(G) = 4$ . Let  $q(G) \geq 5$  and  $xy \in E(G)$  be an edge with edge-degree  $2r$ , where  $r = q(G) - 3$ . Let the neighborhood sets of  $x$  and  $y$  in  $G$  be  $N(x) = \{y, 1', 2', \dots, r'\}$  and  $N(y) = \{x, 1'', 2'', \dots, r''\}$ , respectively. By Lemma 1, we have  $i' \neq j''; i, j = 2, 3, \dots, r$ . Therefore,  $G$  contains at least one of  $S_r^2, S_r^{2,1}$  as a subgraph.

Conversely, if  $G$  contains either  $S_r^2$  or  $S_r^{2,1}$ , where  $r = q(G) - 3$ , as a subgraph, then  $e\text{-deg}_G^{\max} \geq 2r$ . From Theorem 6 and Theorem 7 for  $q(G) \geq 4$ , we have  $e\text{-deg}_G^{\max} \leq 2q(G) - 6 = 2r$ . Hence the theorem holds.  $\square$

Here with the help of Theorem 9, we give a necessary and sufficient conditions for connected graphs having  $q(G) \in \{5, 6, 7, 8, 9\}$  and  $\lambda_{\min}(Q(G)) \notin (0, 1)$  to contain  $S_{q(G)-3}^2$  as a subgraph but not  $S_{q(G)-3}^{2,1}$ .

**Lemma 2.** Let  $G$  be a connected graph having  $q(G) = 5$  and  $\lambda_{\min}(Q(G)) \notin (0, 1)$ . Then  $e\text{-deg}_G^{\max} = 4$  if and only if  $G$  is  $S_2^{2,1}$ -free and contains  $S_2^2$  as a subgraph.

*Proof.* Let  $e\text{-deg}_G^{\max} = 4$ . On the contrary, suppose  $G$  has either  $S_2^{2,1}$  as a subgraph or is  $S_2^2$ -free. By Theorem 9, in both the cases  $S_2^{2,1}$  is a subgraph of  $G$ . Let  $V(S_2^{2,1}) = \{x, y, 1', 2', 2''\}$ , with  $d_G(x) = d_G(y) = 3, N(x) = \{y, 1', 2'\}$  and  $N(y) = \{x, 1', 2''\}$ . Clearly,  $G$  is non-bipartite as it contains a triangle and hence by Theorem 3, we have  $\lambda_{\min}(Q(G)) \geq 1$ . Now the principal submatrix  $Q_p(V(S_2^{2,1}))$  of  $Q(G)$  is given by

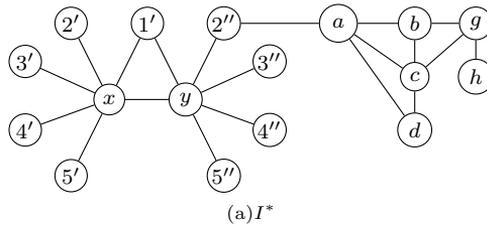
$$Q_p(V(S_2^{2,1})) = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 & 1 \\ 1 & 1 & d_G(1') & a_{1'2'} & a_{1'2''} \\ 1 & 0 & a_{1'2'} & d_G(2') & a_{2'2''} \\ 0 & 1 & a_{1'2''} & a_{2'2''} & d_G(2'') \end{bmatrix}, \tag{3.1}$$

where  $1 \leq d_G(\cdot) \leq 3, d_G(1') \geq 2$  and  $a_{..} \in \{0, 1\}$ . Using MATLAB computation, we find that the only possible values in (3.1) are  $d_G(1') = d_G(2') = d_G(2'') = 2, a_{1'2'} = a_{1'2''} = a_{2'2''} = 0$ , to have  $\rho(Q_p(V(\mathcal{S}_2^{2,1}))) \leq 5$  and  $\lambda_{\min}(Q_p(V(\mathcal{S}_2^{2,1}))) \geq 1$ . Further, in this case, the spectral radius of  $Q_p(V(\mathcal{S}_2^{2,1}))$  is equal to 5 and hence by Perron-Frobenius Theorem, we have  $Q_p(V(\mathcal{S}_2^{2,1})) = Q(G)$ , which is not true since  $Q_p(V(\mathcal{S}_2^{2,1}))$  is not a signless Laplacian matrix. Therefore  $G$  is  $\mathcal{S}_2^{2,1}$ -free and thus by Theorem 9,  $\mathcal{S}_2^2$  is a subgraph of  $G$ .

Conversely, if  $G$  is  $\mathcal{S}_2^{2,1}$ -free containing  $\mathcal{S}_2^2$  as a subgraph, then  $e\text{-deg}_G^{\max} \geq 4$ . Also,  $e\text{-deg}_G^{\max} \leq 4$  when  $q(G) = 5$ . Thus,  $e\text{-deg}_G^{\max} = 4$ . Hence, the lemma holds.  $\square$

**Remark 2.** For  $q(G) = 6$  and 7, the above theorem was proved for  $Q$ -integral graphs in [11] and [12], respectively. But if we relax the condition on the graph to be  $Q$ -integral and instead having only the maximum  $Q$ -eigenvalue to be an integer and  $\lambda_{\min}(Q(G)) \notin (0, 1)$ , we get an analogous version of the above theorem. That is, for a connected graph  $G$  having  $q(G) = 6$  and  $\lambda_{\min}(Q(G)) \notin (0, 1)$ , the same proof given in ([11], Lemma 3.13) can be used to show that if  $e\text{-deg}_G^{\max} = 6$ , then  $\mathcal{S}_3^2$  is a subgraph of  $G$ . The converse can be easily verified by using Theorem 6 and Theorem 7.

Similarly, when  $q(G) = 7$ , the same proof given in ([12], Lemma 4.3, Lemma 4.4), will work to show that  $e\text{-deg}_G^{\max} = 8$  if and only if  $\mathcal{S}_4^2$  is a subgraph of  $G$ .



**Figure 4.** Graph  $I^*$  having  $e\text{-deg}_{I^*}^{\max} = 10$  and  $\lambda_{\min}(Q(I^*)) = 0.2192$ , where  $q(I^*) = 8$

**Remark 3.** Note that in the above lemma and Remark 2, the condition  $\lambda_{\min}(Q(G)) \notin (0, 1)$  can not be relaxed. For example, let  $I^*$  be as given in Figure 4. Here, we have  $e\text{-deg}_{I^*}^{\max} = 10, q(I^*) = 8$  and  $\lambda_{\min}(Q(I^*)) = 0.2192$  while  $\mathcal{S}_5^2$  is not its subgraph but  $\mathcal{S}_5^{2,1}$  is a subgraph. Similarly, one can construct graphs for  $q(G) = 5, 6$  and 7.

