

Graceful coloring of some corona graphs - An algorithmic approach

Laavanya D[†], Devi Yamini S^{*}

Vellore Institute of Technology, Chennai, Tamil Nadu, India.

[†]laavanya.d2020@vitstudent.ac.in

^{*}deviyamini.s@vit.ac.in

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Abstract: A graceful k -coloring of a non-empty graph G is a proper vertex coloring with k colors that induces a proper edge coloring, where the color for an edge uv is the absolute difference between the colors assigned to the vertices u and v . The minimum k for which G admits a graceful k -coloring is called the graceful chromatic number of G ($\chi_g(G)$). The problem of determining the graceful chromatic number for some corona graphs with the related coloring algorithms are studied in this paper.

Keywords: graceful coloring, graceful chromatic number, corona graphs.

AMS Subject classification: 05C15, 05C78

1. Introduction

All graphs $G(V, E)$ examined in this paper are simple, connected and finite. The study of graph labeling was introduced by Alexander Rosa in 1967 [10], which is an assignment of integers to the vertices, edges (or both) of a graph G under certain conditions. There are several types of graph labeling in which β -labeling is one of the eminent labeling and are widely studied in the survey by Gallian [5]. It was referred as graceful labeling by Golomb [6]. A function f is a graceful labeling of G with q number of edges, if f is an injection from $V(G)$ to the set $\{0, 1, 2, \dots, q\}$ such that the induced function f^* is bijective from $E(G)$ to the set $\{1, 2, \dots, q\}$ defined as $f^*(xy) = |f(x) - f(y)|$, for every edge xy in G . A graph G possessing a graceful labeling is a graceful graph.

* Corresponding Author

With the origin of four color problem, the field of graph colorings has been developed into one of the most popular areas of graph theory. Vertex coloring and edge coloring received the most attention in the area of graph coloring. In such colorings, any two adjacent vertices or any two adjacent edges of G are assigned with distinct colors. The minimum number of colors needed in proper vertex coloring of the graph G is called the chromatic number of G ($\chi(G)$), while the minimum number of colors needed in proper edge coloring of the graph G is called the chromatic index of G ($\chi'(G)$) [11].

The concept of graceful chromatic number was introduced by Gary Chartrand [2, 3], as an extension from graceful labeling. A graceful k -coloring of a non-empty graph $G(V, E)$ is a proper vertex coloring $f : V(G) \rightarrow \{1, 2, \dots, k\}$, where $k \geq 2$, which induces a proper edge coloring $f^* : E(G) \rightarrow \{1, 2, \dots, k-1\}$ defined by $f^*(uv) = |f(u) - f(v)|$, where $u, v \in V(G)$ [2]. The graceful chromatic number $\chi_g(G)$ of the graph G is the minimum k for which G has a graceful k -coloring. We define $[e, f] = \{e, e+1, \dots, f-1, f\}$, for $\{e, f\} \subset \mathbb{Z}^+$ with $e < f$.

In the fundamental paper of the graceful coloring [2], the graceful chromatic number for some well known graphs were computed.

Theorem 1. For a subgraph G' of G , $\chi_g(G') \leq \chi_g(G)$.

Theorem 2. Let $f : V(G) \rightarrow \{1, 2, \dots, k\}$, $k \geq 2$ be a coloring of a nontrivial connected graph G . Then f is a graceful coloring of G if and only if

(i) for each vertex v of G , the vertices in the closed neighborhood $N[v]$ are assigned distinct colors by f and

(ii) for each path (x, y, z) in G , $f(y) \neq \frac{f(x)+f(z)}{f(y)}$.

Theorem 3. For a nontrivial connected graph G , $\chi_g(G) \geq \Delta + 1$, where Δ represents the maximum degree of G .

Theorem 4. For a cycle C_n , $n \geq 4$, $\chi_g(C_n) = \begin{cases} 4, & \text{if } n \neq 5 \\ 5, & \text{if } n = 5. \end{cases}$

Theorem 5. For a path P_n , $n \geq 5$, $\chi_g(P_n) = 4$.

Theorem 6. For a wheel graph W_n , $n \geq 6$, $\chi_g(W_n) = n$.

Theorem 7. If T is a tree with maximum degree Δ , then $\chi_g(T) \leq \lceil \frac{5\Delta}{3} \rceil$.

Theorem 8. If G is an r -regular graph, then $\chi_g(G) \geq r + 2$, where $r \geq 2$.

The graceful chromatic number of caterpillars were investigated along with a characterization in [2]. The graceful chromatic number for some subclasses of the following graphs have been established in the literature: unicyclic graphs [1]; graphs with diameter at least 2 [2]; rooted trees $T_{\Delta,h}$ [4]. Also, the graceful chromatic number for few variants of ladder graphs [9] and for a subclass of trees [8] are discussed in the literature. The following result in [9] is useful in proving our main results.

Observation 9. ([9]) If $[1, \Delta + i]$, $i \in \mathbb{Z}^+$ colors are used in the graceful coloring of a graph G , then the possible colors for any vertex of maximum degree (Δ) are the first and last i colors from $[1, \Delta + i]$.

Graceful coloring for many graph products are not yet explored. So we concentrate in evaluating the graceful chromatic number for some corona product of graphs which was introduced by Roberto Frucht together with Frank Harary and was defined as follows:

The corona product of two graphs G and H is the graph $G \odot H$ obtained by taking one copy of G which has p_i vertices and p_i copies of H , and then joining the i^{th} vertex of G by an edge to every vertex in the i^{th} copy of H [7], an illustration shown in Figure 1.

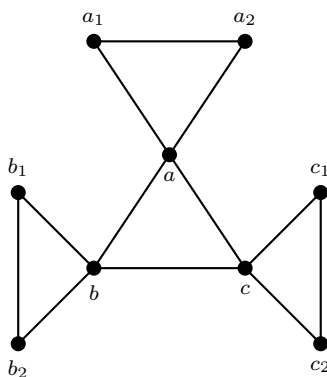


Figure 1. $K_3 \odot K_2$

The graceful chromatic number for corona product of graphs such as star ($K_{1,n}$), path (P_n), cycle (C_n), wheel (W_n) with path (P_m) are evaluated in this paper. Also, we present two main algorithms to assign the colors for the vertices of P_m which are used to run the graceful coloring algorithms for each case.

