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Research Article

Seidel energy of a graph with self-loops

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Abstract: Let G_S be a graph obtained by attaching a self-loop to each vertex of $S \subseteq V$ of a graph $G(V, E)$. The Seidel matrix of G_S is $S(G_S) = [s_{ij}]$, where $s_{ij} = -1$ if v_i and v_j are adjacent and $v_i \in S$, $s_{ij} = 1$ if v_i and v_j are non-adjacent, and it is zero if $i = j$ and $v_i \notin S$. If $\theta_i(G_S)$, $i = 1, 2, \ldots, n$, are the eigenvalues of the Seidel matrix, then the Seidel energy of the graph G_S, containing n vertices and σ self-loops, is defined as $\sum_{i=1}^{n} \left|\theta_i(G_S) + \frac{\sigma}{n}\right|$. In this paper, some basic properties of Seidel energy of graphs containing self-loops are established.

Keywords: seidel energy (of graph), seidel matrix; energy (of graph), graph with self-loops

AMS Subject classification: 05C50, 05C92

1. Introduction

The concept of graph energy was introduced in the 1970s and since then became a popular subject of mathematical investigation, resulting in over one thousand of published papers [9]. Until quite recently, only graphs without self-loops were considered. The first paper on the energy of graphs with self-loops appeared in 2022 [7], and was followed by a few other articles on the same theme [3, 8, 13, 14]. Although the Seidel energy was studied in detail in dozens of publications, see e.g. [1, 2, 6, 11, 12], until

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now the Seidel energy of graphs with self-loops was not considered. The present paper is aimed at filling this gap.

Let $G(V,E)$ be a simple graph of order n and size m. Let S be a subset of V of order σ. The graph G_S is obtained by attaching a self-loop at each vertex of S. The complement of the graph G, denoted by $\overline{G}$ is the graph with same vertex set as that of G, such that two vertices are adjacent if and only they are non-adjacent in G.

The adjacency matrix $A(G)$ of G on n vertices is a square matrix of order n with elements 1, if the corresponding vertices are adjacent and 0, if the corresponding vertices are non-adjacent. The Seidel matrix $S(G)$ of G of order n is a square matrix of order n with elements $s_{ij} = -1$ if v_i is adjacent to v_j, $s_{ij} = 1$ if v_i is non-adjacent to v_j, and $s_{ij} = 0$ if $i = j$.

2. Preliminaries

Definition 1. [7] The adjacency matrix $A(G_S)$ of a graph G_S of order n with self-loops is an $n \times n$ square matrix with elements,

$$(a_{ij})_S = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent,} \\ 1 & \text{if } i = j \text{ and } v_i \in S, \\ 0 & \text{if } i = j \text{ and } v_i \notin S. \end{cases}$$

Let $\lambda_i(G_S)\,,\ 1 \leq i \leq n$, be the eigenvalues of $A(G_S)$. Then the energy of G_S is defined as

$$E(G_S) = \sum_{i=1}^{n} \left| \lambda_i(G_S) - \frac{\sigma}{n} \right|$$

where σ is the number of self-loops. If $\sigma = 0$, then the above energy reduces to the ordinary graph energy [9], i.e., to the sum of absolute values of the eigenvalues.

Definition 2. The Seidel matrix $S(G_S)$ of a graph G_S of order n with self-loops is an $n \times n$ square matrix with elements,

$$(s_{ij})_S = \begin{cases} -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent,} \\ -1 & \text{if } i = j \text{ and } v_i \in S, \\ 0 & \text{if } i = j \text{ and } v_i \notin S. \end{cases}$$

Let $\theta_i(G_S)$, $i = 1, 2, \ldots, n$, be the be the eigenvalues of $S(G_S)$, pertaining to a graph G_S with $|S| = \sigma$ self-loops. Then directly from Definition 2, it follows

$$\sum_{i=1}^{n} \theta_i(G_S) = -\sigma.$$

Bearing this in mind, the Seidel energy of G_S is defined as

$$SE(G_S) = \sum_{i=1}^{n} \left| \theta_i(G_S) + \frac{\sigma}{n} \right| .$$

If $\sigma = 0$, then the above energy reduces to the ordinary Seidel energy, i.e., to the sum of absolute values of the Seidel eigenvalues.
Note that for a graph G_S of order n,

$$S(G_S) = A(\overline{G}) - A(G_S) = A(\overline{G}) - A(G) + J_S = S(G) + J_S \,.$$

Here, J_S is the square matrix of order n, whose off-diagonal elements are zero, diagonal elements are -1 if the corresponding vertex has a self-loop and it is 0 if the corresponding vertex has no self-loop.

