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Research Article

Additive closedness in subsets of $\mathbb{Z}_n$

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Abstract: The r-value in subsets of finite abelian groups serves as a metric for evaluating the degree of closedness within these subsets. The notion of the r-value is intricately linked to other mathematical constructs such as sum-free sets, Sidon sets, and Schur triples. We extend the definition of r-value of a subset in a finite abelian group and investigate the r-values of subsets of $\mathbb{Z}_n$, by constructing a formula for r-values of intervals consist of consecutive residue classes modulo n.

Keywords: r-values, sum-free sets, integers modulo n.

AMS Subject classification: 11B75, 11B13

1. Introduction

A subset A with cardinality d of a finite abelian group G, is said to be r-closed if among d^2 possible ordered pairs (a, b) with $a, b \in A$, there are exactly r pairs such that $a + b \in A$. This notion of r-closed set is defined by Sophie *et al.* in 2009 [10]. The r-value of the r-closed set A is denoted by $r(A)$. Sophie *et al.* obtained the fundamental results on r-closed sets and r-values of subsets of $\mathbb{Z}_p$ in [10].

The concept of r-value set is generalized from sum-free sets of integers. The sum-free set is a set in which there is no triple (a, b, c) such that $a + b = c$. In the context of r-closed sets, if a set is 0-closed then it is a sum-free set. Initially, research

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was focused on the cardinality of sum-free sets, specifically on the maximal sum-free subsets of a given set (see [1, 8]). Renato presented some properties of sum-free abelian group subsets in 2023, particularly investigating the number of 0-valued subsets and maximum cardinality of 0-valued subsets in an abelian group [6]. Terence and Vu in [15] studied sum-free sets in detail and addressed several questions raised by Erdos in his survey Extremal problems in number theory (Proceedings of the Symp. Pure Math. VIII AMS) in 1965. Sum-free sets establish a connection in the determination of the diameter of a finite abelian group concerning its various generating sets [11]. The shape of all sum-free sets is determined in the integer lattice grid (see [7, 12]). In their work, Carlet and Picek [2] introduced the essential criteria pertaining to the concepts of sidon sets and sum-free sets in the field of additive combinatorics. These criteria are crucial for identifying exponents that have the potential to be Almost Perfect Nonlinear (APN) and can expedite the process of discovering new APN exponents. Timmons [16] proved the existence of regular F-saturated graphs for a given graph F when F is a complete graph K_4 and K_5, and they also presented some partial results on the large complete graph using 0-valued sets.

A Sidon set A is a set in which every pair of elements has a distinct sum. Suppose A has cardinality d, and among d^2 ordered pairs, if some r pairs have sum in A, then the Sidon set A is an r-closed set [1]. The generalization of Sidon's problem on Sidon set counts the number of pairs where the total of the elements in each pair is fixed (see [4, 13]). When counting the number of pairs in a r-closed set A, the counting can be thought of as counting the number of pairs (a, b) in $A \times A$ with $a + b = c$ for a fixed $c \in A$, which is similar to Sidon's problem. By enumerating overall such $c \in A$, we get the r-value of A. In [5], Datskovsky discussed the relationship between an r-closed set A of a finite group G and the enumeration of monochromatic Schur triples modulo n. Mullin [14] introduced a comprehensive mathematical framework that explores the concept of a relative anti-closure property for subsets of algebraic systems. Furthermore, they have developed a mathematical model to analyze population dynamics within the framework of the biological concept of mutation. This model considers Mutant sets, which are algebraically connected to r-closed sets through the anti-closure property.

In 2014, Sophie extended the study of r-closed sets to $\mathbb{N}$ and obtained r-values of subsets of interval $[1, N]$ in $\mathbb{N}$ [9]. In a subsequent study conducted by Ostap *et al.* in 2019, they examined the minimum number of additive tuples in groups of prime order [3].

The definition of r-value of a set A in a finite abelian group G counts the number of triplets $(a, b, a+b) \in A \times A \times A$. If A is a disjoint union of two or more subsets of G, say $A = \cup_{i=1}^{n} A_i$, then the counting of number of triplets can be divided into n^3 cases as $(a, b, a+b) \in A_i \times A_j \times A_k$, where $1 \leq i, j, k \leq n$. This motivated us to extend the definition of r-value of a set in a finite abelian group. Moreover, we observe some fundamental properties, particularly in $\mathbb{Z}_n$, where n is a positive integer. In Section 3, we derive the formula for the r-values of subsets of $\mathbb{Z}_n$ based on the various possible subsets form, as discussed in Section 2. Throughout, we will assume that G is a finite additive abelian group.

2. r-values of subsets of $\mathbb{Z}_n$

Definition 1. For any subsets A, B, C of G, we define $r(A, B, C)$ as the cardinality of the set $\{(a, b) \mid a \in A,\ b \in B,\ a + b \in C\}$.

If $A = B$, then we denote $r(A, B, C) = r(A, C)$ and if $A = B = C$ then $r(A, B, C) = r(A)$, which reduce to r-value of A.

Proposition 1. *For any subsets A, B, C of G, $r(A, B, C) = r(B, A, C)$.*

Proof. Let $X = \{(a, b) \mid a \in A,\ b \in B,\ a + b \in C\}$. Since G is abelian $(b, a) \in X$ whenever $(a, b) \in X$, $r(A, B, C) = |X| = r(B, A, C)$. □

Example 1. For subsets $A = \{0, 1, 3\}$, $B = \{2, 3, 8, 11\}$ and $C = \{0, 4, 8, 10, 11, 12\}$ of $\mathbb{Z}_{16}$, $r(A, B, C) = r(B, A, C) = 5$.