With the above remark, we next prove that  $\mathcal{S}_5^{2,1}$  cannot be a subgraph of  $G$  for which  $q(G) = 8$  and  $\lambda_{\min}(Q(G)) \notin (0, 1)$ .

**Lemma 3.** Let  $G$  be a connected graph having  $q(G) = 8$  and  $\lambda_{\min}(Q(G)) \notin (0, 1)$ . Then  $e\text{-deg}_G^{\max} = 10$  if and only if  $G$  is  $\mathcal{S}_5^{2,1}$ -free and contains  $\mathcal{S}_5^2$  as a subgraph.

*Proof.* Let  $e\text{-deg}_G^{\max} = 10$ . On the contrary, assume that either  $G$  is  $\mathcal{S}_5^2$ -free or contains  $\mathcal{S}_5^{2,1}$  as a subgraph. In both cases,  $\mathcal{S}_5^{2,1}$  is a subgraph of  $G$  by Theorem 9.

Let  $V(S_5^{2,1}) = \{x, y, 1', 2', 3', 4', 5', 2'', 3'', 4'', 5''\}$ , with  $N(x) = \{y, 1', 2', 3', 4', 5'\}$  and  $N(y) = \{x, 1'', 2'', 3'', 4'', 5''\}$ . The principal submatrix  $Q_p(V(S_5^{2,1}))$  of  $Q(G)$  is given by

$$Q_p(V(S_5^{2,1})) = \begin{bmatrix} 6 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & d_G(1') & a_{1'2'} & a_{1'3'} & a_{1'4'} & a_{1'5'} & a_{1'2''} & a_{1'3''} & a_{1'4''} & a_{1'5''} \\ 1 & 0 & d_G(2') & a_{2'3'} & a_{2'4'} & a_{2'5'} & a_{2'2''} & a_{2'3''} & a_{2'4''} & a_{2'5''} \\ 1 & 0 & a_{1'3'} & d_G(3') & a_{3'4'} & a_{3'5'} & a_{3'2''} & a_{3'3''} & a_{3'4''} & a_{3'5''} \\ 1 & 0 & a_{1'4'} & a_{2'4'} & d_G(4') & a_{4'5'} & a_{4'2''} & a_{4'3''} & a_{4'4''} & a_{4'5''} \\ 1 & 0 & a_{1'5'} & a_{2'5'} & a_{3'5'} & d_G(5') & a_{5'2''} & a_{5'3''} & a_{5'4''} & a_{5'5''} \\ 0 & 1 & a_{1'2''} & a_{2'2''} & a_{3'2''} & a_{4'2''} & d_G(2'') & a_{2''3''} & a_{2''4''} & a_{2''5''} \\ 0 & 1 & a_{1'3''} & a_{2'3''} & d_G(3'') & a_{3''4''} & a_{2''3''} & d_G(3'') & a_{3''4''} & a_{3''5''} \\ 0 & 1 & a_{1'4''} & a_{2'4''} & a_{3'4''} & d_G(4'') & a_{2''4''} & a_{3''4''} & d_G(4'') & a_{4''5''} \\ 0 & 1 & a_{1'5''} & a_{2'5''} & a_{3'5''} & a_{4'5''} & a_{2''5''} & a_{3''5''} & a_{4''5''} & d_G(5'') \end{bmatrix}, \tag{3.2}$$

where  $1 \leq d_G(\cdot) \leq 6, d_G(1') \geq 2, a_{..} \in \{0, 1\}$ . Note that  $d_G(1') \in \{2, 3\}$ , otherwise  $\rho(Q_p(V(S_5^{2,1}))) > 8$ . Suppose  $d_G(1') = 3$ , then we find by computation, the following holds:

- (i)  $a_{1'i'} = a_{1'i''} = 0; \forall i = 2, 3, 4, 5;$
- (ii)  $a_{i'j'} = a_{i''j''} = a_{i'l''} = 0; \forall i, j, l = 2, 3, 4, 5; i \neq j;$
- (iii)  $d_G(i'), d_G(i'') \leq 2; \forall i = 2, 3, 4, 5.$

For each of the possible choices of  $d_G(\cdot)$ , we have either  $\lambda_{\min}(Q_p(V(S_5^{2,1}))) < 1$  or  $\rho(Q_p(V(S_5^{2,1}))) > 8$ , which is a contradiction to the fact that  $G$  is non-bipartite,  $q(G) = 8$ , and  $\lambda_{\min}(Q(G)) \notin (0, 1)$ . Thus  $d_G(1') = 2$ .

Now, the edge set of the induced subgraph  $G[N(x) \setminus \{y\}]$  is either empty or contains exactly one edge, namely  $\{2'3'\}$  (up to isomorphism), otherwise  $\rho(Q_p(V(S_5^{2,1}))) > 8$ . Suppose  $E(G[N(x) \setminus \{y\}]) = \{2'3'\}$ , then  $E(G[V(S_5^{2,1}) \setminus \{x, y\}]) = \{2'3'\}$ . Now for each possible choices of  $d_G(\cdot)$ , either  $\lambda_{\min}(Q_p(V(S_5^{2,1}))) < 1$  or  $\rho(Q_p(V(S_5^{2,1}))) > 8$ , which is a contradiction to the fact that  $G$  is a non-bipartite graph with  $q(G) = 8, \lambda_{\min}(Q(G)) \notin (0, 1)$ . Due to the symmetric structure of  $S_5^{2,1}$ , we have  $E(G[N(x) \setminus \{y\}]) = E(G[N(y) \setminus \{x\}]) = \{\phi\}$ .