Theorem 1. [4] *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers. If there exist real constants x, y, X and Y such that for each i, $i = 1, 2, \ldots, n$, $x \le x_i \le X$ and $y \le y_i \le Y$, then*

$$\left| n \sum_{i=1}^{n} x_i \, y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \right| \le \alpha(n)(X-x)(Y-y)$$

where $\alpha(n) = n \left[\frac{n}{2}\right] \left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)$. Equality holds if and only if $x_i = x_j$ and $y_i = y_j$ for all $1 \le i, j \le n$.

Theorem 2. [5] *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers. If there exist real constants r and R such that for each i, $i = 1, 2, \ldots, n$, $r\, x_i \le y_i \le R\, x_i$, then*

$$\sum_{i=1}^{n} y_i^2 + rR \sum_{i=1}^{n} x_i^2 \le (r+R) \sum_{i=1}^{n} x_i \, y_i \,.$$

Equality holds if $r\, x_i = y_i = R\, x_i$ for at least one i.

3. Main results

Theorem 3. *Let $(K_n)_S$ be a complete graph with σ self-loops. Then,*

$$SE((K_n)_S) = \frac{n^2 - n - 2\sigma}{n} + \sqrt{(n-1)^2 + 4\sigma} \,. \tag{3.1}$$

Proof. By row and column operation, $\det\big(\lambda I - S((K_n)_S)\big)$ is reduced to

$$(-\lambda)^{\sigma-1}(1-\lambda)^{n-\sigma-1} \begin{vmatrix} -\lambda & -1 \\ -\sigma & 1-\lambda-n \end{vmatrix}$$

and thus the Seidel spectrum of $(K_n)_S$ is

$$\begin{pmatrix} 0 & 1 & \frac{(1-n)+\sqrt{(n-1)^2+4\sigma}}{2} & \frac{(1-n)-\sqrt{(n-1)^2+4\sigma}}{2} \\ \sigma-1 & n-\sigma-1 & 1 & 1 \end{pmatrix}$$

whereas the Seidel energy is

$$\begin{aligned} SE((K_n)_S) &= (\sigma-1)\left(\frac{\sigma}{n}\right) + (n-\sigma-1)\left(1+\frac{\sigma}{n}\right) \\ &\quad + \frac{1-n+\sqrt{(n-1)^2+4\sigma}}{2} + \frac{\sigma}{n} + \frac{n-1+\sqrt{(n-1)^2+4\sigma}}{2} - \frac{\sigma}{n} \end{aligned}$$

and Eq. (3.1) follows. □

Note that the Seidel energy of the complete graph $(K_n)_S$ with σ self-loops is equal to the Seidel energy of the ordinary complete graph if and only if either $\sigma = 0$ or $\sigma = n$.

Theorem 4. *Let $(K_{a,b})_S$, $a \leq b$ be a complete bipartite graph with σ self-loops and $a+b=n$ vertices, $a,b \geq 1$. Then, independently of the actual values of a and b, and independently of the distribution of self-loops,*

$$SE((K_{a,b})_S) = \frac{n^2-3n+2\sigma}{n} + \sqrt{(n-3)^2 - 4(\sigma+2-2n)}\,. \tag{3.2}$$

Proof. Let V_1, V_2 be the partition of the vertex set of $(K_{a,b})_S$ and σ_1, σ_2 be the number vertices having a self-loop in V_1 and V_2, respectively. Let $\sigma = \sigma_1 + \sigma_2$.
The Seidel matrix of $(K_{a,b})_S$ is then a block matrix $\begin{bmatrix} A_{a\times a} & C_{a\times b} \\ C^T_{b\times a} & B_{b\times b} \end{bmatrix}$, where, A and B are square matrices with off-diagonal entries 1 and diagonal entries 1 if the corresponding vertex has a self-loop and 0 if the corresponding vertex does not have a self-loop. C is a matrix with all entries are equal to -1.
By row and column operations, $\det\big(\lambda I - S(K_{a,b})_S\big)$ is reduced to

$$(-\lambda-1)^{n-\sigma_1-\sigma_2-2}(-\lambda-2)^{\sigma_1+\sigma_2-2} \begin{vmatrix} -2-\lambda & -1 & -1 & -1 \\ 0 & -1-\lambda & -3-2\lambda & -3-2\lambda \\ 0 & 0 & -2-\lambda & -1 \\ -\sigma_1 & -a & \sigma_2 - a & b-a-1-\lambda \end{vmatrix}.$$

