

Seidel energy of a graph with self-loops

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Abstract: Let G_S be a graph obtained by attaching a self-loop to each vertex of $S \subseteq V$ of a graph $G(V, E)$. The Seidel matrix of G_S is $S(G_S) = [s_{ij}]$, where $s_{ij} = -1$ if v_i and v_j are adjacent and $v_i \in S$, $s_{ij} = 1$ if v_i and v_j are non-adjacent, and it is zero if $i = j$ and $v_i \notin S$. If $\theta_i(G_S)$, $i = 1, 2, \dots, n$, are the eigenvalues of the Seidel matrix, then the Seidel energy of the graph G_S , containing n vertices and σ self-loops, is defined as $\sum_{i=1}^n |\theta_i(G_S) + \frac{\sigma}{n}|$. In this paper, some basic properties of Seidel energy of graphs containing self-loops are established.

Keywords: seidel energy (of graph), seidel matrix; energy (of graph), graph with self-loops

AMS Subject classification: 05C50, 05C92

1. Introduction

The concept of graph energy was introduced in the 1970s and since then became a popular subject of mathematical investigation, resulting in over one thousand of published papers [9]. Until quite recently, only graphs without self-loops were considered. The first paper on the energy of graphs with self-loops appeared in 2022 [7], and was

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followed by a few other articles on the same theme [3, 8, 13, 14]. Although the Seidel energy was studied in detail in dozens of publications, see e.g. [1, 2, 6, 11, 12], until now the Seidel energy of graphs with self-loops was not considered. The present paper is aimed at filling this gap.

Let $G(V, E)$ be a simple graph of order n and size m . Let S be a subset of V of order σ . The graph G_S is obtained by attaching a self-loop at each vertex of S . The complement of the graph G , denoted by \overline{G} is the graph with same vertex set as that of G , such that two vertices are adjacent if and only they are non-adjacent in G .

The adjacency matrix $A(G)$ of G on n vertices is a square matrix of order n with elements 1, if the corresponding vertices are adjacent and 0, if the corresponding vertices are non-adjacent. The Seidel matrix $S(G)$ of G of order n is a square matrix of order n with elements $s_{ij} = -1$ if v_i is adjacent to v_j , $s_{ij} = 1$ if v_i is non-adjacent to v_j , and $s_{ij} = 0$ if $i = j$.

2. Preliminaries

Definition 1. [7] The adjacency matrix $A(G_S)$ of a graph G_S of order n with self-loops is an $n \times n$ square matrix with elements,

$$(a_{ij})_S = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent,} \\ 1 & \text{if } i = j \text{ and } v_i \in S, \\ 0 & \text{if } i = j \text{ and } v_i \notin S. \end{cases}$$

Let $\lambda_i(G_S)$, $1 \leq i \leq n$, be the eigenvalues of $A(G_S)$. Then the energy of G_S is defined as

$$E(G_S) = \sum_{i=1}^n \left| \lambda_i(G_S) - \frac{\sigma}{n} \right|$$

where σ is the number of self-loops. If $\sigma = 0$, then the above energy reduces to the ordinary graph energy [9], i.e., to the sum of absolute values of the eigenvalues.

Definition 2. The Seidel matrix $S(G_S)$ of a graph G_S of order n with self-loops is an $n \times n$ square matrix with elements,

$$(s_{ij})_S = \begin{cases} -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent,} \\ -1 & \text{if } i = j \text{ and } v_i \in S, \\ 0 & \text{if } i = j \text{ and } v_i \notin S. \end{cases}$$

Let $\theta_i(G_S)$, $i = 1, 2, \dots, n$, be the be the eigenvalues of $S(G_S)$, pertaining to a graph G_S with $|S| = \sigma$ self-loops. Then directly from Definition 2, it follows

$$\sum_{i=1}^n \theta_i(G_S) = -\sigma.$$

Bearing this in mind, the Seidel energy of G_S is defined as

$$SE(G_S) = \sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right|.$$

If $\sigma = 0$, then the above energy reduces to the ordinary Seidel energy, i.e., to the sum of absolute values of the Seidel eigenvalues.

Note that for a graph G_S of order n ,

$$S(G_S) = A(\overline{G}) - A(G_S) = A(\overline{G}) - A(G) + J_S = S(G) + J_S.$$

Here, J_S is the square matrix of order n , whose off-diagonal elements are zero, diagonal elements are -1 if the corresponding vertex has a self-loop and it is 0 if the corresponding vertex has no self-loop.