Definition 2. [10] An interval in $\mathbb{Z}_n$ is defined to be a subset of $\mathbb{Z}_n$ consisting of consecutive residue classes modulo n. The notation $[a, b]$ will denote the interval $\{a, a + 1, \ldots, b - 1, b\} \subseteq \mathbb{Z}_n$.

For the purpose of computation, we treat elements of $\mathbb{Z}_n$ as least non-negative integers, and computations carried under modulo n. We use the Definition 2 of an interval in $\mathbb{Z}_n$ as defined by Sophie *et al.* [10]. Also note that we consider interval $[a, b]$ in $\mathbb{Z}_n$ with $a \leq b$ in $\mathbb{Z}$. The following Lemma 1 gives the representation of any subset of $\mathbb{Z}_n$ in terms of intervals of $\mathbb{Z}_n$.

Lemma 1. *Every subset A of $\mathbb{Z}_n$ can be uniquely written as,*

$$A = \bigcup_{i=1}^{m} [a_i, b_i]$$

with $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_{m-1} \leq b_{m-1} < a_m \leq b_m$, where $a_i, b_i \in A \subseteq \mathbb{Z}_n$, for all $i = 1, 2, \ldots, m$ and $a_i \geq b_{i-1} + 2$ for all $i = 2, 3, \ldots, m$.

Proof. Suppose A is an interval then $A = [a, b]$, where $a = min(A)$ and $b = max(A)$. Suppose A is not an interval then $A = [a, b] \backslash \{t_1, t_2, \ldots, t_l\}$ with $a = min(A)$, $b = max(A)$ and $a < t_1 < t_2 < \cdots < t_l < b$. If $l = 1$ then $A = [a, b] \backslash \{t_1\} = [a, t_1 - 1] \cup [t_1 + 1, b]$. If $l = 2$ then $A = [a, b] \backslash \{t_1, t_2\}$. If $t_2 = t_1 + 1$ then $A = [a, b] \backslash \{t_1, t_2\} = [a, t_1 - 1] \cup [t_2 + 1, b]$. If $t_2 \neq t_1 + 1$ then $A = [a, b] \backslash \{t_1, t_2\} = [a, t_1 - 1] \cup [t_1 + 1, t_2 - 1] \cup [t_2 + 1, b]$. Let $l \geq 2$ and assume that the result is true for $l - 1$. Now suppose $A = [a, b] \backslash \{t_1, t_2, \ldots, t_l\} = B \backslash \{t_l\}$, where $B = [a, b] \backslash \{t_1, t_2, \ldots, t_{l-1}\}$. From

assumption $B = \bigcup_{i=1}^{k} [a_i, b_i]$ with $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_{k-1} \leq_{k-1} < a_k \leq b_k$, where $a_i, b_i \in B \subseteq \mathbb{Z}_n$, for all $i = 1, 2, \ldots, k$ and $a_i \geq b_{i-1} + 2$ for all $i = 2, 3, \ldots, k$. Now $A = B \backslash \{t_l\} = \left(\cup_{i=1}^{k} [a_i, b_i]\right) \backslash \{t_l\}$. Then $t_l \in [a_i, b_i]$ for exactly one i. Thus rewriting this interval using $l = 1$ case, we get A as the finite disjoint union of intervals as defined.
Suppose $A = \cup_{i=1}^{k} [a_i, b_i] = \cup_{j=1}^{l} [p_j, b_j]$. Then by the construction, we have $a_1 = min(A) = p_1$. Now for b_1, if $b_1 < q_1$, then $[a_1, b_1] \subset [p_1, q_1]$ and $[p_1, q_1] \backslash [a_1, b_1] = \{b_1 + 1, \ldots, q_1\} \subset [a_2, b_2]$. Thus we have $b_1 + 2 \leq a_2 \leq b_1 + 1$ a contradiction. Similar arguments holds if $b_1 > q_1$, and hence $b_1 = q_1$.
By continuing similarly, we can deduce that the representations are equal. □

Example 2. The subset $A = \{1, 2, 4, 5, 6, 9, 10, 11, 12\}$ and $B = \{0, 3, 4, 5, 8, 11, 12, 13, 14\}$ of $\mathbb{Z}_{16}$ can be written as $A = [1, 2] \cup [4, 6] \cup [9, 12]$ and $B = [0, 0] \cup [3, 5] \cup [8, 8] \cup [11, 14]$ respectively.

Using the above Lemma 1 we investigate the r-value of subsets of $\mathbb{Z}_n$ by using r-values of intervals.

Lemma 2. *For every subset A of $\mathbb{Z}_n$, using representation of A as in Lemma 1, r-value of A is given by*

$$r(A) = \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} r([a_i, b_i], [a_j, b_j], [a_k, b_k]).$$

Proof. Let $S = \{(a, b) | a, b \in A,\ a + b \in A\}$. We can write,

$$\begin{aligned} S &= \bigcup_{a \in A} \bigcup_{b \in A} \{(a, b) | a + b \in A\} \\ &= \bigcup_{a \in A} \bigcup_{b \in A} \{(a, b) | a + b \in \bigcup_{i=1}^{m} [a_i, b_i]\} \\ &= \bigcup_{b \in A} \bigcup_{a \in A} \bigcup_{k=1}^{m} \{(a, b) | a + b \in [a_k, b_k]\}. \end{aligned}$$

Now using the union representation of A, a and b can be varied over all the intervals of A in the above expression. We get $S = \bigcup_{i=1}^{m} \bigcup_{j=1}^{m} \bigcup_{k=1}^{m} \{(a, b) | a \in [a_i, b_i], b \in [a_j, b_j],\ a + b \in [a_k, b_k]\}$. Hence $r(A) = \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} r([a_i, b_i], [a_j, b_j], [a_k, b_k])$. □