Expanding the above determinant we get

$$\begin{aligned} & (\lambda+1)(\lambda+2)\Big[\lambda^2 + (3-a-b)\lambda + \sigma_1 + \sigma_2 - 2a - 2b + 2\Big] \\ = \; & (\lambda+1)(\lambda+2)\Big[\lambda^2 + (3-n)\lambda + \sigma - 2n + 2\Big] \end{aligned}$$

from which it follows that the remaining four Seidel eigenvalues are

$$-1\,,\ -2\,,\ \frac{1}{2}\left[(n-3)\pm\sqrt{(n-3)^2-4(\sigma+2-2n)}\right].$$

As a somewhat unexpected finding, the entire Seidel spectrum of $(K_{a,b})_S$ depends only on the sum of a and b, and only on the sum of σ_1 and σ_2, i.e., on n and σ, respectively. Therefore, the Seidel spectrum of the complete bipartite graph $(K_{a,b})_S$ with σ self-loops is,

$$\begin{pmatrix} -1 & -2 & \frac{(n-3)+\sqrt{(n-3)^2-4(\sigma+2-2n)}}{2} & \frac{(n-3)-\sqrt{(n-3)^2-4(\sigma+2-2n)}}{2} \\ n-\sigma-1 & \sigma-1 & 1 & 1 \end{pmatrix}.$$

Its Seidel energy is then

$$\begin{aligned} SE((K_{p,q})_S) &= (\sigma-1)\left(\frac{2n-\sigma}{n}\right)+(n-\sigma-1)\left(\frac{n-\sigma}{n}\right) \\ &\quad+\frac{1}{2}\left[n-3+\sqrt{(n-3)^2-4(\sigma+2-2n)}\right]+\frac{\sigma}{n} \\ &\quad+\frac{1}{2}\left[3-n+\sqrt{(n-3)^2-4(\sigma+2-2n)}\right]-\frac{\sigma}{n} \end{aligned}$$

and Eq. (3.2) follows. □

Note that the Seidel energy of the complete bipartite graph $(K_{a,b})_S$ with σ self-loops is equal to the Seidel energy of the ordinary complete graph if and only if either $\sigma=0$ or $\sigma=n$, where $n=a+b$.

Lemma 1. *Let G_S be a graph containing $|S|=\sigma$ self-loops. Then $SE(G_S)=SE(G)$, if $\sigma=0$ and $\sigma=n$.*

Proof. If $\sigma=0$, then $G_S\cong G$ and therefore, $SE(G_S)=SE(G)$. If $\sigma=n$, then $\theta_i(G_S)=\theta_i(G)-1$ since $S(G_S)=S(G)-I_n$. Therefore,

$$SE(G_S)=\sum_{i=1}^{n}\left|\theta_i(G)-1+\frac{\sigma}{n}\right|=\sum_{i=1}^{n}|\theta_i(G)|=SE(G).$$

□

Lemma 2. *Let $\theta_i(G_S)$, $i=1,2,\ldots,n$, be the Seidel eigenvalues of the graph G_S with $|S|=\sigma$. Then,*

$$\sum_{i=1}^{n}\theta_i^2(G_S)=n(n-1)+\sigma.$$

Proof. Let G_S be a graph obtained from G by adding $|S| = \sigma$ self-loops. Let m, $\overline{m}$ be the number of edges of G and $\overline{G}$ respectively. Consider,

$$\begin{aligned}
\sum_{i=1}^{n} \theta_i^2(G_S) &= \sum_{i=1}^{n} (S(G) + J_S)_{ii}^2 = \sum_{i=1}^{n} \big(A(\overline{G}) - A(G) + J_S\big)_{ii}^2 \\
&= \sum_{i=1}^{n} (A(\overline{G}))_{ii}^2 - \sum_{i=1}^{n} (A(\overline{G})A(G))_{ii} + \sum_{i=1}^{n} (A(\overline{G})J_S)_{ii} \\
&- \sum_{i=1}^{n} (A(G)A(\overline{G}))_{ii} + \sum_{i=1}^{n} ((A(G))^2)_{ii} - \sum_{i=1}^{n} (A(G)J_S)_{ii} \\
&+ \sum_{i=1}^{n} (J_S A(\overline{G}))_{ii} + \sum_{i=1}^{n} (J_S A(G))_{ii} - \sum_{i=1}^{n} (J_S)_{ii}^2
\end{aligned}$$