2. Main Results

Theorem 10. $\chi_g(K_{1,n} \odot P_m) = n + m + 1$, for $n \geq 2$, $m \geq 2$.

Proof. Let $u_i \in V(K_{1,n})$, for $0 \leq i \leq n$. By the definition of corona product of $K_{1,n}$ and P_m , there are $(n + 1)$ copies of P_m which are represented as P_m^i , $0 \leq i \leq n$ such that each u_i is adjacent to all the vertices of P_m^i . Let the vertices and edges of $K_{1,n} \odot P_m$ be

$$V(K_{1,n} \odot P_m) = \{u_i : 0 \leq i \leq n\} \cup \{u_{i,j} : 0 \leq i \leq n, 1 \leq j \leq m\}$$

$$E(K_{1,n} \odot P_m) = \{(u_0 u_i) : 1 \leq i \leq n\} \cup \{(u_i u_{i,j}) : 0 \leq i \leq n, 1 \leq j \leq m\} \cup \{(u_{i,j} u_{i,j+1}) : 0 \leq i \leq n, 1 \leq j \leq m - 1\}.$$

The representation of $K_{1,n} \odot P_m$ is provided in Figure 2.

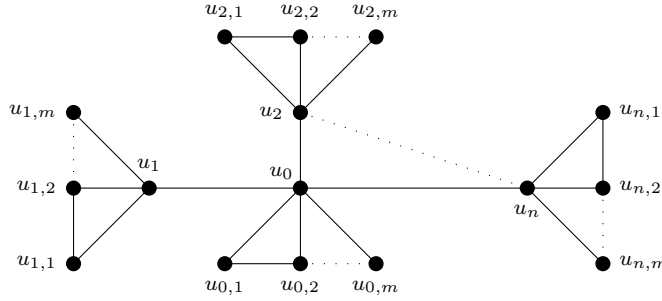


Figure 2. $K_{1,n} \odot P_m$

Note that, the maximum degree vertex in $K_{1,n} \odot P_m$ is u_0 with $d(u_0) = n + m$. Therefore, $\chi_g(K_{1,n} \odot P_m) \geq n + m + 1$ (by Theorem 3). To prove $\chi_g(K_{1,n} \odot P_m) \leq n + m + 1$, we show a graceful coloring of $K_{1,n} \odot P_m$ with $n + m + 1$ colors using Algorithm 1 which uses the two Algorithms 2 and 3. \square

Algorithm 1 Graceful coloring of $G = K_{1,n} \odot P_m$

Input: $G = K_{1,n} \odot P_m$

Output: Graceful $(n + m + 1)$ -coloring of G

colors for the vertices of $K_{1,n}$

- 1: $f(u_0) \leftarrow n + m + 1$
- 2: $f(u_1) \leftarrow 1$
- 3: $f^*(u_0, u_1) \leftarrow n + m$
- 4: **if** $n == 2$ and $m == 3$ **then**
- 5: $f(u_2) \leftarrow 2$
- 6: $f^*(u_0, u_2) \leftarrow 4$
- 7: **else**
- 8: **for** $s_i \in N_i$ **do**
- 9: $f(u_i) \leftarrow m + i$
- 10: $f^*(u, u_i) \leftarrow n - i + 1$
- 11: **end for**
- 12: **end if**

colors for $u_{i,j}$

- 13: **if** $n == 2$ and $m == 3$ **then**

```

14:    $A = \{4, 3, 5\}$ 
15:   for  $j = 1$  to 3 do
16:      $f(u_{0,j}) \leftarrow A[j - 1]$ 
17:     Update the corresponding edge colors and the edge colors between  $u_0$  and
        $u_{0,j} (1 \leq j \leq m)$ 
18:   end for
19: end if
20: if  $n == 2$  and  $m == 4$  then
21:    $A = \{5, 4, 2, 3\}$ 
22:   for  $j = 1$  to 3 do
23:      $f(u_{0,j}) \leftarrow A[j - 1]$ 
24:     Update the corresponding edge colors and the edge colors between  $u_0$  and
        $u_{0,j} (1 \leq j \leq m)$ 
25:   end for
26: else
27:   for  $i = 0$  to  $n$  do
28:     AssignColors $\{u_{i,j}\}$ 
29:   end for
30: end if

```

Algorithm 2 Colors $\{u_{i,j}\}$

Input: Integers from 1 to $\chi_g(G)$, i

Output: Colors for $u_{i,j} \in P_m^i$

```

1:  $A \leftarrow \{1, 2, \dots, \chi_g(G)\}$ 
2:  $X \leftarrow \{ \text{set of vertices at distance 2 from } u_{i,j} \}$ 
3:  $Y \leftarrow \{ \text{set of colors of the vertices at distance 2 from } u_{i,j} \}$ 
4:  $B \leftarrow A \setminus (f(u_i) \cup Y)$ 
5: if  $f(u_i) = 1$  or  $f(u_i) = \chi_g(G)$  then
6:   return B
7: else
8:   Let  $D \leftarrow \{ \text{set of edge colors between } u_i \text{ and } X \}$ 
9:   for every  $a \in B$  do
10:    if  $|f(u_i) - a| \in D$  then
11:       $C \leftarrow B \setminus \{a\}$ 
12:    end if
13:  end for
14: end if
15: if  $\{f(u_i) + 1, f(u_i) - 1\} \subseteq C$  then
16:    $E \leftarrow C \setminus (f(u_i) + 1)$ 
17:   return E
18: else
19:   return C
20: end if