Theorem 1. [4] *Let x_1, x_2, \dots, x_n and y_1, y_2, \dots, y_n be real numbers. If there exist real constants x, y, X and Y such that for each $i, i = 1, 2, \dots, n, x \leq x_i \leq X$ and $y \leq y_i \leq Y$, then*

$$\left| n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i \right| \leq \alpha(n)(X - x)(Y - y)$$

where $\alpha(n) = n \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right)$. Equality holds if and only if $x_i = x_j$ and $y_i = y_j$ for all $1 \leq i, j \leq n$.

Theorem 2. [5] *Let x_1, x_2, \dots, x_n and y_1, y_2, \dots, y_n be real numbers. If there exist real constants r and R such that for each $i, i = 1, 2, \dots, n, r x_i \leq y_i \leq R x_i$, then*

$$\sum_{i=1}^n y_i^2 + rR \sum_{i=1}^n x_i^2 \leq (r + R) \sum_{i=1}^n x_i y_i.$$

Equality holds if $r x_i = y_i = R x_i$ for at least one i .

3. Main results

Theorem 3. *Let $(K_n)_S$ be a complete graph with σ self-loops. Then,*

$$SE((K_n)_S) = \frac{n^2 - n - 2\sigma}{n} + \sqrt{(n - 1)^2 + 4\sigma}. \tag{3.1}$$

Proof. By row and column operation, $\det(\lambda I - S((K_n)_S))$ is reduced to

$$(-\lambda)^{\sigma-1} (1 - \lambda)^{n-\sigma-1} \begin{vmatrix} -\lambda & -1 \\ -\sigma & 1 - \lambda - n \end{vmatrix}$$

and thus the Seidel spectrum of $(K_n)_S$ is

$$\begin{pmatrix} 0 & 1 & \frac{(1-n)+\sqrt{(n-1)^2+4\sigma}}{2} & \frac{(1-n)-\sqrt{(n-1)^2+4\sigma}}{2} \\ \sigma-1 & n-\sigma-1 & 1 & 1 \end{pmatrix}$$

whereas the Seidel energy is

$$\begin{aligned} SE((K_n)_S) &= (\sigma-1) \left(\frac{\sigma}{n}\right) + (n-\sigma-1) \left(1 + \frac{\sigma}{n}\right) \\ &+ \frac{1-n+\sqrt{(n-1)^2+4\sigma}}{2} + \frac{\sigma}{n} + \frac{n-1+\sqrt{(n-1)^2+4\sigma}}{2} - \frac{\sigma}{n} \end{aligned}$$

and Eq. (3.1) follows. □

Note that the Seidel energy of the complete graph $(K_n)_S$ with σ self-loops is equal to the Seidel energy of the ordinary complete graph if and only if either $\sigma = 0$ or $\sigma = n$.

Theorem 4. *Let $(K_{a,b})_S$, $a \leq b$ be a complete bipartite graph with σ self-loops and $a + b = n$ vertices, $a, b \geq 1$. Then, independently of the actual values of a and b , and independently of the distribution of self-loops,*

$$SE((K_{a,b})_S) = \frac{n^2 - 3n + 2\sigma}{n} + \sqrt{(n-3)^2 - 4(\sigma + 2 - 2n)}. \tag{3.2}$$

Proof. Let V_1, V_2 be the partition of the vertex set of $(K_{a,b})_S$ and σ_1, σ_2 be the number vertices having a self-loop in V_1 and V_2 , respectively. Let $\sigma = \sigma_1 + \sigma_2$.

The Seidel matrix of $(K_{a,b})_S$ is then a block matrix $\begin{bmatrix} A_{a \times a} & C_{a \times b} \\ C_{b \times a}^T & B_{b \times b} \end{bmatrix}$, where, A and B are square matrices with off-diagonal entries 1 and diagonal entries 1 if the corresponding vertex has a self-loop and 0 if the corresponding vertex does not have a self-loop. C is a matrix with all entries are equal to -1 .

By row and column operations, $\det(\lambda I - S(K_{a,b})_S)$ is reduced to

$$(-\lambda - 1)^{n-\sigma_1-\sigma_2-2} (-\lambda - 2)^{\sigma_1+\sigma_2-2} \begin{vmatrix} -2-\lambda & -1 & -1 & -1 \\ 0 & -1-\lambda & -3-2\lambda & -3-2\lambda \\ 0 & 0 & -2-\lambda & -1 \\ -\sigma_1 & -a & \sigma_2-a & b-a-1-\lambda \end{vmatrix}.$$