and note that

$$\begin{aligned}
&\sum_{i=1}^{n} (A(\overline{G}))_{ii}^2 = 2\overline{m} \quad , \quad \sum_{i=1}^{n} (A(G))_{ii}^2 = 2m \quad , \quad \sum_{i=1}^{n} (J_S)_{ii}^2 = \sigma \\
&\sum_{i=1}^{n} (A(\overline{G})A(G))_{ii} = \sum_{i=1}^{n} (A(G)A(\overline{G}))_{ii} = 0 \\
&\sum_{i=1}^{n} (A(\overline{G})J_S)_{ii} = \sum_{i=1}^{n} (J_S A(\overline{G}))_{ii} = 0 \\
&\sum_{i=1}^{n} (A(G)J_S)_{ii} = \sum_{i=1}^{n} (J_S A(G))_{ii} = 0\,.
\end{aligned}$$

Lemma 2 follows now by recalling that $m + \overline{m} = \binom{n}{2}$. □

Lemma 3. *Let $\theta_i(G_S)$, $i = 1, 2, \ldots, n$, be the Seidel eigenvalues of the graph G_S with $|S| = \sigma$. Then,*

$$\sum_{i=1}^{n} \left| \theta_i(G_S) + \frac{\sigma}{n} \right|^2 = n(n-1) + \sigma - \frac{\sigma^2}{n}\,.$$

Proof.

$$\sum_{i=1}^{n} \left| \theta_i(G_S) + \frac{\sigma}{n} \right|^2 = \sum_{i=1}^{n} \left(\theta_i^2(G_S) + 2\theta_i(G_S)\frac{\sigma}{n} + \frac{\sigma^2}{n^2} \right) = 2\overline{m} + 2m + \sigma - \frac{\sigma^2}{n}\,.$$

□

Theorem 5. *For the graph G_S, obtained by attaching σ self-loops to the vertices of G on n vertices,*

$$SE(G_S) \le n\sqrt{n - 1 + \frac{\sigma}{n} - \left(\frac{\sigma}{n}\right)^2}.$$

Proof. The quantity $\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\theta_i(G_S)+\frac{\sigma}{n}\right|-\left|\theta_j(G_S)+\frac{\sigma}{n}\right|\right)^2$ is non-negative. Therefore,

$$n\sum_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|^2+n\sum_{j=1}^{n}\left|\theta_j(G_S)+\frac{\sigma}{n}\right|^2-2\sum_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|\sum_{j=1}^{n}\left|\theta_j(G_S)+\frac{\sigma}{n}\right|\geq 0.$$

By Lemma 3 and from the definition of Seidel energy of G_S, it follows that

$$2\,SE(G_S)^2\leq 2n\left(n(n-1)+\sigma-\frac{\sigma^2}{n}\right)$$

which implies Theorem 5. □

Theorem 6. *Let G_S be a graph obtained by attaching σ self-loops to the vertices of G on n vertices. Then,*

$$SE(G_S)\geq\sqrt{n\left(n(n-1)+\sigma-\frac{\sigma^2}{n}\right)+n(n-1)D^{2/n}}$$

where $D=\left|\det\left(S(G_S)+\frac{\sigma}{n}I_n\right)\right|$.