Corollary 1. *For every subset A of $\mathbb{Z}_n$, using representation of A as in Lemma 1, r-value of A is given by*

$$r(A) = \sum_{i=1}^{m} r([a_i,b_i]) + \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} r([a_i,b_i],[a_j,b_j]) + 2\sum_{i=1}^{m}\sum_{j=i+1}^{m}\sum_{k=1}^{m} r([a_i,b_i],[a_j,b_j],[a_k,b_k]).$$

To determine $r(A)$ in $\mathbb{Z}_n$, it is enough to compute $r([a_i,b_i],[a_j,b_j],[a_k,b_k])$ for all combinations of intervals in the union representation $A = \bigcup_{i=1}^{k} [a_i,b_i]$ along with with the above result. The formula for each of these $r([a_i,b_i],[a_j,b_j],[a_k,b_k])$ will be discussed in Section 3.

3. r-values interms of intervals

We derive the formula for each r-values that appeared in Lemma 2, by characterising intervals based on their end points. The following theorem gives the r-value of an interval in $\mathbb{Z}_n$.

Theorem 1. *Let $[k,l]$ be an interval in $\mathbb{Z}_n$ such that $0 \le k \le l \le n-1$. Then*

$$r([k,l]) = \tfrac{1}{2}(l(l+1)+(k-1)k) - (2k-1)(l-k+1) + (2l-n+1)(l-\tfrac{n}{2}-k+1).$$

Proof. Let $a,b \in [k,l]$ with $(a+b)(mod\ n) = i \in [k,l]$. Now for fixed i in $[k,l]$ and $b \equiv i - a(mod\ n)$ it is sufficient find all $a \in [k,l]$ such that $a+b \equiv i(mod\ n)$ and $b \in [a,b]$. We have $a+b = i$ or $a+b = n+i$. Now from $k \le b = i-a \le l$ and $k \le a \le l$ we have

$$max\{i-l,k\} \le a \le min\{i-k,l\}. \tag{3.1}$$

Since $k \le i \le l$ we have $i-l \le k$ and $i-k \le l$. Therefore inequalities in 3.1 becomes $k \le a \le i-k$. That is number of a's for a fixed i with $a+b=i$ is $i-2k+1$ but note that $i-2k+1$ is positive only if $i \ge 2k$. Hence i varies from $2k$ to l. Similarly when $a+b = n+i$ we get $i+n-l \le a \le l$ and hence number of a's for a fixed i is $2l-n+1-i$. Note that $2l-n+1-i$ is positive only if $2l-n \ge i$. Hence i varies from k to $2l-n$. Therefore number of $a's$ with $(a+b) \equiv i(mod\ n)$ whenever $i \in [k,l]$ is given by

$$r([k,l]) = \sum_{i=2k}^{l}(i-2k+1) + \sum_{i=k}^{2l-n}(2l-n+1-i). \tag{3.2}$$

This completes the proof. □

Note that each terms inside the summation in the expression (3.2) is greater than zero. Thus $r([k,l]) = 0$ whenever $l < 2k$ and $2l-n < k$. Hence $[k,l]$ is a sum-free interval in $\mathbb{Z}_n$ if and only if $l < min\{2k, \frac{k+n}{2}\}$.

Theorem 2 to Theorem 5 provide the values of $r(A, B, C)$ when two intervals A, B and C are identical. The following theorem gives $r([k, l], [k, l], [p, q]) = r([k, l], [p, q])$ whenever $0 \leq k \leq l < p \leq q \leq n - 1$.

Theorem 2. *Let $[k, l]$ and $[p, q]$ are two interval in $\mathbb{Z}_n$ such that $0 \leq k \leq l < p \leq q \leq n-1$. Then*

$$r([k,l],[p,q]) = \begin{cases} \sum_{i=max\{2k,p\}}^{q} (i-2k+1) & \text{if } q \leq k+l, \\ \sum_{i=p}^{min\{q,2l\}} (2l+1-i) & \text{if } k+l \leq p. \\ \sum_{i=max\{2k,p\}}^{k+l-1} (i-2k+1) + \sum_{i=k+l}^{min\{q,2l\}} (2l+1-i) & \text{if } p < k+l < q. \end{cases}$$

Proof. Let $i \in [p, q]$. The aim is to estimate number of pairs $(a, b) \in [k, l] \times [k, l]$ such that $a + b \equiv i(mod\ n)$. Since i is fixed and $b \equiv i - a \pmod n$ it is sufficient find all $a \in [k, l]$ such that $a + b \equiv i \pmod n$ and $b \in [k, l]$. We have $a + b = i$ or $a + b = n + i$. First we show that $b = n + i - a$ is not possible. As if $k \leq n + i - a \leq l$, then $l < i + (n - l) \leq a \leq l$ a contradiction, hence $a + b = i$.
Now from $k \leq b = i - a \leq l$ and $k \leq a \leq l$ we have

$$max\{i - l, k\} \leq a \leq min\{i - k, l\}. \tag{3.3}$$

The rest of the proof is divided into three cases.

Case 1. $q \leq k + l$.
Then $i \leq k + l$. Hence inequalities in (3.3) become $k \leq a \leq i - k$. That is number of a's for a fixed i is $i - 2k + 1$ but note that $i - 2k + 1$ is positive only if $i \geq 2k$. Hence $r([k, l], [p, q]) = \sum_{i=max\{2k,p\}}^{q} (i - 2k + 1)$.

