Communications in Combinatorics and Optimization
Vol. 11, No. 2 (2026), pp. 419-432
https://doi.org/10.22049/cco.2024.29138.1860

CCO
Commun. Comb. Optim.

Research Article

Constant-Ratio polynomial time approximation of the asymmetric minimum weight cycle cover problem with limited number of cycles

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Received: 1 November 2023; Accepted: 29 July 2024
Published Online: 5 August 2024

Abstract: We consider polynomial time approximation of the minimum cost cycle cover problem for an edge-weighted digraph, where feasible covers are restricted to have at most k disjoint cycles. In the literature this problem is referred to as Minimum-weight k-Size Cycle Cover Problem. The problem is closely related to the classic Traveling Salesman Problem and Vehicle Routing Problem and has many important applications in algorithms design and operations research. Unlike its unconstrained variant, the studied problem is strongly NP-hard even on undirected graphs and remains intractable in very specific settings. For any metric, the problem can be approximated in polynomial time within ratio 2, while in fixed-dimensional Euclidean spaces it admits polynomial time approximation schemes. In the same time, approximation of the more general asymmetric variant of the problem still remained weakly studied. In this paper, we propose the first constant-ratio approximation algorithm for this problem, which extends the recent breakthrough results of Svensson-Tarnawski-Végh and Traub-Vygen for the Asymmetric Traveling Salesman Problem.

Keywords: approximation algorithm, constant approximation ratio, asymmetric traveling salesman problem, cycle cover problem.

AMS Subject classification: 68W25, 68Q25

1. Introduction

The Cycle Cover Problem (CCP) is a well-known combinatorial optimization problem with important applications in operations research and algorithms design for other combinatorial problems, e.g. the Traveling Salesman Problem (TSP) [2, 5], the Vehicle

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Routing Problem (VRP) [11], several versions of the Stacker Crane Problem (SCP) [4, 8], etc.

Most of the studied CCP settings can be considered as extensions of the classical Linear Sum Assignment Problem (LSAP) formulated on subsets of the symmetric group S_n. For each such a setting, the objective is a *permutation cost* coinciding with the total weight of the corresponding routes in a given graph, while the set of feasible permutations is constrained in terms of the properties of their *cycle decomposition* including length or amount of the cycles. For instance, Bläser and Siebert [3] proposed an approximation algorithms for minimum cost graph covering problems by cycles of length at least k. Later (see, e.g. [17]) these results were extended to the cheapest covers by cycles whose lengths belong to a given set $L \subset \mathbb{N}$.

In this paper, we are focused on polynomial time approximation of the Minimum-weight k-Size Cycle Cover Problem (Min-k-SCCP), where the goal is to construct a minimum cost cover of a given (di)graph by at most k disjoint cycles [7, 13]. In terms of applications, each cycle on the cover can be treated as a route for a vehicle visiting the corresponding set of clients. From this point of view, Min-k-SCCP is close to the classical Vehicle Routing Problem [24]. Furthermore, interest in this topic is confirmed by the *intermediate* complexity status of this problem between the strongly NP-hard TSP (which is Min-k-SCCP for $k = 1$) and the polynomial-time solvable LSAP (which is Min-k-SCCP for $k = n$).

1.1. Related work

For any fixed $k \geqslant 1$, the symmetric version of the Min-k-SCCP formulated on undirected graphs inherits complexity and main approximation properties of the classical TSP [14, 15]. Thus, the problem is strongly NP-hard in the general case and remains intractable even on the Euclidean plane [19]. The metric Min-k-SCCP is APX-complete, while its Euclidean settings formulated in $\mathbb{R}^d$ admit a Polynomial Time Approximation Scheme (PTAS) for an arbitrary fixed dimension d.

In addition, for the Min-k-SCCP settings on random graphs and for the Euclidean Maximum-weight k-Size Cycle Cover Problem (Max-k-SCCP) there exist asymptotically exact algorithms introduced in [7].

On the other hand, polynomial time approximability of the asymmetric Min-k-SCCP (as for asymmetric variants of other well-known combinatorial problems) remain weakly studied for a long time.