Expanding the above determinant we get

$$\begin{aligned} &(\lambda + 1)(\lambda + 2) \left[\lambda^2 + (3 - a - b)\lambda + \sigma_1 + \sigma_2 - 2a - 2b + 2 \right] \\ &= (\lambda + 1)(\lambda + 2) \left[\lambda^2 + (3 - n)\lambda + \sigma - 2n + 2 \right] \end{aligned}$$

from which it follows that the remaining four Seidel eigenvalues are

$$-1, -2, \frac{1}{2} \left[(n-3) \pm \sqrt{(n-3)^2 - 4(\sigma+2-2n)} \right].$$

As a somewhat unexpected finding, the entire Seidel spectrum of $(K_{a,b})_S$ depends only on the sum of a and b , and only on the sum of σ_1 and σ_2 , i.e., on n and σ , respectively. Therefore, the Seidel spectrum of the complete bipartite graph $(K_{a,b})_S$ with σ self-loops is,

$$\begin{pmatrix} -1 & -2 & \frac{(n-3)+\sqrt{(n-3)^2-4(\sigma+2-2n)}}{2} & \frac{(n-3)-\sqrt{(n-3)^2-4(\sigma+2-2n)}}{2} \\ n-\sigma-1 & \sigma-1 & 1 & 1 \end{pmatrix}.$$

Its Seidel energy is then

$$\begin{aligned} SE((K_{p,q})_S) &= (\sigma - 1) \left(\frac{2n - \sigma}{n} \right) + (n - \sigma - 1) \left(\frac{n - \sigma}{n} \right) \\ &\quad + \frac{1}{2} \left[n - 3 + \sqrt{(n - 3)^2 - 4(\sigma + 2 - 2n)} \right] + \frac{\sigma}{n} \\ &\quad + \frac{1}{2} \left[3 - n + \sqrt{(n - 3)^2 - 4(\sigma + 2 - 2n)} \right] - \frac{\sigma}{n} \end{aligned}$$

and Eq. (3.2) follows. □

Note that the Seidel energy of the complete bipartite graph $(K_{a,b})_S$ with σ self-loops is equal to the Seidel energy of the ordinary complete graph if and only if either $\sigma = 0$ or $\sigma = n$, where $n = a + b$.

Lemma 1. *Let G_S be a graph containing $|S| = \sigma$ self-loops. Then $SE(G_S) = SE(G)$, if $\sigma = 0$ and $\sigma = n$.*

Proof. If $\sigma = 0$, then $G_S \cong G$ and therefore, $SE(G_S) = SE(G)$. If $\sigma = n$, then $\theta_i(G_S) = \theta_i(G) - 1$ since $S(G_S) = S(G) - I_n$. Therefore,

$$SE(G_S) = \sum_{i=1}^n \left| \theta_i(G) - 1 + \frac{\sigma}{n} \right| = \sum_{i=1}^n |\theta_i(G)| = SE(G).$$

□

Lemma 2. *Let $\theta_i(G_S)$, $i = 1, 2, \dots, n$, be the Seidel eigenvalues of the graph G_S with $|S| = \sigma$. Then,*

$$\sum_{i=1}^n \theta_i^2(G_S) = n(n-1) + \sigma.$$

Proof. Let G_S be a graph obtained from G by adding $|S| = \sigma$ self-loops. Let m, \bar{m} be the number of edges of G and \bar{G} respectively. Consider,

$$\begin{aligned} \sum_{i=1}^n \theta_i^2(G_S) &= \sum_{i=1}^n (S(G) + J_S)_{ii}^2 = \sum_{i=1}^n (A(\bar{G}) - A(G) + J_S)_{ii}^2 \\ &= \sum_{i=1}^n (A(\bar{G}))_{ii}^2 - \sum_{i=1}^n (A(\bar{G})A(G))_{ii} + \sum_{i=1}^n (A(\bar{G})J_S)_{ii} \\ &\quad - \sum_{i=1}^n (A(G)A(\bar{G}))_{ii} + \sum_{i=1}^n ((A(G))^2)_{ii} - \sum_{i=1}^n (A(G)J_S)_{ii} \\ &\quad + \sum_{i=1}^n (J_S A(\bar{G}))_{ii} + \sum_{i=1}^n (J_S A(G))_{ii} - \sum_{i=1}^n (J_S)_{ii}^2 \end{aligned}$$

and note that

$$\begin{aligned} \sum_{i=1}^n (A(\bar{G}))_{ii}^2 &= 2\bar{m} \quad , \quad \sum_{i=1}^n (A(G))_{ii}^2 = 2m \quad , \quad \sum_{i=1}^n (J_S)_{ii}^2 = \sigma \\ \sum_{i=1}^n (A(\bar{G})A(G))_{ii} &= \sum_{i=1}^n (A(G)A(\bar{G}))_{ii} = 0 \\ \sum_{i=1}^n (A(\bar{G})J_S)_{ii} &= \sum_{i=1}^n (J_S A(\bar{G}))_{ii} = 0 \\ \sum_{i=1}^n (A(G)J_S)_{ii} &= \sum_{i=1}^n (J_S A(G))_{ii} = 0. \end{aligned}$$