Computationally, one can find that  $|E(G[\{N(x) \cup N(y)\} \setminus \{x, y\}])| \leq 1$  otherwise the spectral radius of the corresponding principal submatrix in (3.2) is greater than 8. Suppose without loss of generality,  $E(G[\{N(x) \cup N(y)\} \setminus \{x, y\}]) = \{5'5''\}$ . Observe that, for the spectral radius of the corresponding  $Q_p(V(S_5^{2,1}))$  in (3.2) to be 8, the admissible values of  $d_G(\cdot)$  are  $d_G(5') \leq 3, d_G(5'') = 2$ , and  $d_G(i'), d_G(i'') \leq 2$  for  $i = 2, 3, 4$ . However, for these choices of values of  $d_G(\cdot)$ , the least eigenvalue of  $Q_p(V(S_5^{2,1}))$  is less than 1, which is a contradiction. Thus we have  $E(G[\{N(x) \cup N(y)\} \setminus \{x, y\}]) = \{\phi\}$ . Therefore,  $Q_p(V(S_5^{2,1}))$  in (3.2) becomes

$$Q_p(V(\mathcal{S}_5^{2,1})) = \begin{bmatrix} 6 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & d_G(2') & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & d_G(3') & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & d_G(4') & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & d_G(5') & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & d_G(2'') & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(3'') & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(4'') & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(5'') \end{bmatrix}, \quad (3.3)$$

where  $1 \leq d_G(\cdot) \leq 6$ . The only possible choice of  $d_G(\cdot)$  for which  $\lambda_{\min}(Q_p(V(\mathcal{S}_5^{2,1}))) \geq 1$  and  $\rho(Q_p(V(\mathcal{S}_5^{2,1}))) \leq 8$  for the matrix in (3.3) is  $d_G(i') = d_G(i'') = 2$  ( $i = 2, 3, 4, 5$ ). However, in this case, the spectral radius of this matrix equals 8. By Perron-Frobenius Theorem,  $Q(G) = Q_p(V(\mathcal{S}_5^{2,1}))$ , which is a contradiction since the matrix in (3.3) is not a signless Laplacian matrix.

Thus  $G$  is  $\mathcal{S}_5^{2,1}$ -free and hence  $G$  contains  $\mathcal{S}_5^2$  as a subgraph. Conversely, if  $G$  is  $\mathcal{S}_5^{2,1}$ -free containing  $\mathcal{S}_5^2$  as a subgraph, then  $e\text{-deg}_G^{\max} \geq 10$ . Since  $q(G) = 8$ , we have  $e\text{-deg}_G^{\max} \leq 10$ . Hence the lemma holds.  $\square$

**Lemma 4.** *Let  $G$  be a connected graph with  $q(G) = 9$  and  $\lambda_{\min}(Q(G)) \notin (0, 1)$ . Then  $e\text{-deg}_G^{\max} = 12$  if and only if  $G$  is  $\mathcal{S}_6^{2,1}$ -free and contains  $\mathcal{S}_6^2$  as a subgraph of  $G$ .*

*Proof.* Suppose  $e\text{-deg}_G^{\max} = 12$ . Assume that  $G$  is either  $\mathcal{S}_6^2$ -free or contains  $\mathcal{S}_6^{2,1}$  as a subgraph. By Theorem 9,  $\mathcal{S}_6^{2,1}$  is a subgraph of  $G$  in both the cases. Suppose  $V(\mathcal{S}_6^{2,1}) = \{x, y, 1', 2', 3', 4', 5', 6', 2'', 3'', 4'', 5'', 6''\}$ , with  $N(x) = \{y, 1', 2', 3', 4', 5', 6'\}$  and  $N(y) = \{x, 1', 2'', 3'', 4'', 5'', 6''\}$ .

Let  $H = V(\mathcal{S}_6^{2,1}) = N(x) \cup N(y)$  and  $\Gamma = H \setminus \{x, y\}$ . Suppose  $Q_p(H)$  is a principal submatrix of  $Q(G)$  corresponding to  $H = \{x, y, 1', 2', 3', 4', 5', 6', 2'', 3'', 4'', 5'', 6''\}$ . Since  $G$  contains a triangle,  $G$  is non-bipartite and thus least eigenvalue of  $Q_p(H)$  is at least 1 by Theorem 3. Then  $1'$  is not adjacent to any vertices of  $\Gamma$  and  $d_G(1') \in \{2, 3\}$  otherwise  $\rho(Q_p(H)) > 9$ . If  $d_G(1') = 3$ , then  $a_{i'j''} = 0; i, j = 2, \dots, 6$ , for  $\rho(Q_p(H))$  to be at most 9. Computationally, we observed that for every admissible choices of  $1 \leq d_G(\cdot) \leq 7$  in  $Q_p(H)$ , either  $\rho(Q_p(H)) > 9$  or  $\lambda_{\min}(Q_p(H)) < 1$ , which is a contradiction to the fact that  $G$  is a non-bipartite graph having  $q(G) = 9$ . Therefore,  $d_G(1') = 2$ .

Now we have the following claims.

**Claim (i).** All the edges in  $G[\Gamma]$  are disjoint.

If the above claim is not true, then  $G[\Gamma]$  must contain  $P_3$  as a subgraph. Due to the symmetric structure of  $\mathcal{S}_6^{2,1}$ , we have the following choices for  $P_3$  as a subset of  $E(G[\Gamma])$ : (i)  $\{2'3', 3'4'\}$ , (ii)  $\{2'2'', 2'3''\}$ , (iii)  $\{2'2'', 2'3''\}$ . In each of these 3 cases, we have  $\rho(Q_p(H)) > 9$ , a contradiction to our assumption that  $q(G) = 9$ . Thus all the edges of  $G[\Gamma]$  are disjoint.

**Claim (ii).**  $|E(G[\Gamma])| = \phi$ .

We have following choices for the subset of  $E(G[\Gamma])$  (up to symmetry):

(i)  $\{2'2'', 3'3'', 4'4''\}$ , (ii)  $\{2'2'', 3'3'', 4'5'\}$ , (iii)  $\{2'2'', 3'4', 3''4''\}$ , (iv)  $\{2'3', 4'5', 6'6''\}$ , (v)  $\{2'3', 4'5', 2''3''\}$ , (vi)  $\{2'3', 4'5'\}$ , (vii)  $\{2'3', 2''3''\}$ , (viii)  $\{2'3', 6'6''\}$ , (ix)  $\{2'2'', 3'3''\}$ , (x)  $\{2'3'\}$ , (xi)  $\{2'2''\}$ . For all the admissible choices of  $1 \leq d_G(\cdot) \leq 7$ , either the spectral radius of the corresponding  $Q_p(H)$  is greater than 9 or the least eigenvalue is less than 1, which is a contradiction. Hence  $|E(G[\Gamma])| = \phi$ .