Case 2. $k + l \leq p$.
Then $k \leq i - l$ and $l \leq i - k$. Hence inequalities in (3.3) become $i - l \leq a \leq l$. That is number of a's for a fixed i is $2l + 1 - i$ but note that $2l + 1 - i$ is positive only if $i \leq 2l$. Hence $r([k, l], [p, q]) = \sum_{i=p}^{min\{q,2l\}} (2l + 1 - i)$.

Case 3. $p < k + l < q$.
This case follows from the last two cases depending on $p \leq i \leq k + l - 1$ and $k + l \leq i \leq q$. This completes the proof. □

The closed form of expressions in Theorem 2 is computed in each case and tabulated in Table 3. A similar table can be constructed for the forthcoming theorems to obtain closed form expressions. The following theorem gives $r([p, q], [p, q], [k, l]) = r([p, q], [k, l])$ whenever $0 \leq k \leq l < p \leq q \leq n - 1$.

Table 1. Closed form of the expressions in Theorem 2.

	$q \le k+l$	$k+l \le p$	$p < k+l < q$
$2k \le p$ and $2l \le q$	$\frac{1}{2}(q(q+1)-(p-1)p)$ $-(q-p+1)(2k-1)$	$(l-p+1)(2l+1)$ $-\frac{1}{2}(p-1)p$	$(k+l-1)(k+l)-\frac{1}{2}(p-1)p$ $-(k+l-p)(2k-1)+(1-k)(2l+1)$
$2k \le p$ and $2l > q$	$\frac{1}{2}(q(q+1)-(p-1)p)$ $-(q-p+1)(2k-1)$	$(q-p+1)(2l+1)$ $-\frac{1}{2}(q(q+1)-(p-1)p)$	$(k+l-1)(k+l)-\frac{1}{2}((p-1)p+(q+1)q)$ $-(k+l-p)(2k-1)+(q-k-l+1)(2l+1)$
$2k > p$ and $2l \le q$	$\frac{1}{2}q(q+1)$ $-(2k-1)(q-k+1)$	$(l-p+1)(2l+1)$ $-\frac{1}{2}(p-1)p$	$(k+l-1)(k+l)-l(2k-1)+(1-k)(2l+1)$
$2k > p$ and $2l > q$	$\frac{1}{2}q(q+1)$ $-(2k-1)(q-k+1)$	$(q-p+1)(2l+1)$ $-\frac{1}{2}(q(q+1)-(p-1)p)$	$(k+l-1)(k+l)-\frac{1}{2}q(q+1)$ $-l(2k-1)+(q-k-l+1)(2l+1)$

Theorem 3. *Let $[k, l]$ and $[p, q]$ are two interval in $\mathbb{Z}_n$ such that $0 \le k \le l < p \le q \le n-1$. Then*

$$r([p,q],[k,l]) = \begin{cases} \sum\limits_{i=max\{2p-n,k\}}^{l} (i-2p+n+1) & \text{if } l \le p+q-n, \\ \sum\limits_{i=k}^{min\{l,2q-n\}} (2q-n+1-i) & \text{if } p+q-n \le k, \\ \sum\limits_{i=max\{2p-n,k\}}^{p+q-n-1} (i-2p+n+1) + \sum\limits_{i=p+q-n}^{min\{l,2q-n\}} (2q-n+1-i) & \text{if } k < p+q-n < l. \end{cases}$$

Proof. Let $i \in [k, l]$. The aim is to estimate number of pairs $(a, b) \in [p, q] \times [p, q]$ such that $a + b \equiv i(mod\ n)$. Since i is fixed and $b \equiv i - a \pmod n$ it is sufficient find all $a \in [p, q]$ such that $a + b \equiv i \pmod n$ and $b \in [p, q]$. We have $a + b = i$ or $a + b = n + i$. First we show that $b = i - a$ is not possible. Using $p \le i - a \le q$ and $i - p < 0$, we get $a < i - p < 0$ a contradiction, hence $a + b = n + i$.
Now from $p \le b = i + n - a \le q$ and $p \le a \le q$ we have

$$max\{i+n-q, p\} \le a \le min\{i+n-p, q\}. \tag{3.4}$$

The rest of the proof is divided into three cases.
Case 1. $l \le p + q - n$.
Then $i \le p + q - n$. Hence inequalities in (3.4) become $p \le a \le i + n - p$. That is number of a's for a fixed i is $i - 2p + n + 1$ but note that $i - 2p + n + 1$ is positive only if $i \ge 2p - n$. Hence $r([p, q], [k, l]) = \sum\limits_{i=max\{2p-n,k\}}^{l} (i - 2p + n + 1)$.
Case 2. $p + q - n \le k$.
Then $p \le i+n-q$ and $q \le i+n-p$. Hence inequalities in (3.4) become $i+n-q \le a \le q$. That is number of a's for a fixed i is $2q - n + 1 - i$ but note that $2q - n + 1 - i$ is positive only if $i \le 2q - n$. Hence $r([p, q], [k, l]) = \sum\limits_{i=p}^{min\{l,2q-n\}} (2q - n + 1 - i)$.

Case 3. $k < p+q-n < l$. This case follows from the last two cases depending on $k \le i \le p+q-n-1$ and $p+q-n \le i \le l$. This completes the proof. □

The following proposition provides a condition for 0-value in a two-interval case.

Proposition 2. *Let $[k, l]$ and $[p, q]$ are two interval in $\mathbb{Z}_n$ such that $0 \le k \le l < p \le q \le n-1$.*

1. *If $q < 2k$, then $r([k, l], [p, q]) = 0$.*
2. *If $2l < p$, then $r([k, l], [p, q]) = 0$.*
3. *If $l < 2p-n$, then $r([p, q], [k, l],) = 0$.*
4. *If $2q-n < k$, then $r([p, q], [k, l]) = 0$.*

The proof directly follows from Theorem 2 and Theorem 3.