Proof. By the arithmetic-geometric mean inequality,

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i\neq j}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|\left|\theta_j(G_S)+\frac{\sigma}{n}\right| &\geq\left(\prod_{i\neq j}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|\left|\theta_j(G_S)+\frac{\sigma}{n}\right|\right)^{\frac{1}{n(n-1)}}\\ &=\left(\prod_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}\\ &=\left(\prod_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|\right)^{2/n}\\ &=\left|\prod_{i=1}^{n}\left(\theta_i(G_S)+\frac{\sigma}{n}\right)\right|^{2/n}\\ &=\left|\det\left(S(G_S)+\frac{\sigma}{n}I_n\right)\right|^{2/n}=D^{2/n}.\end{aligned}$$

Therefore,

$$\sum_{i\neq j}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|\left|\theta_j(G_S)+\frac{\sigma}{n}\right|\geq n(n-1)D^{2/n}. \tag{3.3}$$

Now consider,

$$\begin{aligned} SE(G_S)^2 &= \left(\sum_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|\right)^2 \\ &= \sum_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|^2 + \sum_{i\neq j}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|\left|\theta_j(G_S)+\frac{\sigma}{n}\right|. \end{aligned} \tag{3.4}$$

Theorem 6 follows from Lemma 3 by substituting Eq. (3.3) into Eq. (3.4). □

It should be noted that Theorems 5 and 6 are obtained by reasonings analogous to those used in the theory of ordinary graph energy [9, 10].

Theorem 7. *Let $\theta_1(G_S), \theta_2(G_S), \ldots, \theta_n(G_S)$ be the Seidel eigenvalues of the graph G_S containing σ self-loops. Then,*

$$SE(G_S) \geq n\sqrt{n-1+\frac{\sigma}{n}-\left(\frac{\sigma}{n}\right)^2-\frac{1}{4}\left(\left|\theta_1(G_S)+\frac{\sigma}{n}\right|-\left|\theta_n(G_S)+\frac{\sigma}{n}\right|\right)^2}. \tag{3.5}$$

Proof. Let $\left|\theta_1(G_S)+\frac{\sigma}{n}\right| \geq \left|\theta_2(G_S)+\frac{\sigma}{n}\right| \geq \cdots \geq \left|\theta_n(G_S)+\frac{\sigma}{n}\right|$. By substituting $x_i = y_i = \left|\theta_i(G_S)+\frac{\sigma}{n}\right|$, $x = y = \left|\theta_n(G_S)+\frac{\sigma}{n}\right|$, and $X = Y = \left|\theta_1(G_S)+\frac{\sigma}{n}\right|$ in Theorem 1 and noting that $\alpha(n) \leq \frac{n^2}{4}$, we get

$$\left|n\sum_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|^2-\left(\sum_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|\right)^2\right| \leq \frac{n^2}{4}\left(\left|\theta_1(G_S)+\frac{\sigma}{n}\right|-\left|\theta_n(G_S)+\frac{\sigma}{n}\right|\right)^2.$$

Noting that

$$\sum_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right| = SE(G_S)$$

and

$$\sum_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|^2 = 2\overline{m}+2m+\sigma-\frac{\sigma^2}{n} = n(n-1)+\sigma-\frac{\sigma^2}{n} \tag{3.6}$$

we obtain

$$n\left(n(n-1)+\sigma-\frac{\sigma^2}{n}\right)-(SE(G_S))^2 \leq \frac{n^2}{4}\left(\left|\theta_1(G_S)+\frac{\sigma}{n}\right|-\left|\theta_n(G_S)+\frac{\sigma}{n}\right|\right)^2.$$

This implies the inequality (3.5). □

Theorem 8. *Using the same notation as in Theorem 7,*

$$SE(G_S) \geq \frac{n(n-1)+\sigma-\frac{\sigma^2}{n}+\left|\theta_1(G_S)+\frac{\sigma}{n}\right|\left|\theta_n(G_S)+\frac{\sigma}{n}\right|}{\left|\theta_1(G_S)+\frac{\sigma}{n}\right|+\left|\theta_n(G_S)+\frac{\sigma}{n}\right|}.$$

Proof. Let, as before, $\left|\theta_1(G_S)+\frac{\sigma}{n}\right| \geq \left|\theta_2(G_S)+\frac{\sigma}{n}\right| \geq \cdots \geq \left|\theta_n(G_S)+\frac{\sigma}{n}\right|$. By substituting $x_i=1$, $y_i=\left|\theta_i(G_S)+\frac{\sigma}{n}\right|$, $r=\left|\theta_n(G_S)+\frac{\sigma}{n}\right|$, and $R=\left|\theta_1(G_S)+\frac{\sigma}{n}\right|$ in Theorem 2, we get

$$\sum_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|^2+\left|\theta_1(G_S)+\frac{\sigma}{n}\right|\left|\theta_n(G_S)+\frac{\sigma}{n}\right|\sum_{i=1}^{n}1^2$$
$$\leq\left(\left|\theta_1(G_S)+\frac{\sigma}{n}\right|+\left|\theta_n(G_S)+\frac{\sigma}{n}\right|\right)\sum_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|.$$