The matrix  $Q_p(H)$  becomes

$$Q_p(H) = \begin{bmatrix} 7 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & d_G(2') & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & d_G(3') & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & d_G(4') & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & d_G(5') & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(6') & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(2'') & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(3'') & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(4'') & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(5'') \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(6'') \end{bmatrix} \tag{3.4}$$

where  $1 \leq d_G(\cdot) \leq 7$ . Now the only admissible choices for which  $\rho(Q_p(H)) \leq 9$  and  $\lambda_{\min}(Q_p(H)) \geq 1$  of the corresponding  $Q_p(H)$  in (3.4) is  $d_G(i') = d_G(i'') = 2, i = 2, \dots, 6$ . Moreover, here we have  $\rho(Q_p(H)) = 9$  and thus by Perron-Frobenius Theorem, we have  $Q_p(H) = Q(G)$  which is a contradiction since the matrix in (3.4) does not represent a signless Laplacian matrix. Therefore  $G$  is  $\mathcal{S}_6^{2,1}$ -free and hence  $\mathcal{S}_6^2$  is a subgraph of  $G$  by Theorem 9.

Conversely, suppose  $G$  is  $\mathcal{S}_6^{2,1}$ -free containing  $\mathcal{S}_6^2$  as a subgraph. Then  $e\text{-deg}_G^{\max} \geq 12$ . From Theorem 7, we have  $e\text{-deg}_G^{\max} \leq 12$ . Hence the lemma holds.  $\square$

Now we combine the above results from Lemma 2 to Lemma 4 in the following theorem.

**Theorem 10.** *Let  $G(\neq K_{1,3}, C_3)$  be a connected graph with  $q(G) \in \{4, 5, 6, 7, 8, 9\}$  and  $\lambda_{\min}(Q(G)) \notin (0, 1)$ . Then  $e\text{-deg}_G^{\max} = 2q(G) - 6$  if and only if  $G$  is  $\mathcal{S}_{q(G)-3}^{2,1}$ -free and contains  $\mathcal{S}_{q(G)-3}^2$  as a subgraph.  $\square$*

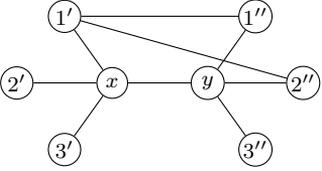


Figure 5.  $S^*$

From now, for simplicity, we use  $\Gamma$  to denote the set of vertices  $N(x) \cup N(y) \setminus \{x, y\}$  in  $\mathcal{S}_r^2, \mathcal{S}_r^{2,1}$  (given in Figure 3). Now, we will study the structure of the graphs under the condition when either  $\mathcal{S}_{q(G)-3}^2$  or,  $\mathcal{S}_{q(G)-3}^{2,1}$  is a subgraph of  $G$ .

**Theorem 11.** *Let  $G$  be a connected graph with integral  $Q$ -spectral radius  $q = q(G) \geq 6$ . If  $\mathcal{S}_{q-3}^2$  is a subgraph of  $G$ , then the following hold.*

- (1)  $G[N(x)], G[N(y)]$  are  $C_{q-t}$ -free,  $q \geq t + 3, 3 \leq t \leq 5$ .
- (2) Either  $G = \mathcal{S}^*$  (in Figure 5) or  $d_{G[\mathcal{S}_{q-3}^2]}(v) \leq q - 5, \forall v \in \Gamma$ .

*Proof.* For  $r = q - 3$ , the principal submatrix  $Q_p(V(\mathcal{S}_r^2))$  corresponding to the vertices  $V(\mathcal{S}_r^2) = \{x, y, 1', \dots, r', 1'', \dots, r''\}$  of  $Q(G)$ , is

$$Q_p(V(\mathcal{S}_r^2)) = \begin{bmatrix} q-2 & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 1 & q-2 & 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & 0 & d_G(1') & \dots & a_{1'r'} & a_{1'1''} & \dots & a_{1'r''} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & a_{1'r'} & \dots & d_G(r') & a_{r'1''} & \dots & a_{r'r''} \\ 0 & 1 & a_{1'1''} & \dots & a_{r'1''} & d_G(1'') & \dots & a_{1''r''} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & a_{1'r''} & \dots & a_{r'r''} & a_{1''r''} & \dots & d_G(r'') \end{bmatrix} \tag{3.5}$$

where  $1 \leq d_G(\cdot) \leq q - 2, a_{..} \in \{0, 1\}$ .

(1) We first prove that  $G[N(x)]$  is  $C_{q-t}$ -free for  $q \geq t + 3, 3 \leq t \leq 5$ . Analogously,  $G[N(y)]$  is  $C_{q-t}$ -free follows. Suppose  $G[N(x)]$  is not  $C_{q-t}$ -free, for  $q \geq t + 3; t \in \{3, 4, 5\}$ . The matrix  $Q_p(V(\mathcal{S}_{q-3}^2))$  in (3.5) dominates the following matrix

$$M = \begin{bmatrix} q-2 & 1 & J & J & O \\ 1 & q-2 & O & O & J \\ J & O & Q(C_{q-t}) + I_{q-t} & O & O \\ J & O & O & I_{t-3} & O \\ O & J & O & O & I_{q-3} \end{bmatrix}. \tag{3.6}$$

Note that when  $t = 3, I_{t-3}$  becomes  $I_0$  i.e., the rows and columns corresponding to  $I_0$  does not exist in  $M$ . Therefore, the equitable quotient matrix of  $M$  in (3.6) is given by

$$\mathcal{E}_M = \begin{bmatrix} q-2 & 1 & q-3 & 0 \\ 1 & q-2 & 0 & q-3 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The characteristics polynomial of  $\mathcal{E}_M$  is

$$\mathcal{P}_{\mathcal{E}_M}(x) = x^4 - (2q + 2)x^3 + (q^2 + 6q - 10)x^2 + (-4q^2 + 10q - 4)x + 4q - 12.$$

Note that when  $x = q$ , we have

$$\mathcal{P}_{\mathcal{E}_M}(q) = -12 < 0.$$

Also when  $x = q + 1$ , we get

$$\mathcal{P}_{\mathcal{E}_M}(q + 1) = 3q^2 - 8q - 27 > 0, \quad \text{for } q \geq 6.$$

Thus  $\mathcal{P}_{\mathcal{E}_M}(x)$  has a root in  $(q, q + 1)$  and therefore by interlacing theorem, the spectral radius  $q(G) \geq \rho(Q_p(V(\mathcal{S}_{q-3}^2))) \geq \rho(M) = \rho(\mathcal{E}_M) > q$ , which leads to a contradiction. For  $t = 4, 5$ , using similar techniques, we get  $q(G) > q$ , a contradiction to our assumption.