```

Algorithm 3 AssignColors $\{u_{i,j}\}$ **Input:** $i, f(u_i), m, \text{Colors}\{u_{i,j}\}$ **Output:** Graceful coloring for the respective $u_{i,j}$

```

1: Set  $r$  as zero and  $X \leftarrow \text{Colors}\{u_{i,j}\}$ 
2: for  $j = 1$  to  $m$  increased by 2 do
3:    $f(u_{i,j}) \leftarrow X[r]$ 
4:    $f^*(u_i, u_{i,j}) \leftarrow |(f(u_i) - f(u_{i,j}))|$ 
5:    $X \leftarrow X \setminus X[r]$ 
6: end for
7: if  $m$  is even then
8:    $k = m$ 
9: else
10:   $k = m - 1$ 
11: end if
12: for  $j = k$  to 2 decremented by 2 do
13:   $SC \leftarrow \emptyset$ 
14:  if  $X \neq \emptyset$  then
15:     $f(u_{i,j}) \leftarrow X[r]$ 
16:    Update the corresponding edge colors
17:     $X \leftarrow X \setminus X[r]$ 
18:    if  $j \neq 2$  and adjacent edges of  $u_{i,j}$  receives same color then
19:       $SC \leftarrow SC \cup f(u_{i,j})$ 
20:       $f(u_{i,j}) \leftarrow X[r]$ 
21:      Update the corresponding edge colors
22:    end if
23:  else
24:    for  $t = 1$  to  $|SC|$  do
25:       $f(u_{i,j}) \leftarrow SC[t]$ 
26:      Update the corresponding edge colors
27:       $SC \leftarrow SC \setminus SC[t]$ 
28:    end for
29:  end if
30: end for
31: if  $f^*(u_{i,2}, u_i) == f^*(u_{i,2}, u_{i,3})$  then
32:   $f(u_{i,2}) \leftarrow$  interchange the colors of  $u_{i,1}$  and  $u_{i,2}$ 
33:  Update the corresponding edge colors
34: end if
35: if  $f^*(u_{i,2}, u_i) == f^*(u_{i,2}, u_{i,1})$  then
36:   $f(u_{i,2}) \leftarrow$  interchange the colors of  $u_{i,2}$  and  $u_{i,3}$ 
37:  Update the corresponding edge colors
38: end if

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39: for  $j = 1$  to  $m - 2$  do
40:   if  $f^*(u_{i,j}, u_{i,j+1}) == f^*(u_{i,j+1}, u_{i,j+2})$  then
41:     Interchange  $f(u_{i,j+1})$  and  $f(u_{i,j+2})$ 
42:     Update the corresponding edge colors
43:   end if
44: end for

```

An illustration for graceful 6-coloring of $K_{1,3} \odot P_2$ is shown in Figure 3.

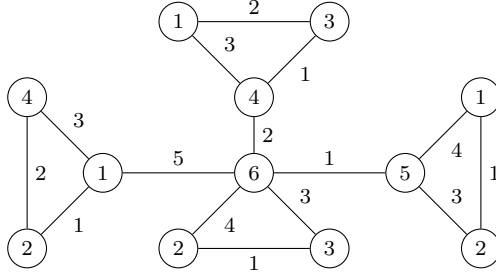


Figure 3. $\chi_g(K_{1,3} \odot P_2) = 6$

Corollary 1. For $n \geq 2$, $m \geq 2$, $\chi_g(K_{1,n} \odot mK_1) = n + m + 1$.

Next, we determine the graceful chromatic number of corona product of path P_n with another path P_m .

Theorem 11. For $n \geq 2$, $m \geq 2$, $\chi_g(P_n \odot P_m) = \begin{cases} \Delta + 1, & n = 2, 3, 4 \\ \Delta + 2, & n \geq 5. \end{cases}$

Proof. Let $u_i \in V(P_n)$, for $1 \leq i \leq n$ and $u_{i,j} \in V(P_m^i)$, for $1 \leq i \leq n$, $1 \leq j \leq m$. Thus $V(P_n \odot P_m) = \{u_1, u_2, \dots, u_n\} \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$
 $E(P_n \odot P_m) = \{(u_i u_{i+1}) : 1 \leq i \leq n - 1\} \cup \{(u_i u_{i,j}) : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{(u_{i,j} u_{i,j+1}) : 1 \leq i \leq n, 1 \leq j \leq m - 1\}$.

See Figure 4 for a representation of $P_n \odot P_m$.

Case 1. $n = 2, 3, 4$.

Based on Theorem 3, $\chi_g(P_n \odot P_m) \geq \Delta + 1$. To prove $\chi_g(P_n \odot P_m) \leq \Delta + 1$, we show a graceful coloring of $P_n \odot P_m$ with $\Delta + 1$ colors using Algorithm 4.

Algorithm 4 Graceful $(\Delta + 1)$ -coloring of $G = P_n \odot P_m$

Input: $G = P_n \odot P_m$, $n = 2, 3, 4$

Output: Graceful $(\Delta + 1)$ -coloring of G

```

1:                                     # colors for the vertices in  $P_n$ 
2: if  $n == 2$  then
3:    $f(u_1) \leftarrow 1$ 
4:    $f(u_2) \leftarrow \Delta + 1$ 