For instance, while constant-ratio approximation algorithms for the metric TSP, metric Capacitated Vehicle Routing Problem (CVRP) and their numerous symmetric versions have been known since the late 1970s, thanks to the seminal results of Christofides [6], Serdyukov [22], Wolsey [26], and Haimovich and Rinnooy Kan [10], until 2018 the Asymmetric Traveling Salesman Problem (ATSP) could only be approximated within the factor $O(\log n/\log\log n)$ [1].

Recently, Svensson, Tarnawski, and Végh [23] and Traub and Vygen [25] introduced a breakthrough approach to the polynomial time approximation of the ATSP with triangle inequality, which led to the first constant-ratio approximation algorithms for that problem. For the sake of convenience, in the sequel we refer to this approach and the state-of-the-art $(22+\varepsilon)$-approximation algorithm proposed in [25] as $\mathrm{S(TV)}^2$-framework and $\mathrm{S(TV)}^2$-algorithm, respectively.

Relying on this algorithm as a building block, the first proofs of polynomial time approximation within constant factors for several related asymmetric problems including Steiner Cycle Problem (SCP), Rural Postman Problem (RPP), CVRP with unsplittable demands [16] and Prize Collecting ATSP [20] were obtained. Summary of these results is presented in [18].

On the other hand, for many asymmetric routing problems including Min-k-SCCP, constant-ratio approximation algorithms based on cost-preserving reductions to single or multiple auxiliary ATSP instances and straightforward application of results obtained in [25] have still not been designed. To approximate these problems within constant ratios, one might need to design a more deep adaptation of $\mathrm{S(TV)}^2$-framework itself.

1.2. Contribution

In this paper

(i) we propose the first extension of $\mathrm{S(TV)}^2$-framework to the class of routing problems including Min-k-SCCP for an arbitrary fixed $k \geqslant 1$;

(ii) as a consequence, for any $\varepsilon > 0$, we develop a polynomial time $(24 + \varepsilon)$-approximation algorithm for the Min-k-SCCP.

The rest of the paper is structured as follows. In Section 2, we recall a mathematical formulation of Min-k-SCCP.

Further, we show that construction of a constant-ratio approximation algorithm for the Min-k-SCCP can be successively reduced to the similar task for more structured instances of this problem, i.e. Min-k-SCCP_S (in Section 3), and strongly laminar instances (in Section 4). In Section 5, we propose our extension of $\mathrm{S(TV)}^2$-framework for the Min-k-SCCP_S, describe $(24 + \varepsilon)$-approximation algorithm for this problem and prove its performance guarantee. Finally, in Section 6, we summarize our paper.

2. Problem statement

Suppose we are given by a strongly connected digraph $G = (V, E)$ and weighting function $c\colon E \to \mathbb{R}_+$ that specifies transportation cost $c_e = c(e) = c(v, w)$ for a transition along each arc $e = (v, w) \in E$ of the graph G. Hereinafter, we assume that the *triangle inequality*

$$c(v, w) \leqslant c(v, u) + c(u, w) \tag{2.1}$$

holds for any arcs (v, u), (u, w), and (v, w). To any multi-set of arcs F, we assign the incidence vector $x = \chi^F$, $x\colon E \to \mathbb{Z}_+$ and cost $c(F) = \sum_{e\in E} c_e x_e$.

Definition 1. An instance of the Min-k-SCCP is defined as follows. For an ordered pair (G, c), it is required to compute a spanning Eulerian submultigraph $G_F = (V, F)$ of the digraph G, such that

(i) G_F has no isolated nodes,

(ii) G_F has at most k connected components,

(iii) F has the minimum cost $c(F)$.

Although the given statement slightly generalizes the known formulation of the Min-k-SCCP studied in previous papers [7, 15], these formulations coincide with each other for complete graphs. Furthermore, the classical ATSP is Min-k-SCCP for $k = 1$.

3. Restricted Min-k-SCCP

In this section, we reduce approximation of the Min-k-SCCP to the same task as the restricted version of this problem, which we call Min-k-SCCP$_S$.

Definition 2. An instance of the Min-k-SCCP$_S$ is given by a triple (G, c, S), where $S \subset V$, $|S| = k$. It is required to find a spanning Eulerian submultigraph G_F, which along with conditions (i)–(iii) satisfies an additional constraint: $V(D) \cap S \neq \varnothing$ for any connected component D of G_F.