Therefore we conclude that  $G[N(x)]$  is  $C_{q-t}$ -free where  $q(G) \geq t + 3; t \in \{3, 4, 5\}$ . Hence (1) holds.

(2) Suppose there exist a vertex, say  $1' \in \Gamma$  such that  $d_{G[\mathcal{S}_{q-3}^2]}(1') \geq q - 4$ . Assume that  $1'$  is adjacent to at least  $m$  ( $0 \leq m \leq q - 4$ ) vertices of  $N(y) \cap \Gamma$ , say  $1'', 2'', \dots, m''$  and at least  $q - m - 4$  vertices of  $N(x) \cap \Gamma$ , say  $2', \dots, l'$ , where  $l = q - m - 3$ . For any admissible choices of  $d_G(\cdot), a_{..}$ , the following matrix  $M$  is dominated by the principal submatrix  $Q_p(V(\mathcal{S}_{q-3}^2))$

$$M = \begin{bmatrix} q-2 & 1 & 1 & J & J & O & O \\ 1 & q-2 & 0 & O & O & J & J \\ 1 & 0 & q-3 & J & O & J & O \\ J & O & J & 2I_{q-m-4} & O & O & O \\ J & O & O & O & I_m & O & O \\ O & J & J & O & O & 2I_m & O \\ O & J & O & O & O & O & I_{q-m-3} \end{bmatrix}. \tag{3.7}$$

Now we have the following cases according to the values of  $m$ .

**Case 2.1.**  $m = 0$ .

Using similar techniques as in (1), we get that the equitable quotient matrix  $\mathcal{E}_M$  of  $M$  in (3.7) has an eigenvalue greater than  $q$ . Thus  $\rho(Q_p(V(\mathcal{S}_{q-3}^2))) \geq \rho(M) = \mathcal{E}_M > q$ , which is a contradiction to  $\rho(Q(G)) = q$ . Hence this case is not possible.

**Case 2.2.**  $1 \leq m \leq q - 5$ .

The equitable quotient matrix of  $M$  given in (3.7) is

$$\mathcal{E}_M = \begin{bmatrix} q-2 & 1 & 1 & q-m-4 & m & 0 & 0 \\ 1 & q-2 & 0 & 0 & 0 & m & q-m-3 \\ 1 & 0 & q-3 & q-m-4 & 0 & m & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The characteristics polynomial of  $\mathcal{E}_M$  is

$$\begin{aligned} \mathcal{P}_{\mathcal{E}_M}(x) = & x^7 + (1 - 3q)x^6 + (3q^2 + q - 4)x^5 + (-q^3 - 5q^2 + 6q + 2m + 14)x^4 \\ & + (3q^3 - 8q - 2m^2 - 14m - 36)x^3 + (-2q^3 - 2mq^2 - 10q^2 + 2m^2q \\ & + 12mq + 56q + 2m^2 + 16m - 8)x^2 + (2mq^2 + 12q^2 - 2m^2q - 4mq \\ & - 44q - 8m^2 - 44m)x - 8mq - 8q + 8m^2 + 40m + 32. \end{aligned}$$

When  $x = q$ ,

$$\mathcal{P}_{\mathcal{E}_M}(q) \leq -4q^4 + 44q^3 - 104q^2 - 108q + 32 < 0, \quad \text{for } q \geq 6.$$

At  $x = q + 1$ ,

$$\mathcal{P}_{\mathcal{E}_M}(q + 1) \geq q^4 + 2mq^3 + 55q^3 + 6mq^2 + 19q^2 + 4mq - 75q > 0, \quad \text{for } q \geq 6.$$

Thus  $\mathcal{E}_M$  has an eigenvalue in  $(q, q + 1)$ . Therefore we have  $\rho(Q_p(V(\mathcal{S}_{q-3}^2))) > q$ , a contradiction to our assumption that  $\rho(Q(G)) = q$ . Hence Case 2.2 is not a possible choice.

**Case 2.3.**  $m = q - 4$ .

Similar to Case 2.2, for  $q \geq 7$ , we arrive at a contradiction to our assumption that  $\rho(Q(G)) = q$ . When  $q = 6$ ,  $M$  represents the matrix  $Q(\mathcal{S}^*)$  and here  $\rho(M) = \rho(\mathcal{E}_M) = 6$ . Since  $Q(G), Q_p(V(\mathcal{S}_{q-3}^2)), M$  are non-negative real symmetric matrices and  $Q_p(V(\mathcal{S}_{q-3}^2)) \geq M$ , by Perron-Frobenius Theorem, we have  $6 = \rho(Q(G)) \geq \rho(Q_p(V(\mathcal{S}_{q-3}^2))) \geq \rho(M) = \rho(\mathcal{E}_M) = 6$ . So, this implies  $Q(G) = Q_p(V(\mathcal{S}_{q-3}^2)) = M = Q(\mathcal{S}^*)$ . Therefore  $G = \mathcal{S}^*$ .

From Cases 2.1-2.3, we conclude that either  $d_{G[\Gamma]}(1') \leq q - 5$  or  $G = \mathcal{S}^*$ . Hence (2) holds. □

**Theorem 12.** *Let  $G$  be a connected graph with integral  $q = q(G) \geq 10$ . If  $\mathcal{S}_{q-3}^{2,1}$  is a subgraph of  $G$ , then the following hold.*

(1)  $G[N(x) \setminus \{1'\}], G[N(y) \setminus \{1'\}]$  are  $C_{q-4}$ -free.

(2)  $d_{G[\mathcal{S}_{q-3}^{2,1}]}(v) \leq q - 4, \forall v \in \Gamma$ .

*Proof.* For  $r = q - 3$ , the principal submatrix  $Q_p(\mathcal{S}_r^{2,1})$  of  $Q(G)$  corresponding to the vertex set  $V(\mathcal{S}_r^{2,1}) = \{x, y, 1', 2', \dots, r', 2'', \dots, r''\}$  is

$$Q_p(V(\mathcal{S}_r^{2,1})) = \begin{bmatrix} q-2 & 1 & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 1 & q-2 & 1 & 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & 1 & d_G(1') & a_{1'2'} & \dots & a_{1'r'} & a_{1'2''} & \dots & a_{1'r''} \\ 1 & 0 & a_{1'2'} & d_G(2') & \dots & a_{2'r'} & a_{2'2''} & \dots & a_{2'r''} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & a_{1'r'} & a_{2'r'} & \dots & d_G(r') & a_{r'2''} & \dots & a_{r'r''} \\ 0 & 1 & a_{1'2''} & a_{2'2''} & \dots & a_{r'2''} & d_G(2'') & \dots & a_{2'r''} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & a_{1'r''} & a_{2'r''} & \dots & a_{r'r''} & a_{2'r''} & \dots & d_G(r'') \end{bmatrix}, \tag{3.8}$$

where  $1 \leq d_G(\cdot) \leq q - 2, d_G(1') \geq 2, a_{..} \in \{0, 1\}$ .