The following theorem gives $r([k, l], [p, q], [k, l])$ whenever $0 \le k \le l < p \le q \le n-1$.

Theorem 4. *Let $[k, l]$ and $[p, q]$ are two interval in $\mathbb{Z}_n$ such that $0 \le k \le l < p \le q \le n-1$. Then $r([k, l], [p, q], [k, l]) =$*

$$\begin{cases} (l+q-n+1)(\frac{1}{2}(l+q-n)-(k-1))+\frac{1}{2}(k-1)k & \text{if } l+p-n \le k, \\ 0 & \text{if } l \le l+p-n, \\ (q-p+1)((l-n)+p+q-k+1)-\frac{1}{2}(q^2-p^2+p+q) & \text{if } k < l+p-n < l. \end{cases}$$

Proof. Let $i \in [k, l]$. We count the number of pairs $(a, b) \in [k, l] \times [p, q]$ such that $a+b \equiv i (mod\ n)$. Since i is fixed and $b \equiv i-a \pmod n$ it is sufficient find all $a \in [k, l]$ such that $a+b \equiv i \pmod n$ and $b \in [p, q]$. We have $a+b = i$ or $a+b = n+i$. First we show that $b = i-a$ is not possible. Using $p \le i-a \le q$ and $i-p < 0$, we get $a < i-p < 0$ a contradiction, hence $a+b = n+i$.
Now from $p \le b = i+n-a \le q$ and $k \le a \le l$ we have

$$max\{i+n-q, k\} \le a \le min\{i+n-p, l\}. \tag{3.5}$$

Now using $k+q \le i+n$ inequalities in (3.5) become $i+n-q \le a \le min\{i+n-p, l\}$. The rest of the proof is divided into three cases.

Case 1. $l+p-n \le k$.
Then $l \le i+n-p$. Hence inequalities in (3.5) become $i+n-q \le a \le l$. That is number of a's for a fixed i is $l+q-n+1-i$ but note that $l+q-n+1-i$ is positive only if $i \le l+q-n$. Hence $r([k, l], [p, q], [k, l]) = \sum_{i=k}^{l+q-n} (l+q-n+1-i)$.

Case 2. $l \le l+p-n$.
Then $n < p$ a contradiction. Hence $l \le l+p-n$ is not possible.

Case 3. $k < l + p - n < l$.
Then for all $i \in [k, l]$, we have either $k \le i \le l+p-n-1$ or $l+p-n \le i \le l$. Whenever $k \le i \le l+p-n-1$, then inequalities in (3.5) become $i+n-q \le a \le i+n-p$. That is number of a's for a fixed i is $q-p+1$. The case $l+p-n \le i \le l$ is follows from the case $l+p-n \le k$. Hence $r([k, l], [p, q], [k, l]) = \sum_{i=k}^{l+p-n-1} (q-p+1) + \sum_{i=l+p-n}^{l+q-n} (l+q-n+1-i)$.
This completes the proof. □

The following theorem gives $r([k, l], [p, q], [p, q])$ whenever $0 \le k \le l < p \le q \le n-1$.

Theorem 5. *Let $[k, l]$ and $[p, q]$ are two interval in $\mathbb{Z}_n$ such that $0 \le k \le l < p \le q \le n-1$. Then $r([k, l], [p, q], [p, q]) =$*

$$\begin{cases} \frac{1}{2}q(q+1) - (p+k-1)(q - \frac{1}{2}(p+k)+1) & \text{if } q \le p+l, \\ (q-p+1)(1-k) & \text{if } p+l \le p, \\ \frac{1}{2}(p+l-1)(p+l) - (p+k-1)(\frac{1}{2}(p-k)+l) + (q-p-l+1)(l-k+1) & \text{if } p < p+l < q. \end{cases}$$

Proof. Let $i \in [p, q]$. We count the number of pairs $(a, b) \in [k, l] \times [p, q]$ such that $a+b \equiv i(mod\ n)$. Since i is fixed and $b \equiv i-a \pmod n$ it is sufficient find all $a \in [k, l]$ such that $a + b \equiv i \pmod n$ and $b \in [p, q]$. We have $a + b = i$ or $a + b = n + i$. Now from $k \le a = n + i - b \le l$ and $p \le b \le q$ we have

$$max\{n+i-l, p\} \le b \le min\{n+i-k, q\}. \tag{3.6}$$

Since $i - k > 0$ and $i - l > 0$, inequalities in (3.6) become $n + i - l \le b \le q$. But then $q < n < n + (i - l) \le q$. Hence the case $a + b = i + n$ is not possible. Now from $k \le a = i - b \le l$ and $p \le b \le q$ we have

$$max\{i-l, p\} \le b \le min\{i-k, q\}. \tag{3.7}$$

Note that $i - k < q$. Therefore inequalities in (3.7) become $max\{i-l, p\} \le b \le i-k$. The rest of the proof is divided into three cases.

Case 1. $q \le p + l$.
Then $i \le p + l$. Hence inequalities in (3.7) become $p \le b \le i - k$. That is number of a's for a fixed i is $i - k - p + 1$ but note that $i - k - p + 1$ is positive only if $i \ge k + p$. Hence $r([k, l], [p, q], [p, q]) = \sum_{i=k+p}^{q} (i - k - p + 1)$.