```

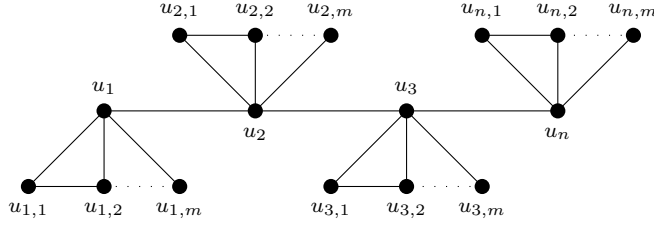


Figure 4. $P_n \odot P_m$

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5:    $f^*(u_1, u_2) \leftarrow \Delta$ 
6: end if
7: if  $n == 3$  and  $m == 3$  then
8:    $f(u_1) \leftarrow 5$ 
9:    $f(u_2) \leftarrow 1$ 
10:   $f(u_3) \leftarrow 6$ 
11:  Update the corresponding edge colors
12: end if
13: if  $n == 4$  and  $m == 3$  then
14:    $f(u_1) \leftarrow 5$ 
15:    $f(u_2) \leftarrow 1$ 
16:    $f(u_3) \leftarrow 6$ 
17:    $f(u_4) \leftarrow 2$ 
18:   Update the corresponding edge colors
19: else
20:    $f(u_1) \leftarrow 2$ 
21:    $f(u_2) \leftarrow 1$ 
22:
23:    $f(u_3) \leftarrow \Delta + 1$ 
24:
25:    $f(u_4) \leftarrow \Delta$ 
26:
27:   Update the corresponding edge colors
28: end if
                                     # colors for  $u_{i,j}$ 
29:
30: if  $n == 2$  and  $m == 4$  then
31:    $A = \{5, 2, 4, 3\}$ 
32:    $B = \{4, 1, 3, 2\}$ 
33:    $C = \{3, 2, 1\}$ 
34:   for  $j = 1$  to 4 do
35:      $f(u_{1,j}) \leftarrow A[j - 1]$ 
36:      $f^*(u_1, u_{1,j}) \leftarrow B[j - 1]$ 
37:   end for
38:   for  $j = 1$  to 3 do
39:      $f^*(u_{1,j}, u_{1,j+1}) \leftarrow C[j - 1]$ 
40:   end for
41: end if
42: if  $n == 2$  and  $m == 5$  then

```



```

43:    $A = \{5, 2, 4, 3, 6\}$ 
44:    $B = \{4, 1, 3, 2, 5\}$ 
45:    $C = \{3, 2, 1, 3\}$ 
46:   for  $j = 1$  to 5 do
47:      $f(u_{1,j}) \leftarrow A[j - 1]$ 
48:      $f^*(u_1, u_{1,j}) \leftarrow B[j - 1]$ 
49:   end for
50:   for  $j = 1$  to 4 do
51:      $f^*(u_{1,j}, u_{1,j+1}) \leftarrow C[j - 1]$ 
52:   end for
53: end if
54: if  $n == 4$  and  $m = 4$  then
55:    $A = \{5, 2, 4, 3\}$ 
56:    $B = \{2, 5, 3, 4\}$ 
57:    $C = \{4, 1, 3, 2\}$ 
58:    $D = \{3, 2, 1\}$ 
59:   for  $j = 1$  to 4 do
60:      $f(u_{3,j}) \leftarrow A[j - 1]$ 
61:      $f^*(u_3, u_{3,j}) \leftarrow B[j - 1]$ 
62:      $f(u_{4,j}) \leftarrow C[j - 1]$ 
63:      $f^*(u_4, u_{4,j}) \leftarrow B[j - 1]$ 
64:   end for
65:   for  $j = 1$  to 3 do
66:      $f^*(u_{3,j}, u_{3,j+1}) \leftarrow D[j - 1]$ 
67:      $f^*(u_{4,j}, u_{4,j+1}) \leftarrow D[j - 1]$ 
68:   end for
69: else
70:   for  $i = 1$  to  $n$  do
71:     AssignColors $\{u_{i,j}\}$ 
72:   end for
73: end if

```

A graceful 5-coloring of $P_2 \odot P_3$ and $P_3 \odot P_2$ are shown in Figure 5 (a) and (b) respectively.

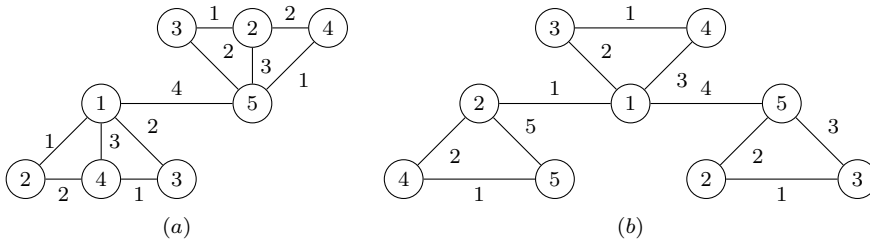


Figure 5. Graceful 5-coloring of (a) $P_2 \odot P_3$ and (b) $P_3 \odot P_2$

Case 2. $n \geq 5$.

Since $P_4 \odot P_m$ is a subgraph of $P_n \odot P_m$, then from Theorem 1, $\chi_g(P_n \odot P_m) \geq \chi_g(P_4 \odot P_m) = \Delta + 1$. We show that, it is not possible to use $\Delta + 1$ colors for the graceful coloring of $P_n \odot P_m$ for $n \geq 5$. Let S be the set of all maximum degree vertices in $P_n \odot P_m$. That is, $S = \{u_2, u_3, \dots, u_{n-1}\}$. By Observation 9, all the vertices in S are to be colored only with

$\{1, \Delta + 1\}$. Since $n \geq 5$, we obtain a contradiction to proper edge coloring. Thus, at least one more color is needed for the graceful coloring of $P_n \odot P_m$. Hence, $\chi_g(P_n \odot P_m) \geq \Delta + 2$. To prove $\chi_g(P_n \odot P_m) \leq \Delta + 2$, we show a graceful $(\Delta + 2)$ -coloring of $P_n \odot P_m$ using Algorithm 5.

Algorithm 5 Graceful $(\Delta + 2)$ -coloring of $G = P_n \odot P_m$, $n \geq 5$

Input: $G = P_n \odot P_m$, $n \geq 5$

Output: Graceful $(\Delta + 2)$ -coloring of G

```

# colors for  $u_i$ 
1: for  $i = 1$  to  $n$  incremented by 4 do
2:    $f(u_i) \leftarrow 2$ 
3: end for
4: for  $i = 2$  to  $n$  incremented by 4 do
5:    $f(u_i) \leftarrow \Delta + 2$ 
6: end for
7: for  $i = 3$  to  $n$  incremented by 4 do
8:    $f(u_i) \leftarrow 1$ 
9: end for
10: for  $i = 4$  to  $n$  incremented by 4 do
11:    $f(u_i) \leftarrow \Delta + 1$ 
12: end for
13: for  $i = 1$  to  $n - 1$  do
14:    $f^*(u_i, u_{i+1}) \leftarrow |f(u_i) - f(u_{i+1})|$ 
15: end for