Taking into account

$$\sum_{i=1}^{n}\left|\theta_i(G_S)+\frac{\sigma}{n}\right|=SE(G_S)$$

and Eq. (3.6), we get

$$n(n-1)+\sigma-\frac{\sigma^2}{n}+n\left|\theta_1(G_S)+\frac{\sigma}{n}\right|\left|\theta_n(G_S)+\frac{\sigma}{n}\right|$$
$$\leq\left(\left|\theta_1(G_S)+\frac{\sigma}{n}\right|+\left|\theta_n(G_S)+\frac{\sigma}{n}\right|\right)SE(G_S)$$

from which the inequality in Theorem 8 straightforwardly follows. □

4. Conclusion

In this article we have obtained some basic results on the Seidel energy of graphs with self-loops, together with a few lower and upper bounds. The Seidel energy of complete and complete bipartite graphs with self-loops are also calculated.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] S. Akbari, J. Askari, and K.C. Das, *Some properties of eigenvalues of the seidel matrix*, Linear Multilinear Algebra **70** (2022), no. 11, 2150–2161.
https://doi.org/10.1080/03081087.2020.1790481.

[2] S. Akbari, M. Einollahzadeh, M.M. Karkhaneei, and M.A. Nematollahi, *Proof of a conjecture on the seidel energy of graphs*, European J. Combin. **86** (2020), Article ID: 103078.
https://doi.org/10.1016/j.ejc.2019.103078.

[3] D.V. Anchan, S. D'Souza, H.J. Gowtham, and P.G. Bhat, *Laplacian energy of a graph with self-loops*, MATCH Commun. Math. Comput. Chem. **90** (2023), no. 1, 247–258.
https://doi.org/10.46793/match.90-1.247V.

[4] M. Biernacki, H. Pidek, and C. Ryll-Nardzewski, *Sur une inégalité entre des intégrales définies*, Ann. Univ. Mariae Curie-Sklodowska **4** (1950), 1–4.

[5] J.B. Díaz and F.T. Metcalf, *Stronger forms of a class of inequalities of G. Pólya-G. Szegö, and L.V. Kantorovich*, Bull. Am. Math. Soc. **69** (1963), 415–418.

[6] M. Einollahzadeh and M.A. Nematollahi, *An improved lower bound for the seidel energy of trees*, Discrete Appl. Math. **320** (2022), 381–386.
https://doi.org/10.1016/j.dam.2022.06.012.

[7] I. Gutman, I. Redžepović, B. Furtula, and A. Sahal, *Energy of graphs with self-loops*, MATCH Commun. Math. Comput. Chem. **87** (2022), 645–652.

[8] I. Jovanović, E. Zogić, and E. Glogić, *On the conjecture related to the energy of graphs with self-loops*, MATCH Commun. Math. Comput. Chem. **89** (2023), no. 2, 479–488.
https://doi.org/10.46793/match.89-2.479J.

[9] X. Li, Y. Shi, and I. Gutman, *Graph Energy*, Springer, New York, 2012.

[10] B.J. McClelland, *Properties of the latent roots of a matrix: The estimation of π-electron energies*, J. Chem. Phys. **54** (1971), no. 2, 640–643.
https://doi.org/10.1063/1.1674889.

[11] M.R. Oboudi, *Energy and seidel energy of graphs*, MATCH Commun. Math. Comput. Chem. **75** (2016), 291–303.

[12] M.R. Oboudi and M.A. Nematollah, *Improving a lower bound for seidel energy of graphs*, MATCH Commun. Math. Comput. Chem. **89** (2023), no. 2, 489–502.
https://doi.org/10.46793/match.89-2.489O.

[13] K.M. Popat and K.R. Shingala, *Some new results on energy of graphs with self loops*, J. Math. Chem. **61** (2023), no. 6, 1462–1469.
https://doi.org/10.1007/s10910-023-01467-7.

[14] U.P. Preetha, M. Suresh, and E. Bonyah, *On the spectrum, energy and laplacian energy of graphs with self-loops*, Heliyon **9** (2023), no. 7, #e17001.
https://doi.org/10.1016/j.heliyon.2023.e17001.