Case 2. $p + l \le p$.
Then $p \le i - l$ and hence inequalities in (3.7) become $i \le b \le i - k$. That is number of a's for a fixed i is $l - k + 1$. Also using $p + l \le p$ we must have $l = 0$. Hence $r([k, l], [p, q], [p, q]) = (q - p + 1)(1 - k)$.

Case 3. $p < p + l < q$.
This case follows from the last two cases depending on $p \le i \le p+l-1$ and $p+l \le i \le q$.
This completes the proof. □

We derive the formula of $r([k,l],[p,q],[s,t])$ in the following theorem.

Theorem 6. *Let $[k,l],[p,q]$ and $[s,t]$ are three interval in $\mathbb{Z}_n$ such that $0 \leq k \leq l < p \leq q < s \leq t \leq n-1$.*

1. *If $t \leq k+q$, then $r([k,l],[p,q],[s,t]) =$*

$$\begin{cases} \sum_{i=max\{s,p+k\}}^{t} (i-p-k+1) & \text{if } t \leq p+l, \\ (t-s+1)(l-k+1) & \text{if } p+l \leq s, \\ \sum_{i=max\{s,p+k\}}^{p+l-1} (i-p-k+1) + (t-p-l+1)(l-k+1) & \text{if } s < p+l < t. \end{cases}$$

2. *If $k+q \leq s$, then $r([k,l],[p,q],[s,t]) =$*

$$\begin{cases} (t-s+1)(q-p+1) & \text{if } t \leq p+l, \\ \sum_{i=s}^{min\{l+q,t\}} (l+q+1-i) & \text{if } p+l \leq s, \\ (p+l-s)(q-p+1) + \sum_{i=p+l}^{min\{l+q,t\}} (l+q+1-i) & \text{if } s < p+l < t. \end{cases}$$

3. *If $s < k+q < t$, then $r([k,l],[p,q],[s,t]) =$*

$$\begin{cases} \sum_{i=max\{s,p+k\}}^{k+q-1} (i-p-k+1) + (t-k-q+1)(q-p+1) & \text{if } t \leq p+l, \\ (k+q-s)(l-k+1) + \sum_{i=k+q}^{min\{l+p,t\}} (l+q+1-i) & \text{if } p+l \leq s, \\ \sum_{i=max\{s,p+k\}}^{l+p} (i-p-k+1) + (k+q-l-p)(l-k+1) + \sum_{i=k+q+1}^{min\{l+q,t\}} (l+q+1-i) & \text{if } p+l \leq k+q, \\ \sum_{i=max\{s,p+k\}}^{k+q} (i-p-k+1) + (l+p-k-q)(q-p+1) + \sum_{i=l+p+1}^{min\{l+q,t\}} (l+q+1-i) & \text{if } k+q < p+l. \end{cases}$$

Proof. Let $i \in [s,t]$. Goal is to find number of pairs $(a,b) \in [k,l] \times [p,q]$ such that $a+b \equiv i (mod\ n)$. Since i is fixed and $b \equiv i-a (mod\ n)$ it is sufficient find all $a \in [k,l]$ such that $a+b \equiv i(mod\ n)$ and $b \in [p,q]$. We have $a+b = i$ or $a+b = n+i$. First we show that $b = n+i-a$ is not possible. As if $p \leq n+i-a \leq q$, then $k < i+(n-q) \leq a \leq i+(n-p)$ but $l < i+(n-q)$. Thus we have $i+n-q \leq a \leq l < i+n-q$ a contradiction, hence $a+b = i$. Now from $p \leq b = i-a \leq q$ and $k \leq a \leq l$ we have

$$max\{i-q,k\} \leq a \leq min\{i-p,l\}. \tag{3.8}$$

1. Given $t \leq k+q$. Hence $i-q \leq k$ and inequalities in (3.8) become $k \leq a \leq min\{i-p,l\}$. The rest of the proof is divided into three cases.

   Case 1. $t \leq p+l$.
   Then $i-p \leq l$. Hence inequalities in (3.8) become $k \leq a \leq i-p$. That is

number of a's for a fixed i is $i-p-k+1$ but note that $i-p-k+1$ is positive only if $i \geq p+k$. Hence $r([k,l],[p,q],[s,t]) = \sum_{i=max\{s,p+k\}}^{t} (i-p-k+1)$.

Case 2. $p+l \leq s$.
Then $l \leq i-p$. Equivalently a can assume any value in $[k,l]$ that is a has $l-k+1$ choices. Thus $r([k,l],[p,q],[s,t]) = (t-s+1)(l-k+1)$.

Case 3. $s < p+l < t$.
This case follows from the last two cases depending on $s \leq i \leq p+l-1$ and $p+l \leq i \leq t$.

2. Let $k+q \leq s$. Then $k \leq i-q$ and inequalities in (3.8) become $i-q \leq a \leq min\{i-p,l\}$. Now the $min\{i-p,l\}$ is decided based on $p+l$ and $[s,t]$.

   If $t \leq p+l$, then $i-p \leq l$. Hence inequalities in (3.8) become $i-q \leq a \leq i-p$. That is number of a's for a fixed i is $q-p+1$. Hence $r([k,l],[p,q],[s,t]) = (t-s+1)(q-p+1)$. Let $p+l \leq s$. Then $l \leq i-p$. Hence inequalities in (3.8) become $i-q \leq a \leq l$. That is number of a's for a fixed i is $l-i+q-1$ but note that $l-i+q-1$ is positive only if $l+q-1 \geq i$. Hence $r([k,l],[p,q],[s,t]) = \sum_{i=s}^{min\{l+q,t\}} (l-i+q+1)$. Finally let $s < p+l < t$. This case follows from the last two cases depending on $s \leq i \leq p+l-1$ and $p+l \leq i \leq t$.