# colors for  $u_{i,j}$ 
16: for  $i = 1$  to  $n$  do
17:   AssignColors $\{u_{i,j}\}$ 
18: end for

```

□

A graceful 6-coloring of $P_5 \odot P_2$ is shown in Figure 6.

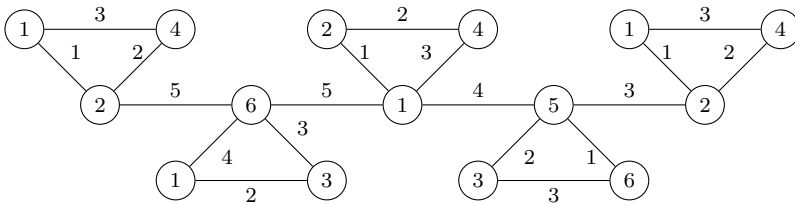


Figure 6. $\chi_g(P_5 \odot P_2) = 6$

Corollary 2. For $n \geq 3$, $m \geq 2$, $\chi_g(P_n \odot mK_1) = \begin{cases} \Delta + 1, & n = 2, 3, 4 \\ \Delta + 2, & n \geq 5. \end{cases}$

Consequently, we determine the graceful chromatic number of corona product of cycles C_n with path P_m .

Theorem 12. For $n \geq 3, m \geq 2$, $\chi_g(C_n \odot P_m) = \begin{cases} \Delta + 2, & n \neq 5 \\ \Delta + 3, & n = 5. \end{cases}$

Proof. Let the vertex set and the edge set of $C_n \odot P_m$ be

$$V(C_n \odot P_m) = \{u_i : 1 \leq i \leq n\} \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$$

$$E(C_n \odot P_m) = \{(u_1, u_n), (u_i, u_{i+1}) : 1 \leq i \leq n-1\} \cup \{(u_i, u_{i,j}) : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{(u_{i,j}, u_{i,j+1}) : 1 \leq i \leq n, 1 \leq j \leq m-1\}.$$

Refer Figure 7 for the representation of $C_n \odot P_m$.

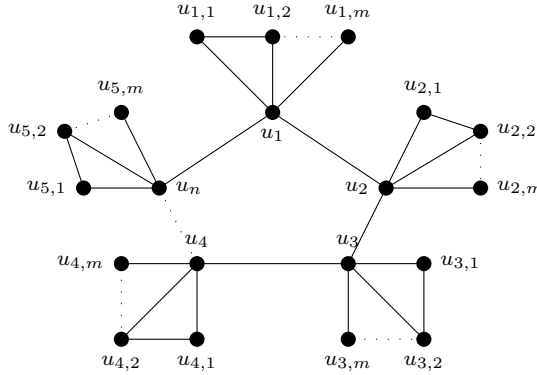


Figure 7. $C_n \odot P_m$

Case 1. $n \neq 5$.

By Theorem 3, $\chi_g(C_n \odot P_m) \geq \Delta + 1$. Assume that, there is a graceful $(\Delta + 1)$ -coloring f of $C_n \odot P_m$. The vertices $\{u_i\}$ which lies on C_n can be colored using only $\{1, \Delta + 1\}$, by Observation 9. Without loss of generality, let $f(u_1) = 1$, so $f(u_2) = \Delta + 1$. Now, the vertex u_3 or u_n cannot be colored gracefully using the colors $\{1, \Delta + 1\}$, which leads to a contradiction to our assumption. Hence, $\chi_g(C_n \odot P_m) \geq \Delta + 2$. It remains to define a graceful $(\Delta + 2)$ -coloring of $C_n \odot P_m$, which is attained in the following Algorithm 6.

Algorithm 6 Graceful $(\Delta + 2)$ -coloring of $G = C_n \odot P_m$, $n \neq 5$

Input: $G = C_n \odot P_m$, $n \neq 5$

Output: Graceful $(\Delta + 2)$ -coloring of G

colors for u_i

- 1: **if** $m == 3$ **then**
- 2: **if** n is a multiple of 3 **then**
- 3: **for** $i = 1$ to n incremented by 3 **do**
- 4: $f(u_i) \leftarrow 1$
- 5: **end for**
- 6: **for** $i = 2$ to n incremented by 3 **do**
- 7: $f(u_i) \leftarrow 6$
- 8: **end for**
- 9: **for** $i = 3$ to n incremented by 3 **do**
- 10: $f(u_i) \leftarrow 2$
- 11: **end for**
- 12: Update the corresponding edge colors and the edge colors between u_i and u_{i+1} ($1 \leq i \leq n-1$)

```

13:  else if
14:      for dothent = 7 to n incremented by 3
15:          if n == t then
16:               $f(u_n) \leftarrow 7$ 
17:              for  $i = 1$  to  $n - 1$  incremented by 3 do
18:                   $f(u_i) \leftarrow 1$ 
19:              end for
20:              for  $i = 2$  to  $n$  incremented by 3 do
21:                   $f(u_i) \leftarrow 6$ 
22:              end for
23:              for  $i = 3$  to  $n$  incremented by 3 do
24:                   $f(u_i) \leftarrow 2$ 
25:              end for
26:              Update the corresponding edge colors and the edge colors between  $u_i$  and
 $u_{i+1}$  ( $1 \leq i \leq n - 1$ )
27:          end if
28:      end for
29:  else if
30:
31:      for  $t = 8$  to  $n$  incremented by 3 do
32:           $A = \{1, 6, 2, 7\}$ 
33:          if  $n == t$  then
34:              for  $i = 1, 2, 3, 4$  do
35:                   $f(u_i) \leftarrow A[i - 1]$ 
36:              end for
37:               $f(u_n) \leftarrow 7$ 
38:              for  $i = 5$  to  $n - 3$  incremented by 3 do
39:                   $f(u_i) \leftarrow 1$ 
40:              end for
41:              for  $i = 7$  to  $n - 1$  incremented by 3 do
42:                   $f(u_i) \leftarrow 2$ 
43:              end for
44:              for  $i = 6$  to  $n - 2$  incremented by 3 do
45:                   $f(u_i) \leftarrow 6$ 
46:              end for
47:              Update the corresponding edge colors and the edge colors between  $u_i$  and
 $u_{i+1}$  ( $1 \leq i \leq n - 1$ )
48:          end if
49:      end for
50:  end if then
51:  else if
52:
53:      if  $n$  is a multiple of 4 then
54:          for  $i = 1$  to  $n - 3$  incremented by 4 do
55:               $f(u_i) \leftarrow 1$ 
56:          end for
57:          for  $i = 2$  to  $n - 2$  incremented by 4 do
58:               $f(u_i) \leftarrow \Delta + 2$ 
59:          end for
60:          for  $i = 3$  to  $n - 1$  incremented by 4 do
61:               $f(u_i) \leftarrow 2$ 

```