We use standard notation $\delta^+(U) = \{(v, w) \colon v \in U, w \in V \setminus U\}$, $\delta^-(U) = \delta^+(V \setminus U)$, and $\delta(U) = \delta^-(U) \cup \delta^+(U)$ for the cuts specified by an arbitrary non-empty subset of nodes $U \subset V$. In addition, we use the abbreviations $\delta(v) = \delta(\{v\})$ and $x(E') = \sum_{e \in E'} x_e$ for any $v \in V$ and subset of arcs $E' \subset E$. By OPT and OPT_S we denote the costs of optimum solutions of the initial Min-k-SCCP instance (G, c) and the corresponding auxiliary Min-k-SCCP$_S$ instances (G, c, S), respectively.

Lemma 1. *For arbitrary $k \geqslant 1$ and $\alpha > 0$, existence of an α-approximation polynomial algorithm for the Min-k-SCCP$_S$ implies polynomial time approximation for the Min-k-SCCP within the same ratio.*

Proof. Let (G, c) be a Min-k-SCCP instance to be solved and $\mathcal{A}_S$ be the α-approximation algorithm for the Min-k-SCCP$_S$. Since G is strongly connected, the instance (G, c) and all the auxiliary instances (G, c, S) for $S \subset V$, $|S| = k$ are feasible and can be solved to optimality. By applying algorithm $\mathcal{A}_S$ to any instance (G, c, S) we assign the spanning Eulerian subgraph $G_S = (V, F_S)$, such that, for each connected component D of the graph G_S, $V(D) \cap S \neq \varnothing$ and $\mathrm{OPT}_S \leqslant c(F_S) \leqslant \alpha\, \mathrm{OPT}_S$.
Further, let $H^* = (V, F^*)$ be an optimal solution of the initial instance (G, c) and $S^* \subset V$ be an arbitrary k-element subset, which has a non-empty intersection with any connected component of the graph H^*. Then, for the subgraph $(V, F) = \arg\min\{c(F_S) \colon S \subset V, |S| = k\}$, we have

$$\mathrm{OPT} \leqslant c(F) \leqslant c(F_{S^*}) \leqslant \alpha \cdot \mathrm{OPT}_{S^*} \leqslant \alpha \cdot c(F^*) = \alpha \cdot \mathrm{OPT}\,.$$

Thus, the Min-k-SCCP has an α-approximation polynomial time algorithm, since $\big|\{S \subset V : |S| = k\}\big| = O(n^k)$. Lemma is proved. □

4. Strongly laminar instances

We proceed with approximation algorithms for the Min-k-SCCP$_S$ by assignment to this problem the following MILP-model

$$\min \sum_{e \in E} c_e x_e \tag{4.1}$$

$$\text{s.t.}\quad x(\delta^-(v)) - x(\delta^+(v)) = 0 \qquad (v \in V), \tag{4.2}$$

$$x(\delta(U)) \geqslant 2 \qquad (U \in \mathcal{V}), \tag{4.3}$$

$$x_e \in \mathbb{Z}_+ \qquad (e \in E), \tag{4.4}$$

where $\mathcal{V} = \{U : \varnothing \neq U \subset V \setminus S\} \cup \{\{u\} : u \in S\}$. Here, equations (4.2) ensure that any feasible submultigraph will be Eulerian while (4.3) provide an absence of the isolated nodes and upper bound for the number of its connected components. In the sequel, we consider LP-relaxation $\mathcal{P}$ of problem (4.1)–(4.4) and its dual $\mathcal{D}$:

$$\begin{aligned} &\max \sum_{U \in \mathcal{V}} 2y_U \\ \text{s.t.}\quad & a_w - a_v + \sum_{U \in \mathcal{V}:\, e \in \delta(U)} y_U \leqslant c(e) \qquad (e = (v,w) \in E) \\ & y_U \geqslant 0 \qquad (U \in \mathcal{V}), \end{aligned}$$

where the dual variables a_w and a_v are induced by the constraints (4.2) and (4.4) while y_U for $U \in \mathcal{V}$ are induced by the inequalities (4.3). Under our assumptions, both problems are solvable and have the same optimal value $\mathcal{P}^* = \mathcal{D}^*$.
In this section, we show that the approximation of the Min-k-SCCP$_S$ can be reduced to the case of structured instances of this problem called *strongly laminar*. Recall that a family of subsets $\mathcal{L}$ of the set V is called *laminar*, if for any $L_1, L_2 \in \mathcal{L}$, $L_1 \cap L_2 \neq \varnothing$ implies either $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$.