(1) Suppose  $G[N(x) \cap (\Gamma \setminus \{1'\})]$  is not  $C_{q-4}$ -free. The principal submatrix  $Q_p(V(\mathcal{S}_{q-3}^{2,1}))$  in (3.8) of  $Q(G)$  dominates the following matrix  $M$ .

$$M = \begin{bmatrix} q-2 & 1 & 1 & J & O \\ 1 & q-2 & 1 & O & J \\ 1 & 1 & 2 & O & O \\ J & O & O & Q(C_{q-4}) + I_{q-4} & O \\ O & J & O & O & I_{q-4} \end{bmatrix}.$$

The equivalent quotient matrix of  $M$  is

$$\mathcal{E}_M = \begin{bmatrix} q-2 & 1 & 1 & q-4 & 0 \\ 1 & q-2 & 1 & 0 & q-4 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The characteristics polynomial of  $\mathcal{E}_M$  is

$$\mathcal{P}_{\mathcal{E}_M}(x) = x^5 - (2q + 4)x^4 + (q^2 + 10q - 6)x^3 + (-6q^2 - 2q + 16)x^2 + (8q^2 - 18q + 13)x - 12q + 28.$$

For  $q \geq 10$ , we have  $\mathcal{P}_{\mathcal{E}_M}(q) < 0$  and  $\mathcal{P}_{\mathcal{E}_M}(q + 1) > 0$ . Therefore  $\mathcal{E}_M$  has an eigenvalue in  $(q, q + 1)$ . Thus  $\rho(Q_p(V(\mathcal{S}_{q-3}^{2,1}))) \geq \rho(M) = \rho(\mathcal{E}_M) > q$ , which is a contradiction to our assumption that  $\rho(Q(G)) = q$ . Hence  $G[N(x) \cap (\Gamma \setminus \{1'\})]$  is  $C_{q-4}$ -free.

Analogously, it can be shown that  $G[N(y) \cap (\Gamma \setminus \{1'\})]$  is also  $C_{q-4}$ -free, and hence (1) holds.

(2) We first prove the statement for the vertex  $1'$  and then prove for any vertex  $v$  in  $\Gamma \setminus \{1'\}$ .

**Case 2.1.**  $d_{G[\Gamma]}(1') \leq q - 6$ .

Suppose  $d_{G[\Gamma]}(1') \geq q - 5$ , that is,  $1'$  is adjacent to at least  $m - 1$  ( $1 \leq m \leq q - 5$ ) vertices of  $N(y) \cap \Gamma$ , say  $2'', \dots, m''$ , and at least  $q - m - 4$  vertices of  $N(x) \cap \Gamma$ , say  $2', \dots, q - m - 3'$ .

For any admissible choices of  $d_G(\cdot), a_{..}$ , the principal submatrix  $Q_p(V(\mathcal{S}_{q-3}^{2,1}))$  in (3.8) corresponding to the vertex set  $V(\mathcal{S}_{q-3}^{2,1}) = N(x) \cup N(y) = \{x, y, 1', 2', \dots, q - m - 3', q - m - 2', \dots, q - 3', 2'', \dots, m'', m + 1'', \dots, q - 3''\}$  dominates the following matrix

$$M = \begin{bmatrix} q-2 & 1 & 1 & J & J & O & O \\ 1 & q-2 & 1 & O & O & J & J \\ 1 & 1 & q-3 & J & O & J & O \\ J & O & J & 2I_{q-m-4} & O & O & O \\ J & O & O & O & I_m & O & O \\ O & J & J & O & O & 2I_{m-1} & O \\ O & J & O & O & O & O & I_{q-m-3} \end{bmatrix}. \tag{3.9}$$

Now we have the following cases according to the values of  $m$ .

**Case 2.1.1.**  $m = 1$ .

The equitable quotient matrix of  $M$  in (3.9) is given by

$$\mathcal{E}_M = \begin{bmatrix} q-2 & 1 & 1 & q-5 & 1 & 0 \\ 1 & q-2 & 1 & 0 & 0 & q-4 \\ 1 & 1 & q-3 & q-5 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which has an eigenvalue in  $(q, q + 2)$ . Therefore  $\rho(Q_p(V(\mathcal{S}_{q-3}^{2,1}))) > q$ , a contradiction to  $q(G) = q$ .

**Case 2.1.2.**  $2 \leq m \leq q - 5$ .

The equitable quotient matrix of  $M$  in (3.9) is

$$\mathcal{E}_M = \begin{bmatrix} q-2 & 1 & 1 & q-m-4 & m & 0 & 0 \\ 1 & q-2 & 1 & 0 & 0 & m-1 & q-m-3 \\ 1 & 1 & q-3 & q-m-4 & 0 & m-1 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The characteristics polynomial of  $\mathcal{E}_M$  is

$$\begin{aligned} \mathcal{P}_{\mathcal{E}_M}(x) = & x^7 + (1 - 3q)x^6 + (3q^2 + q - 3)x^5 + (-q^3 - 5q^2 + 3q + 19)x^4 + (3q^3 + 2q^2 \\ & + 2mq - 14q - 2m^2 - 6m - 26)x^3 + (-2q^3 - 2mq^2 - 14q^2 + 2m^2q + 4mq \\ & + 80q + 2m^2 + 6m - 60)x^2 + (2mq^2 + 14q^2 - 2m^2q + 5mq - 63q - 11m^2 \\ & - 33m + 52)x - 12mq - 4q + 12m^2 + 36m + 16. \end{aligned}$$

We have  $\mathcal{P}_{\mathcal{E}_M}(q) \leq -9q^4 + 79q^3 - 166q^2 - 170q + 136 < 0$ , for  $q \geq 10$ , and  $\mathcal{P}_{\mathcal{E}_M}(q + 2) \geq 9q^4 + 241q^3 + 688q^2 + 280q - 108 > 0$ . Thus  $\mathcal{P}_{\mathcal{E}_M}(x)$  has a root in  $(q, q + 2)$ . Therefore  $\rho(Q_p(V(\mathcal{S}_{q-3}^{2,1}))) > q$ , which is a contradiction to  $q(G) = q$ .