```

62:     end for
63:     for  $i = 4$  to  $n$  incremented by 4 do
64:          $f(u_i) \leftarrow \Delta + 1$ 
65:     end for
66:     Update the corresponding edge colors and the edge colors between  $u_i$  and  $u_{i+1}$  ( $1 \leq$ 
67:      $i \leq n - 1$ ) then
68:         else if
69:             if  $n == 9$  then
70:                 for  $i = 1$  to  $n - 2$  incremented by 3 do
71:                      $f(u_i) \leftarrow 1$ 
72:                 end for
73:                 for  $i = 2$  to  $n - 1$  incremented by 3 do
74:                      $f(u_i) \leftarrow \Delta + 2$ 
75:                 end for
76:                 for  $i = 3$  to  $n$  incremented by 3 do
77:                      $f(u_i) \leftarrow 2$ 
78:                 end for
79:                 Update the corresponding edge colors and the edge colors between  $u_i$  and
80:                  $u_{i+1}$  ( $1 \leq i \leq n - 1$ )
81:             end if then
82:             else if
83:                 for  $t = 13$  to  $n$  incremented by 4 do
84:                     if  $n == t$  then
85:                         for  $i = n - 2, n - 5, n - 8$  do
86:                              $f(u_i) \leftarrow 1$ 
87:                         end for
88:                         for  $i = n - 1, n - 4, n - 7$  do
89:                              $f(u_i) \leftarrow 2$ 
90:                         end for
91:                         for  $i = n, n - 3, n - 6$  do
92:                              $f(u_i) \leftarrow \Delta + 2$ 
93:                         end for
94:                         for  $i = 1$  to  $n - 12$  incremented by 4 do
95:                              $f(u_i) \leftarrow 1$ 
96:                         end for
97:                         for  $i = 2$  to  $n - 11$  incremented by 4 do
98:                              $f(u_i) \leftarrow 2$ 
99:                         end for
100:                        for  $i = 3$  to  $n - 10$  incremented by 4 do
101:                             $f(u_i) \leftarrow \Delta + 2$ 
102:                        end for
103:                        for  $i = 4$  to  $n - 9$  incremented by 4 do
104:                             $f(u_i) \leftarrow \Delta + 1$ 
105:                        end for
106:                    end if
107:                end for
108:                Update the corresponding edge colors and the edge colors between  $u_i$  and
109:                 $u_{i+1}$  ( $1 \leq i \leq n - 1$ ) then
110:            else if

```

```

110:
111:   if  $n == 6$  then
112:     for  $i = 1$  to  $n - 2$  incremented by 3 do
113:        $f(u_i) \leftarrow 1$ 
114:     end for
115:     for  $i = 2$  to  $n - 1$  incremented by 3 do
116:        $f(u_i) \leftarrow \Delta + 2$ 
117:     end for
118:     for  $i = 3$  to  $n$  incremented by 3 do
119:        $f(u_i) \leftarrow \Delta + 1$ 
120:     end for
121:     Update the corresponding edge colors and the edge colors between  $u_i$  and
122:      $u_{i+1}$  ( $1 \leq i \leq n - 1$ )
123:   end if then
124:   else if
125:     for  $t = 10$  to  $n$  incremented by 4 do
126:       if  $n == t$  then
127:         for  $i = n - 2, n - 5$  do
128:            $f(u_i) \leftarrow 1$ 
129:         end for
130:         for  $i = n - 1, n - 4$  do
131:            $f(u_i) \leftarrow 2$ 
132:         end for
133:         for  $i = n, n - 3$  do
134:            $f(u_i) \leftarrow \Delta + 2$ 
135:         end for
136:         for  $i = 1$  to  $n - 9$  incremented by 4 do
137:            $f(u_i) \leftarrow 1$ 
138:         end for
139:         for  $i = 2$  to  $n - 8$  incremented by 4 do
140:            $f(u_i) \leftarrow 2$ 
141:         end for
142:         for  $i = 3$  to  $n - 7$  incremented by 4 do
143:            $f(u_i) \leftarrow \Delta + 2$ 
144:         end for
145:         for  $i = 4$  to  $n - 6$  incremented by 4 do
146:            $f(u_i) \leftarrow \Delta + 1$ 
147:         end for
148:         Update the corresponding edge colors and the edge colors between  $u_i$  and
149:          $u_{i+1}$  ( $1 \leq i \leq n - 1$ )
150:       end if
151:     end for then
152:   else
153:     for  $t = 3$  to  $n$  incremented by 4 do
154:       if  $n == t$  then
155:         for  $i = 1$  to  $n - 2$  incremented by 4 do
156:            $f(u_i) \leftarrow 1$ 
157:         end for
158:         for  $i = 2$  to  $n - 1$  incremented by 4 do
159:            $f(u_i) \leftarrow 2$ 

```

```

159:           end for
160:           for  $i = 3$  to  $n$  incremented by 4 do
161:                $f(u_i) \leftarrow \Delta + 2$ 
162:           end for
163:           for  $i = 4$  to  $n - 3$  incremented by 4 do
164:                $f(u_i) \leftarrow \Delta + 1$ 
165:           end for
166:           Update the corresponding edge colors and the edge colors between  $u_i$  and
            $u_{i+1}$  ( $1 \leq i \leq n - 1$ )
167:           end if
168:       end for
169:   end if
170: end if
                                     # colors for  $u_{i,j}$ 