Lemma 2 follows now by recalling that $m + \bar{m} = \binom{n}{2}$. □

Lemma 3. Let $\theta_i(G_S), i = 1, 2, \dots, n$, be the Seidel eigenvalues of the graph G_S with $|S| = \sigma$. Then,

$$\sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right|^2 = n(n-1) + \sigma - \frac{\sigma^2}{n}.$$

Proof.

$$\sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right|^2 = \sum_{i=1}^n \left(\theta_i^2(G_S) + 2\theta_i(G_S)\frac{\sigma}{n} + \frac{\sigma^2}{n^2} \right) = 2\bar{m} + 2m + \sigma - \frac{\sigma^2}{n}.$$

□

Theorem 5. For the graph G_S , obtained by attaching σ self-loops to the vertices of G on n vertices,

$$SE(G_S) \leq n \sqrt{n-1 + \frac{\sigma}{n} - \left(\frac{\sigma}{n}\right)^2}.$$

Proof. The quantity $\sum_{i=1}^n \sum_{j=1}^n \left(\left| \theta_i(G_S) + \frac{\sigma}{n} \right| - \left| \theta_j(G_S) + \frac{\sigma}{n} \right| \right)^2$ is non-negative. Therefore,

$$n \sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right|^2 + n \sum_{j=1}^n \left| \theta_j(G_S) + \frac{\sigma}{n} \right|^2 - 2 \sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right| \sum_{j=1}^n \left| \theta_j(G_S) + \frac{\sigma}{n} \right| \geq 0.$$

By Lemma 3 and from the definition of Seidel energy of G_S , it follows that

$$2 SE(G_S)^2 \leq 2n \left(n(n-1) + \sigma - \frac{\sigma^2}{n} \right)$$

which implies Theorem 5. □

Theorem 6. Let G_S be a graph obtained by attaching σ self-loops to the vertices of G on n vertices. Then,

$$SE(G_S) \geq \sqrt{n \left(n(n-1) + \sigma - \frac{\sigma^2}{n} \right) + n(n-1)D^{2/n}}$$

where $D = |\det (S(G_S) + \frac{\sigma}{n} I_n)|$.

Proof. By the arithmetic-geometric mean inequality,

$$\begin{aligned} \frac{1}{n(n-1)} \sum_{i \neq j} \left| \theta_i(G_S) + \frac{\sigma}{n} \right| \left| \theta_j(G_S) + \frac{\sigma}{n} \right| &\geq \left(\prod_{i \neq j} \left| \theta_i(G_S) + \frac{\sigma}{n} \right| \left| \theta_j(G_S) + \frac{\sigma}{n} \right| \right)^{\frac{1}{n(n-1)}} \\ &= \left(\prod_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right|^{2(n-1)} \right)^{\frac{1}{n(n-1)}} \\ &= \left(\prod_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right| \right)^{2/n} \\ &= \left| \prod_{i=1}^n \left(\theta_i(G_S) + \frac{\sigma}{n} \right) \right|^{2/n} \\ &= \left| \det \left(S(G_S) + \frac{\sigma}{n} I_n \right) \right|^{2/n} = D^{2/n}. \end{aligned}$$

Therefore,

$$\sum_{i \neq j} \left| \theta_i(G_S) + \frac{\sigma}{n} \right| \left| \theta_j(G_S) + \frac{\sigma}{n} \right| \geq n(n-1)D^{2/n}. \tag{3.3}$$

Now consider,

$$\begin{aligned}
 SE(G_S)^2 &= \left(\sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right| \right)^2 \\
 &= \sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right|^2 + \sum_{i \neq j} \left| \theta_i(G_S) + \frac{\sigma}{n} \right| \left| \theta_j(G_S) + \frac{\sigma}{n} \right|. \tag{3.4}
 \end{aligned}$$

Theorem 6 follows from Lemma 3 by substituting Eq. (3.3) into Eq. (3.4). □

It should be noted that Theorems 5 and 6 are obtained by reasonings analogous to those used in the theory of ordinary graph energy [9, 10].