From Cases 2.1.1-2.1.2, we conclude that  $d_{G[\Gamma]}(1') \leq q - 6$  and that  $1'$  is adjacent to both  $x$  and  $y$  implies that  $d_{G[\mathcal{S}_{q-3}^{2,1}]} \leq q - 4$  and thus the Case 2.1 holds.

**Case 2.2.**  $d_{G[\mathcal{S}_{q-3}^{2,1}]}(v) \leq q - 4, \forall v \in \Gamma \setminus \{1'\}$ .

In fact, we will show that  $d_{G[\Gamma \setminus \{1'\}]}(v) \leq q - 6, \forall v \in \Gamma \setminus \{1'\}$ . Assume that there exists a vertex, say  $2' \in \Gamma \setminus \{1'\}$  such that  $d_{G[\Gamma \setminus \{1'\}]}(2') \geq q - 5$ , that is,  $2'$  is adjacent to at least  $m - 1$  vertices of  $N(y) \cap (\Gamma \setminus \{1'\})$  say,  $2'', \dots, m''$  ( $1 \leq m \leq q - 4$ ) and at least  $q - m - 4$  vertices of  $N(x) \cap (\Gamma \setminus \{1'\})$  say,  $3', \dots, q - m - 2'$ .

For any admissible choices of  $d_G(\cdot), a_{..}$ , the following matrix  $M$  is dominated by the principal submatrix  $Q_p(H)$  in (3.8)

$$M = \begin{bmatrix} q-2 & 1 & 1 & 1 & J & J & O & O \\ 1 & q-2 & 1 & 0 & O & O & J & J \\ 1 & 1 & 2 & 0 & O & O & O & O \\ 1 & 0 & 0 & q-4 & J & O & J & O \\ J & O & O & J & 2I_{q-m-4} & O & O & O \\ J & O & O & O & O & I_{m-1} & O & O \\ O & J & O & J & O & O & 2I_{m-1} & O \\ O & J & O & O & O & O & O & I_{q-m-3} \end{bmatrix}. \tag{3.10}$$

Similar to Case 2.1, for  $1 \leq m \leq q - 4$ , the equitable quotient matrix of  $M$  has an eigenvalue greater than  $q$ . Therefore  $\rho(Q_p(V(\mathcal{S}_{q-3}^{2,1}))) > q$ , a contradiction to  $q(G) = q$ . Thus we conclude that  $d_{G[\Gamma \setminus \{1'\}]}(2') \leq q - 6$ . Also, since  $2'$  is adjacent to  $x$  and may be adjacent to  $1'$ , we have that  $d_{G[\mathcal{S}_{q-3}^{2,1}]}(2') \leq q - 4$ . Since  $2' \in \Gamma \setminus \{1'\}$  is an arbitrary vertex, we have  $d_{G[\mathcal{S}_{q-3}^{2,1}]}(v) \leq q - 4, \forall v \in \Gamma \setminus \{1'\}$ .

Hence, from Cases 2.1- 2.2, we have that  $d_{G[\mathcal{S}_{q-3}^{2,1}]}(v) \leq q - 4, \forall v \in \Gamma$ . □

Finally, we end this section with a result which is an improvement to Theorem 2.5, where we identify all possible graphs when we relax the condition of  $Q$ -integrability and restrict that  $\lambda_{\min}(Q(G)) \notin (0, 1)$  and  $q(G)$  to be an integer.

**Theorem 13.** *If  $G$  is a connected edge-non-regular graph with maximum edge-degree equal to 4 and  $\lambda_{\min}(Q(G)) \notin (0, 1), q(G) \in \mathbb{Z}$ , then  $G$  is one of the following graphs.*

- (a)  $G = \mathcal{H}^*$ , shown in Figure 1(a), is the only non-bipartite graph.
- (b)  $G = K_{1,2} \square K_2$ .
- (c)  $G$  is a bipartite graph having  $\mathcal{S}_2^2$  or  $\gamma_1$  (given in Figure 6(a)) as an induced subgraph.

*Proof.* From Theorem 3 and  $e\text{-deg}_G^{\max} = 4$ , we have  $q(G) = 5$  and thus by Lemma 2,  $\mathcal{S}_2^2$  is a subgraph of  $G$ . Let  $V(\mathcal{S}_2^2) = \{x, y, 1', 2', 1'', 2''\}$ , with  $d_G(x) = d_G(y) = 3$ . Let  $N(x) = \{y, 1', 2'\}$  and  $N(y) = \{x, 1'', 2''\}$ . The principal submatrix  $Q_p(V(\mathcal{S}_2^2))$  of  $Q(G)$  is

$$Q_p(V(\mathcal{S}_2^2)) = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 1 \\ 1 & 0 & d_G(1') & a_{1'2'} & a_{1'1''} & a_{1'2''} \\ 1 & 0 & a_{1'2'} & d_G(2') & a_{2'1''} & a_{2'2''} \\ 0 & 1 & a_{1'1''} & a_{2'1''} & d_G(1'') & a_{1''2''} \\ 0 & 1 & a_{1'2''} & a_{2'2''} & a_{1''2''} & d_G(2'') \end{bmatrix}, \tag{3.11}$$

where  $a_{..} \in \{0, 1\}, 1 \leq d_G(\cdot) \leq 3$ . If  $G[\Gamma]$  contains  $P_3$  as a subgraph, then we have the following two choices for  $|E(G[\Gamma])|$  (up to the symmetry  $\mathcal{S}_2^2$ ): (i)  $\{1'2', 1'1''\}$ ; (ii)  $\{1'1'', 1'2''\}$ . In both of these cases, we have  $\rho(Q_p(V(\mathcal{S}_2^2))) > 5$ , a contradiction to  $q(G) = 5$ . Therefore all the edges in  $G[\Gamma]$  are disjoint.

The possible choices for  $E(G[\Gamma])$  are (up to the symmetry  $\mathcal{S}_2^2$ ): (i)  $\{1'2', 1''2''\}$ , (ii)  $\{1'2'\}$ , (iii)  $\{1'1'', 2'2''\}$ , (iv)  $\{1'1''\}$ , (iv)  $\{\phi\}$ .