171: for  $i = 1$  to  $n$  do
172:     AssignColors  $\{u_{i,j}\}$ 
173: end for

```

Case 2. $n = 5$.

Since $P_n \odot P_m$ is a subgraph of $C_n \odot P_m$, $\chi_g(C_5 \odot P_m) \geq \chi_g(P_5 \odot P_m) = \Delta + 2$ (from Theorem 1 and Theorem 11). We claim that $\chi_g(C_5 \odot P_m) \neq \Delta + 2$. Assume $(C_5 \odot P_m)$ is graceful $(\Delta + 2)$ -colorable. Then the possible colors for the vertices of C_5 are $\{1, 2, \Delta + 1, \Delta + 2\}$ (by Observation 9) which is not possible for the graceful coloring of C_5 . Hence, $\chi_g(C_5 \odot P_m) \geq \Delta + 3$. To show, $\chi_g(C_5 \odot P_m) = \Delta + 3$, we define a graceful $(\Delta + 3)$ -coloring of $C_5 \odot P_m$, which is attained using Algorithm 7.

Algorithm 7 Graceful $(\Delta + 3)$ -coloring of $G = C_n \odot P_m$, $n = 5$

Input: $G = C_n \odot P_m$, $n = 5$

Output: Graceful $(\Delta + 3)$ -coloring of G

```

                                     # colors for  $u_i$ 

1:  $A = \{1, \Delta + 2, \Delta + 3, 2, 3\}$ 
2:  $B = \{\Delta + 1, 1, \Delta + 1, 1\}$ 
3: for  $i = 1$  to 5 do
4:      $f(u_i) \leftarrow A[i - 1]$ 
5: end for
6: for  $i = 1$  to 4 do
7:      $f^*(u_i, u_{i+1}) \leftarrow B[i - 1]$ 
8:      $f^*(u_5, u_1) \leftarrow 2$ 
9: end for
                                     # colors for  $u_{i,j}$ 

10: for  $i = 1$  to  $n$  do
11:     AssignColors  $\{u_{i,j}\}$ 
12: end for

```

□

A graceful $(\Delta + 2)$ -coloring of $C_4 \odot P_3$ is shown in Figure 8.

Corollary 3. For $n \geq 3, m \geq 2$, $\chi_g(C_n \odot mK_1) = \begin{cases} \Delta + 2, & n \neq 5 \\ \Delta + 3, & n = 5. \end{cases}$

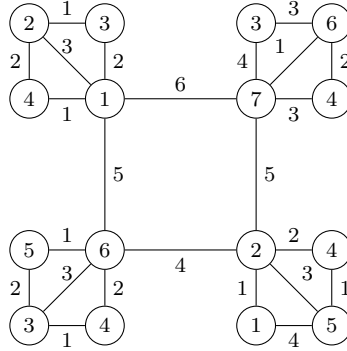


Figure 8. $\chi_g(C_4 \odot P_3) = 7$

The graceful chromatic number of corona product of wheel graph W_n having n vertices with path P_m is evaluated as follows:

Theorem 13. For $n \geq 4, m \geq 4$, $\chi_g(W_n \odot P_m) = \begin{cases} \Delta + 2, & n = 4, 5 \\ \Delta + 1, & n \geq 6. \end{cases}$

Proof. A wheel graph W_n on n vertices is obtained by joining a central vertex to all the vertices of the cycle C_{n-1} . Let u_1 be the central vertex of W_n such that the vertices $\{u_i : 2 \leq i \leq n\}$ are adjacent to u_1 . For $1 \leq i \leq n$ and $1 \leq j \leq m$, $\{u_{i,j}\} \in V(P_m)$ such that each $\{u_{i,j}\}$ is adjacent to u_i . Thus,

$$V(W_n \odot P_m) = \{u_i \cup u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$$

$$E(W_n \odot P_m) = \{(u_1, u_i) : 2 \leq i \leq n\} \cup \{(u_2, u_n), (u_i, u_{i+1}) : 2 \leq i \leq n-1\} \cup \{(u_i, u_{i,j}) : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{(u_{i,j}, u_{i,j+1}) : 1 \leq i \leq n, 1 \leq j \leq m-1\}.$$

Refer Figure 9 for the representation of $W_n \odot P_m$.

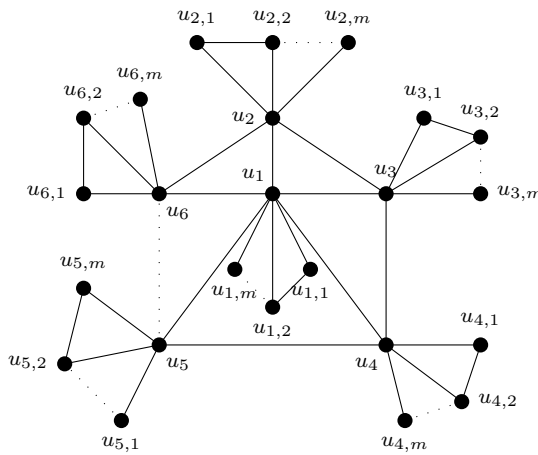


Figure 9. $W_n \odot P_m$

Case 1. $n = 4, 5$.

When $n = 4$, the maximum degree of $W_4 \odot P_m$ is $3 + m$ which is attained at the vertices $\{u_i : 1 \leq i \leq 4\}$. By Theorem 3 $\chi_g(W_4 \odot P_m) \geq \Delta + 1$. We now show that, $\chi_g(W_4 \odot P_m) \neq \Delta + 1$. Suppose on a contrary, if $\Delta + 1$ colors are used in the graceful coloring of $W_4 \odot P_m$, then by Observation 9, $f(u_i) \in \{1, \Delta + 1\}, 1 \leq i \leq 4$. This is not possible, since the vertices u_i are mutually adjacent. Hence, $\chi_g(W_4 \odot P_m) \geq \Delta + 1$.

When $n = 5$, the central vertex u_1 is of maximum degree and the vertices $u_i, 2 \leq i \leq 5$ are of degree $\Delta - 1$. From Theorem 3, $\chi_g(W_5 \odot P_m) \geq \Delta + 1$. Suppose if $\Delta + 1$ colors are used in the graceful coloring of $W_5 \odot P_m$, then for $2 \leq i \leq 5, f(u_i) \in \{1, 2, \Delta, \Delta + 1\}$ (preserve edge coloring). Without loss of generality, let $f(u_1) = 1$ (by Observation 9). Then the available colors for u_2, u_3, u_4, u_5 are $\{2, \Delta, \Delta + 1\}$ which are not sufficient for the graceful coloring, since the vertices $u_i, 1 \leq i \leq n$ are at distance 2 from each other. Hence, at least $\Delta + 2$ colors are required for the graceful coloring of $W_5 \odot P_m$. To claim, $\chi_g(W_n \odot P_m) \leq \Delta + 2, n = 4, 5$, we define a graceful $(\Delta + 2)$ -coloring of $W_n \odot P_m$ in Algorithm 8. Hence, $\chi_g(W_n \odot P_m) = \Delta + 2, n = 4, 5$.

Algorithm 8 Graceful $(\Delta + 3)$ -coloring of $G = C_n \odot P_m, n = 5$

Input: $G = W_n \odot P_m, n = 4, 5$

Output: Graceful $(\Delta + 2)$ - coloring of G