Theorem 7. *Let $\theta_1(G_S), \theta_2(G_S), \dots, \theta_n(G_S)$ be the Seidel eigenvalues of the graph G_S containing σ self-loops. Then,*

$$SE(G_S) \geq n \sqrt{n - 1 + \frac{\sigma}{n} - \left(\frac{\sigma}{n}\right)^2 - \frac{1}{4} \left(\left| \theta_1(G_S) + \frac{\sigma}{n} \right| - \left| \theta_n(G_S) + \frac{\sigma}{n} \right| \right)^2}. \tag{3.5}$$

Proof. Let $\left| \theta_1(G_S) + \frac{\sigma}{n} \right| \geq \left| \theta_2(G_S) + \frac{\sigma}{n} \right| \geq \dots \geq \left| \theta_n(G_S) + \frac{\sigma}{n} \right|$. By substituting $x_i = y_i = \left| \theta_i(G_S) + \frac{\sigma}{n} \right|$, $x = y = \left| \theta_n(G_S) + \frac{\sigma}{n} \right|$, and $X = Y = \left| \theta_1(G_S) + \frac{\sigma}{n} \right|$ in Theorem 1 and noting that $\alpha(n) \leq \frac{n^2}{4}$, we get

$$\left| n \sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right|^2 - \left(\sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right| \right)^2 \right| \leq \frac{n^2}{4} \left(\left| \theta_1(G_S) + \frac{\sigma}{n} \right| - \left| \theta_n(G_S) + \frac{\sigma}{n} \right| \right)^2.$$

Noting that

$$\sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right| = SE(G_S)$$

and

$$\sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right|^2 = 2\bar{m} + 2m + \sigma - \frac{\sigma^2}{n} = n(n - 1) + \sigma - \frac{\sigma^2}{n} \tag{3.6}$$

we obtain

$$n \left(n(n - 1) + \sigma - \frac{\sigma^2}{n} \right) - (SE(G_S))^2 \leq \frac{n^2}{4} \left(\left| \theta_1(G_S) + \frac{\sigma}{n} \right| - \left| \theta_n(G_S) + \frac{\sigma}{n} \right| \right)^2.$$

This implies the inequality (3.5). □

Theorem 8. *Using the same notation as in Theorem 7,*

$$SE(G_S) \geq \frac{n(n - 1) + \sigma - \frac{\sigma^2}{n} + \left| \theta_1(G_S) + \frac{\sigma}{n} \right| \left| \theta_n(G_S) + \frac{\sigma}{n} \right|}{\left| \theta_1(G_S) + \frac{\sigma}{n} \right| + \left| \theta_n(G_S) + \frac{\sigma}{n} \right|}.$$

Proof. Let, as before, $|\theta_1(G_S) + \frac{\sigma}{n}| \geq |\theta_2(G_S) + \frac{\sigma}{n}| \geq \dots \geq |\theta_n(G_S) + \frac{\sigma}{n}|$. By substituting $x_i = 1$, $y_i = |\theta_i(G_S) + \frac{\sigma}{n}|$, $r = |\theta_n(G_S) + \frac{\sigma}{n}|$, and $R = |\theta_1(G_S) + \frac{\sigma}{n}|$ in Theorem 2, we get

$$\begin{aligned} & \sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right|^2 + \left| \theta_1(G_S) + \frac{\sigma}{n} \right| \left| \theta_n(G_S) + \frac{\sigma}{n} \right| \sum_{i=1}^n 1^2 \\ & \leq \left(\left| \theta_1(G_S) + \frac{\sigma}{n} \right| + \left| \theta_n(G_S) + \frac{\sigma}{n} \right| \right) \sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right|. \end{aligned}$$

Taking into account

$$\sum_{i=1}^n \left| \theta_i(G_S) + \frac{\sigma}{n} \right| = SE(G_S)$$

and Eq. (3.6), we get

$$\begin{aligned} & n(n-1) + \sigma - \frac{\sigma^2}{n} + n \left| \theta_1(G_S) + \frac{\sigma}{n} \right| \left| \theta_n(G_S) + \frac{\sigma}{n} \right| \\ & \leq \left(\left| \theta_1(G_S) + \frac{\sigma}{n} \right| + \left| \theta_n(G_S) + \frac{\sigma}{n} \right| \right) SE(G_S) \end{aligned}$$

from which the inequality in Theorem 8 straightforwardly follows. □

4. Conclusion

In this article we have obtained some basic results on the Seidel energy of graphs with self-loops, together with a few lower and upper bounds. The Seidel energy of complete and complete bipartite graphs with self-loops are also calculated.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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