**Case (i).**  $E(G[\Gamma]) = \{1'2', 1''2''\}$ .

Computationally, one can observe that  $d_G(i') = d_G(i'') = 2, i = 1, 2$  otherwise the spectral radius of the corresponding  $Q_p(V(\mathcal{S}_2^2))$  in (3.11) becomes greater than 5. For this choice of  $d_G(\cdot)$ , we get  $\rho(Q_p(V(\mathcal{S}_2^2))) = 5$ . Therefore  $Q(G) = Q_p(V(\mathcal{S}_2^2))$  implying  $G = \mathcal{H}^*$ .

**Case (ii).**  $E(G[\Gamma]) = \{1'2'\}$ .

The only possible choice of  $d_G(\cdot)$  for which the spectral radius of the corresponding  $Q_p(V(\mathcal{S}_2^2))$  in (3.11) is at most 5 and  $\lambda_{\min}(Q_p(V(\mathcal{S}_2^2))) \geq 1$  is  $d_G(i') = 2, d_G(i'') = 3, i = 1, 2$ . Consider a neighbor of  $1''$ , other than  $y$ , in  $G$ , say  $w$ . Computationally, it can be observed that the spectral radius of the corresponding  $Q_p(V(\mathcal{S}_2^2) \cup \{w\})$  is greater than 5, a contradiction to  $q(G) = 5$ . Thus Case (ii) is not possible.

**Case (iii).**  $E(G[\Gamma]) = \{1'1'', 2'2''\}$ .

The principal submatrix  $Q_p(V(\mathcal{S}_2^2))$  in (3.11) becomes

$$Q_p(V(\mathcal{S}_2^2)) = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 1 \\ 1 & 0 & d_G(1') & 0 & 1 & 0 \\ 1 & 0 & 0 & d_G(2') & 0 & 1 \\ 0 & 1 & 1 & 0 & d_G(1'') & 0 \\ 0 & 1 & 0 & 1 & 0 & d_G(2'') \end{bmatrix},$$

where  $d_G(\cdot) \in \{2, 3\}$ . Computationally, one can observe that  $d_G(\cdot) = 2$  for  $\rho(Q_p(V(\mathcal{S}_2^2)))$  to be at most 5. Moreover, in this case  $\rho(Q_p(V(\mathcal{S}_2^2))) = 5$ , and thus by Perron-Frobenius Theorem we have  $Q(G) = Q_p(V(\mathcal{S}_2^2))$  implying  $G = K_{1,2} \square K_2$ .

**Case (iv).**  $E(G[\Gamma]) = \{1'1''\}$ .

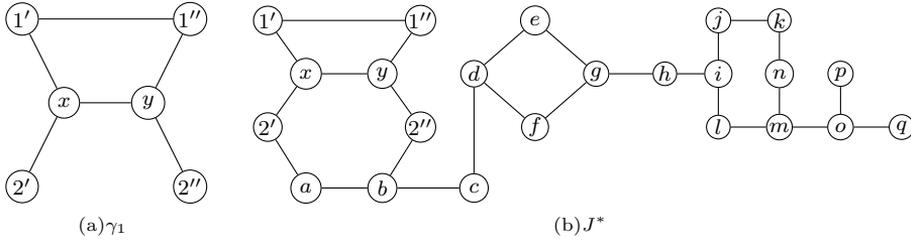
Computationally, one can observe that for each of possible choices of  $d_G(\cdot)$ , the spectral radius of the corresponding  $Q_p(V(\mathcal{S}_2^2))$  in (3.11) becomes less than 1. Thus,  $G$  is bipartite containing  $\gamma_1$  as an induced subgraph.

**Case (v).**  $E(G[\Gamma]) = \{\phi\}$ .

If  $d_G(i') = d_G(i'') = 3, i = 1, 2$ , then the spectral radius of the corresponding

$Q_p(V(\mathcal{S}_2^2))$  in (3.11) is 5, and thus  $Q(G) = Q_p(V(\mathcal{S}_2^2))$ , which is a contradiction. Therefore at least one  $d_G(\cdot) \leq 2$ . However, here the least eigenvalue of the corresponding  $Q_p(V(\mathcal{S}_2^2))$  in (3.11) becomes less than 1 implying  $G$  is bipartite. Also,  $\mathcal{S}_2^2$  is an induced subgraph in this case.

From Cases (i)-(v), we conclude that the theorem holds. □



**Figure 6.** The graphs  $\gamma_1$  and  $J^*$ .

**Remark 4.** There exists at least one bipartite graph as mentioned in Theorem 13(c). We have constructed one such graph  $J^*$ , as shown in Figure 6(b), which has maximum edge-degree = 4,  $q(J^*) = 5$  and whose minimum  $Q$ -eigenvalue is 0. Note that this graph in fact contains both  $\mathcal{S}_2^2$  and  $\gamma_1$  as induced subgraphs.

### 4. Conclusion

In this article, we have studied the structure of simple connected graphs having integral  $Q$ -spectral radius. We have shown that the necessary and sufficient condition for such graphs to contain either a double star  $\mathcal{S}_{q(G)-3}^2$  or its variation  $\mathcal{S}_{q(G)-3}^{2,1}$  as a subgraph is that the maximum edge-degree is  $2q(G) - 6$ .

However, based on our observations and proofs, we propose the following conjectures:

**Conjecture 1.** Every connected  $Q$ -integral graph having  $q(G) \geq 4$  and maximum edge-degree equal to  $2q(G) - 6$  is  $\mathcal{S}_{q(G)-3}^{2,1}$ -free and contains  $\mathcal{S}_{q(G)-3}^2$  as a subgraph.

**Conjecture 2.** Every connected graph with integral  $Q$ -spectral radius  $q(G) \geq 4$ ,  $\lambda_{\min} \notin (0, 1)$  and maximum edge-degree  $2q(G) - 6$  is  $\mathcal{S}_{q(G)-3}^{2,1}$ -free and contains  $\mathcal{S}_{q(G)-3}^2$  as a subgraph.

Observe that Conjecture 2 implies Conjecture 1, that is, if Conjecture 2 is true then Conjecture 1 will also be true. In this context, we have thus shown here that the above conjectures are true when  $4 \leq q(G) \leq 9$ . The above conjectures remain open for  $q(G) \geq 10$ .

In addition, we have also characterized all the connected edge-non-regular graphs having maximum edge-degree equal to 4 whose  $Q$ -spectral radius is an integer and the minimum  $Q$ -eigenvalue does not belong to  $(0, 1)$ .

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