Definition 3. A tuple $\mathcal{I} = (G, \mathcal{L}, k, S, x, y)$ is a strongly laminar Min-k-SCCP$_S$ instance, if

- $G = (V, E)$ is a strongly connected digraph and $|V| > k$;
- S is a subset of V of size k;
- $\mathcal{L}$ is a laminar family of subsets of $V \setminus S$, such that for any $L \in \mathcal{L}$, the induced subgraph $G[L]$ is strongly connected;

- $x : E \to \mathbb{R}_+$ is a feasible solution for (4.2)–(4.3), s.t. $x(\delta(L)) = 2$ for an arbitrary $L \in \mathcal{L}$;

- $y \colon \mathcal{L} \to \mathbb{R}_+$.

Each $\mathcal{I}$ induces the structured instance $(G, \bar{c}, S)$ of the Min-k-SCCP$_S$ with the special weighting function

$$\bar{c}_e = \bar{c}(e) = \sum_{L \in \mathcal{L} \colon e \in \delta(L)} y_L \quad (e \in E). \tag{4.5}$$

Define $y' \colon \mathcal{V} \to \mathbb{R}_+$ as follows:

$$y'_U = \begin{cases} y_U, & \text{if } U \in \mathcal{L}, \\ 0, & \text{otherwise.} \end{cases}$$

By construction, x and $(0, y')$ are optimal solutions of the corresponding linear programs $\mathcal{P}$ and $\mathcal{D}$, for $c \equiv \bar{c}$. In the sequel, we call these problems $\mathcal{P}_{ind}$ and $\mathcal{D}_{ind}$ and use the following notation

$$\mathrm{LP}(\mathcal{I}) = \mathcal{P}^*_{ind} = \sum_{e \in E} \bar{c}_e x_e = \sum_{L \in \mathcal{L}} 2y_L = \mathcal{D}^*_{ind}. \tag{4.6}$$

The concept of strongly laminar Min-k-SCCP$_S$ instance is a natural extension of the known concept of strongly laminar ATSP instance. In [25], those instances were considered for $k = 1$. In our case, we use a simpler notation $\mathcal{I} = (G, \mathcal{L}, x, y)$. Furthermore, as in [23], in the case where $\mathcal{L}$ consists only singletons $\{v\}$, we refer to $\mathcal{I}$ as *singleton instance* of the Min-k-SCCP$_S$.

Lemma 2. *Suppose, for some $\alpha \geqslant 1$, there exists a polynomial time algorithm that, for any strongly laminar instance $\mathcal{I} = (G, \mathcal{L}, l, S, x, y)$, $l \leqslant k$, computes a feasible solution of cost at most $\alpha \cdot \mathrm{LP}(\mathcal{I})$. Then, there exists a polynomial time algorithm that, for an arbitrary instance of Min-k-SCCP$_S$, finds a feasible solution of cost $c(F_S) \leqslant \alpha \cdot \mathcal{P}^*$.*

Proof. Consider an arbitrary Min-k-SCCP$_S$ instance (G, c, S). Let x^* be an optimal solution of the LP-relaxation $\mathcal{P}$. Although $\mathcal{P}$ has an exponential number of constraints (4.3), x^* can be found in polynomial time, e.g. by the ellipsoid method augmented with polynomial time separation oracle [9]. In addition, without loss of generality, we can assume that

$$\left| \left\{ U \in \mathcal{V} \colon x^*(\delta(U)) = 2 \right\} \right| = poly(n). \tag{4.7}$$

By construction, the graph $G' = (V, E')$, where $E' = \{e \in E \colon x^* > 0\}$ has at most k strongly connected components $W_1, \ldots, W_p$ for $p \leqslant k$. Besides that, there exists a partition $S_1 \cup \ldots \cup S_p$ of the set S, such that $S_i \subset W_i$ for each $i = \overline{1,p}$. We denote by $x'[i]$ the restriction of x^* on $E'(W_i)$.