3. Given $s < k+q < t$. Then for all $i \in [s,t]$ we have either $s \leq i \leq k+q-1$ or $k+q \leq i \leq t$. Hence combining cases 1 and 2 we arrive at the desired formula.

□

The following theorem gives $r([k,l],[s,t],[p,q])$ whenever $0 \leq k \leq l < p \leq q < s \leq t \leq n-1$.

Theorem 7. *Let $[k,l],[p,q]$ and $[s,t]$ are three interval in $\mathbb{Z}_n$ such that $0 \leq k \leq l < p \leq q < s \leq t \leq n-1$. Then $r([k,l],[s,t],[p,q]) = 0$.*

Proof. We show that for any $i \in [p,q]$ there is no pairs $(a,b) \in [k,l] \times [s,t]$ such that $a+b \equiv i(mod\ n)$. Let $i \in [p,q]$. Using $s \leq b = i-a \leq t$ and $k \leq a \leq l$ we have

$$max\{i-t,k\} \leq a \leq min\{i-s,l\}. \tag{3.9}$$

Since $i-t < 0$ and $i-s < 0$ inequalities in (3.9) become $k \leq a \leq i-s$. But then $i-s < 0 \leq k$. Hence $a+b = i$ not possible. Similarly using $s \leq b = n+i-a \leq t$ and $k \leq a \leq l$ we have

$$max\{i+n-t,k\} \leq a \leq min\{i+n-s,l\}. \tag{3.10}$$

Then using $k < i+n-t$ and $l < i+n-s$, inequalities in (3.10) become $i+n-t \leq a \leq l$. But then $l < i+n-t$. Hence $a+b = n+i$ not possible. □

Suppose a subset A of $\mathbb{Z}_n$ is represented by $A = \cup_{i=1}^{m}[a_i, b_i]$. In the computation of $r(A)$, using Theorem 7 $r([a_i, b_i], [a_j, b_j], [a_k, b_k])$ are zero whenever $i < k$ and $j > k$ for all k between 2 and $m-1$. Moreover number of such combinations are $\frac{m(m-1)(m-2)}{6}$. Hence these formulas can be directly avoided in the Corollary 1 to compute $r(A)$.

The following Theorem 8 give the r-value $r([p, q], [s, t], [k, l])$, where $[k, l], [p, q]$ and $[s, t]$ are three interval in $\mathbb{Z}_n$ with $0 \le k \le l < p \le q < s \le t \le n-1$.

Theorem 8. *Let $[k, l], [p, q]$ and $[s, t]$ are three interval in $\mathbb{Z}_n$ such that $0 \le k \le l < p \le q < s \le t \le n-1$.*

1. *If $l \le p - n + t$, then $r([p, q], [s, t], [k, l]) =$*
$$\begin{cases} \sum\limits_{i=max\{s+p-n,k\}}^{l} (i + n - s - p + 1) & \text{if } l \le q - n + s, \\ (l - k + 1)(q - p + 1) & \text{if } q - n + s \le k, \\ \sum\limits_{i=max\{s+p-n,k\}}^{q-n+s} (i + n - s - p + 1) + (l - q + n - s)(q - p + 1) & \text{if } k < q - n + s < l. \end{cases}$$

2. *If $p - n + t \le k$, then $r([p, q], [s, t], [k, l]) =$*
$$\begin{cases} (l - k + 1)(t - s + 1) & \text{if } l \le q - n + s, \\ \sum\limits_{i=k}^{min\{l,q-n+t\}} (q - n + t + 1 - i) & \text{if } q - n + s \le k, \\ (q - n + s - k + 1)(t - s + 1) + \sum\limits_{i=q-n+s+1}^{min\{l,q-n+t\}} (q - n + t + 1 - i) & \text{if } k < q - n + s < l. \end{cases}$$

3. *If $k < p - n + t < l$, then $r([p, q], [s, t], [k, l]) =$*
$$\begin{cases} \sum\limits_{i=max\{s+p-n,k\}}^{p-n+t} (i + n - s - p + 1) + (l - p + n - t + 1)(t - s + 1) & \text{if } l \le q - n + s \\ (p - n + t - k + 1)(q - p + 1) + \sum\limits_{i=p-n+t}^{min\{l,q-n+t\}} (q - n + t + 1 - i) & \text{if } q - n + s \le k \\ \sum\limits_{i=max\{s+p-n,k\}}^{q-n+s} (i + n - s - p + 1) + (p + t - q - s)(q - p + 1) + \sum\limits_{i=p-n+t+1}^{min\{l,q-n+t\}} (q - n + t + 1 - i) & \text{if } q - n + s \le p - n + t \\ \sum\limits_{i=max\{s+p-n,k\}}^{p-n+t} (i + n - s - p + 1) + (q + s - p - t)(t - s + 1) + \sum\limits_{i=q-n+s+1}^{min\{l,q-n+t\}} (q - n + t + 1 - i) & \text{if } q - n + s > p - n + t. \end{cases}$$

Proof. Let $i \in [k, l]$. Goal is to find number of pairs $(a, b) \in [p, q] \times [s, t]$ such that $a + b \equiv i (mod\ n)$. Since i is fixed and $b \equiv i - a (mod\ n)$ it is sufficient find all $a \in [p, q]$ such that $a + b \equiv i (mod\ n)$ and $b \in [s, t]$. We have $a + b = i$ or $a + b = n + i$. Note that $i < a$. Therefore $b = i - a$ is not possible. Now from $s \le b = i + n - a \le t$ and $p \le a \le q$ we have

$$max\{i + n - t, p\} \le a \le min\{i + n - s, q\}. \tag{3.11}$$

1. Given $l \le p - n + t$. Hence $i + n - t \le p$ and inequalities in (3.11) become $p \le a \le min\{i + n - s, q\}$. The rest of the proof is divided into three cases.