```

# colors for  $u_i$ 
1:  $f(u_1) \leftarrow 1$ 
2: if  $n == 4$  then
3:    $A = \{2, \Delta + 1, \Delta + 2\}$ 
4:    $B = \{1, \Delta, \Delta + 1\}$ 
5:    $C = \{\Delta, 1\}$ 
6:   for  $i = 2, 3, 4$  do
7:      $f(u_i) \leftarrow A[i - 2]$ 
8:      $f^*(u_1, u_i) \leftarrow B[i - 2]$ 
9:   end for
10:  for  $i = 2, 3$  do
11:     $f^*(u_i, u_{i+1}) \leftarrow C[i - 2]$ 
12:  end for
13:   $f^*(u_2, u_4) \leftarrow \Delta - 1$ 
14: end if
15: if  $n == 5$  then
16:    $A = \{2, \Delta + 1, 3, \Delta + 2\}$ 
17:    $B = \{1, \Delta, 2, \Delta + 1\}$ 
18:    $C = \{\Delta - 1, \Delta - 2, \Delta - 1\}$ 
19:   for  $i = 2, 3, 4, 5$  do
20:      $f(u_i) \leftarrow A[i - 2]$ 
21:   end for
22:    $f^*(u_1, u_i) \leftarrow B[i - 2]$ 
23: end if
24: for  $i = 2, 3, 4$  do
25:    $f^*(u_i, u_{i+1}) \leftarrow C[i - 2]$ 
26: end for
27:  $f^*(u_2, u_5) \leftarrow \Delta$ 
# colors for  $u_{i,j}$ 
28: for  $i = 1$  to  $n$  do
29:   AssignColors $\{u_{i,j}\}$ 

```

30: **end for**

Case 2. $n \geq 6$.

$\chi_g(W_n \odot P_m) \geq \Delta + 1$, for $n \geq 6$ (from Theorem 3). To show $\chi_g(W_n \odot P_m) \leq \Delta + 1$, we define a graceful $(\Delta + 1)$ -coloring of $W_n \odot P_m$ using Algorithm 9. Hence, $\chi_g(W_n \odot P_m) = \Delta + 1$, for $n \geq 6$. \square

A depiction of a graceful $(\Delta + 2)$ -coloring for $W_4 \odot P_4$ can be observed in Figure 10.

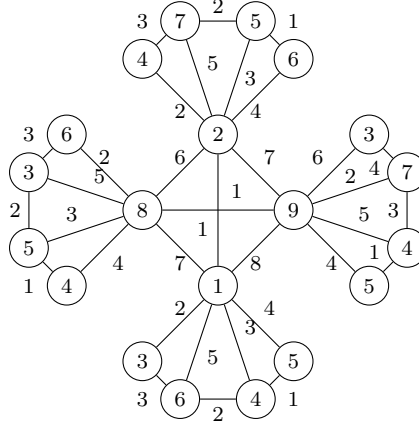


Figure 10. $\chi_g(W_4 \odot P_4) = 9$

Corollary 4. For $n \geq 4, m \geq 4$, $\chi_g(W_n \odot mK_1) = \begin{cases} \Delta + 2, & n = 4, 5 \\ \Delta + 1, & n \geq 6. \end{cases}$

Algorithm 9 Graceful $(\Delta + 1)$ -coloring of $G = W_n \odot P_m$, $n \geq 6$

Input: $G = W_n \odot P_m$, $n \geq 6$

Output: Graceful $(\Delta + 1)$ -coloring of G

colors for the vertices of u_i

- 1: $f(u_1) \leftarrow 1$
- 2: Set t as zero
- 3: $X = \{2, 3, 4, \dots, \Delta + 1\}$
- 4: **for** $i = 2$ to n incremented by 3 **do**
- 5: $f(u_i) \leftarrow X[t]$
- 6: $X \leftarrow X \setminus X[t]$
- 7: **end for**
- 8: Set s as zero
- 9: **for** $i = 3$ to n incremented by 3 **do**
- 10: $f(u_i) \leftarrow \Delta + 1 - s$
- 11: Increment s by 2

```

12: end for
13: Set  $r$  as zero
14: for  $i = 4$  to  $n$  incremented by 3 do
15:    $f(u_i) \leftarrow \Delta - r$ 
16:   Increment  $r$  by 2
17:   Update the corresponding edge colors
18: end for
                                     #colors for  $u_{i,j}$ 
19: for  $i = 1$  to  $n$  do
20:   AssignColors $\{u_{i,j}\}$ 
21: end for

```

3. Conclusion

There has not been much research done on the concept of the graceful coloring of graph products. In this paper, we analyze the graceful chromatic number of corona product of graphs. Also we provide suitable algorithms for the graceful coloring process. Working on the following open problems stated below are interesting and a challenging one.

Problem 1. Characterize the graphs for which (i) $\chi_g(G) = \chi(G) + 1$ (ii) $\chi_g(G) = \chi'(G)$ (iii) $\chi_g(G) = \chi'(G) + 1$.

Problem 2. Evaluate the graceful chromatic number of some general graphs like bipartite graphs, co-bipartite graphs, regular graphs, and split graphs.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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