Let us verify that $x'[i]$ is an optimal solution of the LP-relaxation $\mathcal{P}_i$ of the model (4.1)–(4.4) for the instance (W_i, k_i, S_i), where $k_i = |S_i|$. Indeed, the equations

$$\begin{aligned} x'[i](\delta^-(v)) - x'[i](\delta^+(v)) = 0 \qquad & (v \in V(W_i)) \\ x'[i](\delta(U)) \geqslant 2 \qquad & (U \in \mathcal{V}_i), \end{aligned}$$

for $\mathcal{V}_i = \big\{U\colon \varnothing \neq U \subset V(W_i) \setminus S_i\big\} \cup \big\{\{u\}\colon c \in S_i\big\}$, follows straightforwardly from the choice of $x'[i]$. Next, optimality of $x'[i]$ in problem $\mathcal{P}_i$ follows from the evident decomposition

$$\mathcal{P}^* = \sum_{e\in E} c_e x^*_e = \sum_{i=1}^{p} \sum_{e\in E(W_i)} c_e (x'[i])_e \tag{4.8}$$

and the optimality of x^* in problem $\mathcal{P}$.

For each $i = \overline{1,p}$, find an optimal solution $(a^*[i], y^*[i])$ of the dual $\mathcal{D}_i$. Due to (4.7), these computations can also be carried out in polynomial time. Applying Karzanov's algorithm [12] and following the argument of [25, Lemma 3], to any solution $(a^*[i], y^*[i])$ we assign an optimal solution $(a'[i], y'[i])$ of $\mathcal{D}_i$, such that $\operatorname{supp}(y'[i]) = \mathcal{L}_i = \{U \in \mathcal{V}_i\colon (y'[i])_U > 0\}$ is a laminar family and, for any $L \in \mathcal{L}_i$, the subgraph $W_i[L]$ is strongly connected.

We denote by $y''[i]$ the restriction of $y'[i]$ onto $\mathcal{L}_i$ and introduce the strongly laminar instance $\mathcal{I}_i = (W_i, \mathcal{L}_i, k_i, S_i, x'[i], y''[i])$. By construction, the problems $\mathcal{P}_i$ and $(\mathcal{P}_i)_{ind}$ have the same set of feasible solutions. Furthermore, for the corresponding weighting functions c_e and $\bar{c}_e$, we have $c_e = \bar{c}_e + (a'[i])_w - (a'[i])_v$, which follows from (4.5) and the complementarity conditions. As a consequence, for an arbitrary feasible solution χ of both problems, we have

$$\begin{aligned} \sum_{e\in E'(W_i)} c_e \chi_e &= \sum_{e=(v,w)\in E'(W_i)} (\bar{c}_e + (a'_w[i]) - (a'_v[i]))\chi_e \\ &= \sum_{e\in E'(W_i)} \bar{c}_e \chi_e + \sum_{u\in V(W_i)} (\chi(\delta^-(u)) - \chi(\delta^+(u)))(a'[i])_u \\ &= \sum_{e\in E'(W_i)} \bar{c}_e \chi_e. \end{aligned} \tag{4.9}$$

Finally, by the hypothesis of Lemma 2, for each instance $\mathcal{I}_i$ in polynomial time we can find a solution (W_i, F_i), such that $\bar{c}(F_i) \leqslant \alpha\,\mathrm{LP}(\mathcal{I}_i)$, which implies $c(F_i) \leqslant \alpha\mathcal{P}^*_i$ due to (4.9). Therefore, the subgraph (V, F), where $F_1 \cup \ldots \cup F_p$, is a desired approximate solution for the initial Min-k-SCCP$_S$ instance (G, c, S) of cost

$$c(F) = \sum_{i=1}^{p} c(F_i) \leqslant \alpha \sum_{i=1}^{p} \mathcal{P}^*_i = \alpha \cdot \mathcal{P}^*,$$

where the last equality follows from (4.8). Lemma is proved. □

5. Approximation of a strongly laminar instance

In this section, we propose an approximation algorithm for strongly laminar instances of the Min-k-SCCP$_S$. Consider an arbitrary strongly laminar instance $\mathcal{I} = (G, \mathcal{L}, k, S, x, y)$ with induced weighting function c.