   Case 1. $l \le q - n + s$.
   Then $i + n - s \le q$. Hence inequalities in (3.11) become $p \le a \le i + n - s$. That is number of a's for a fixed i is $i + n - s - p + 1$ but note that $i + n - s - p + 1$ is positive

only if $i \geq s+p-n$. Hence $r([p,q],[s,t],[k,l]) = \sum_{i=max\{s+p-n,k\}}^{l} (i+n-s-p+1)$.

Case 2. $q-n+s \leq k$.
Then $q \leq i+n-s$. Equivalently a can assume any value in $[p,q]$ that is a has $q-p+1$ choices. Thus $r([p,q],[s,t],[k,l]) = (l-k+1)(q-p+1)$.

Case 3. $k < q-n+s < l$.
This case follows from the last two cases depending on $k \leq i \leq q-n+s$ and $q-n+s+1 \leq i \leq l$.

2. Let $p-n+t \leq k$. Then $p \leq i+n-t$ and inequalities in (3.11) become $i+n-t \leq a \leq min\{i+n-s,q\}$. Now the $min\{i+n-s,q\}$ is decided based on $q-n+s$ and $[k,l]$.

   If $l \leq q-n+s$, then $i+n-s \leq q$. Hence inequalities in (3.11) become $i+n-t \leq a \leq i+n-s$. That is number of a's for a fixed i is $t-s+1$. Hence $r([p,q],[s,t],[k,l]) = (l-k+1)(t-s+1)$. Let $q-n+s \leq k$. Then $q \leq i+n-s$. Hence inequalities in (3.11) become $i+n-t \leq a \leq q$. That is number of a's for a fixed i is $q-i-n+t+1$ but note that $q-i-n+t+1$ is positive only if $q-n+t \geq i$. Hence $r([p,q],[s,t],[k,l]) = \sum_{i=k}^{min\{l,q-n+t\}} (q-n+t+1-i)$. Finally let $k < q-n+s < l$. This case follows from the last two cases depending on $k \leq i \leq q-n+s$ and $q-n+s+1 \leq i \leq l$.

3. Given $k < p-n+t < l$. Then for all $i \in [k,l]$ we have either $k \leq i \leq p-n+t-1$ or $p-n+t \leq i \leq l$. Hence combining cases 1 and 2 in three respective sub-cases we arrive the desired formula.

□

The following example illustrates the estimation of the r-value of a subset of $\mathbb{Z}_n$ using the formulas obtained in Section 3.

Example 3. Let $A = \{1,2,4,5,6,9,10,11,12\}$ be a subset of $\mathbb{Z}_{16}$. As A is the union of three intervals $A = [1,2] \cup [4,6] \cup [9,12]$, the r-value of A is obtained by summing all the r-values $r(A_1, A_2, A_3)$, where A_1, A_2, A_3 are all combinations of three intervals. (see Table 3). We get $r(A) = 43$. For instance, $r([9,12],[9,12],[4,6])$ in $\mathbb{Z}_{16}$ is calculated using Theorem 3. Given that $k=4$, $l=6$, $p=9$, $q=12$, $n=16$ and that $k < p+q-n < l$, we can apply 3^{rd} case of the Theorem 3 to obtain $r([9,12],[9,12],[4,6]) = \sum_{i=4}^{4}(i-1) + \sum_{i=5}^{6}(9-i) = 10$. Also note that $r([1,2],[9,12],[4,6])$ and $r([9,12],[1,2],[4,6])$ are not listed in the Table 3 as they vanishes as per Theorem 7.

4. Conclusion

The study determined the r-value of subsets of $\mathbb{Z}_n$ by obtaining the formula for r-value of intervals in $\mathbb{Z}_n$. These formulas can be used to classify subsets based on their

Table 2. **Computation of r-value of A.**

	Interval combinations	r-value
$r([k,l])$	$r([1,2])$	1
	$r([4,6])$	0
	$r([9,12])$	0
$r([k,l],[p,q])$	$r([1,2],[4,6])$	1
	$r([1,2],[9,12])$	0
	$r([4,6],[9,12])$	8
$r([p,q],[k,l])$	$r([4,6],[1,2])$	0
	$r([9,12],[1,2])$	1
	$r([9,12],[4,6])$	10
$r([k,l],[p,q],[k,l])$	$r([1,2],[4,6],[1,2]) = r([4,6],[1,2],[1,2])$	2×0
	$r([1,2],[9,12],[1,2]) = r([9,12],[1,2],[1,2])$	2×0
	$r([4,6],[9,12],[4,6]) = r([9,12],[4,6],[4,6])$	2×0
$r([k,l],[p,q],[p,q])$	$r([1,2],[4,6],[4,6]) = r([4,6],[1,2],[4,6])$	2×3
	$r([1,2],[9,12],[9,12]) = r([9,12],[1,2],[9,12])$	2×5
	$r([4,6],[9,12],[9,12]) = r([9,12],[4,6],[9,12])$	2×0
$r([k,l],[p,q],[s,t])$	$r([1,2],[4,6],[9,12]) = r([4,6],[1,2],[9,12])$	2×0
$r([p,q],[s,t],[k,l])$	$r([4,6],[9,12],[1,2]) = r([9,12],[4,6],[1,2])$	2×3
	Total	43

r-values. Further, these results can be used to construct sum-free subsets of $\mathbb{Z}_n$. In the future, one can use these results to study Sidon sets, Schur triples, etc.

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