5.1. Preliminaries

We start with some necessary additional notation and preliminary technical results. Let $W \in \mathcal{L} \cup \{V\}$ be the minimal subset that contains nodes u and v. The u-v-path $P_{u,v}$ in the strongly connected subgraph $G[W]$ (of the graph G) and visiting each $L \in \mathcal{L}$ at most once is called a *nice* path. As shown in [25], for arbitrary u and v, the nice path $P_{u,v}$ can be found in polynomial time and its cost is defined by the formula

$$c(E(P_{u,v})) = \sum_{L \in \mathcal{L}_W \colon L \cap V(P_{u,v}) \neq \varnothing} 2y_L - \sum_{L \in \mathcal{L}_W \colon u \in L} y_L - \sum_{L \in \mathcal{L}_W \colon v \in L} y_L, \tag{5.1}$$

where $\mathcal{L}_W = \{L \in \mathcal{L} \colon L \subset W\}$. It is useful to assign to each subset W the value

$$D_W = \max\{D_W(u, v) \colon u, v \in W\}, \tag{5.2}$$

where

$$D_W(u, v) = c(E(P_{u,v})) + \sum_{L \in \mathcal{L}_W \colon u \in L} y_L + \sum_{L \in \mathcal{L}_W \colon v \in L} y_L. \tag{5.3}$$

We slightly extend the concept of a *backbone* introduced in [23].

Definition 4. An Eulerian submultigraph without isolated nodes B of the graph G is called S-backbone, if

- $V(B) \cap L \neq \varnothing$ for any $L \in \mathcal{L}_{\geqslant 2} = \{L \in \mathcal{L} \colon |L| \geqslant 2\}$,
- $S \subset V(B)$,
- $V(D) \cap S \neq \varnothing$ holds for an arbitrary connected component D of B.

Definition 5. An ordered pair $(\mathcal{I}, B)$, where $\mathcal{I}$ is a strongly laminar Min-k-SCCP$_S$ instance and B is an S-backbone, is called a vertebrate pair.

The constant-ratio approximation algorithm proposed in [25] for the ATSP relies on an efficient building block called (κ, η)-*algorithm for vertebrate pairs*. For given parameters $\kappa, \eta \geqslant 0$ and arbitrary vertebrate pair $(\mathcal{I}, B)$, this algorithm computes in polynomial time a set of arcs F', such that $(V, E(B) \cup F')$ is a spanning Eulerian connected submultigraph of G and

$$c(F') \leqslant \kappa \, \mathrm{LP}(\mathcal{I}) + \eta \cdot \sum_{v \in V \setminus V(B) \colon \{v\} \in L} 2y_{\{v\}}. \tag{5.4}$$

As follows from [25], for any $\varepsilon > 0$ and $(\mathcal{I}, B)$, where $\mathcal{I}$ is a strongly laminar ATSP instance and B is an arbitrary connected backbone, there exists $(2, 14+\varepsilon)$-algorithm for vertebrate pairs.
We extended (κ, η)-algorithm to the case of vertebrate pairs, where $\mathcal{I}$ is a strongly laminar Min-k-SCCP$_S$ instance and B is an S-backbone.

Lemma 3. *For any $k > 1$, $\varepsilon > 0$, there exists a polynomial time algorithm, which extends the S-backbone B of an arbitrary vertebrate pair $(\mathcal{I}, B)$ to a feasible solution of the Min-k-SCCP$_S$ instance $\mathcal{I}$ by appending a multiset of arcs F', such that (5.4) is valid for $\kappa = 2$ and $\eta = 14 + \varepsilon$.*

The proof of Lemma 3 can be obtained in a similar way to the argument of [25, Theorem 35], where existence of (κ, η)-algorithm for $\eta = 4\alpha + \beta + 1 + \varepsilon$ was proved as a consequence of existence of the polynomial time (α, κ, β)-algorithm for an auxiliary Subtour Cover Problem (SCP). In turn, (α, κ, β)-algorithm for the SCP was developed (in [25, Theorem 16]) on top of the well-known flow rerouting and rounding technique (see, e.g. [21]) applicable if

$$x(\delta(U)) \geqslant 2 \quad (\varnothing \neq U \subset V \setminus V(B)). \tag{5.5}$$

In the ATSP, inequality (5.5) follows straightforwardly from the problem formulation. In our case, we ensure it by introducing S-backbones. For the sake of brevity, we postpone the full rather technical proof to the forthcoming paper.

5.2. Algorithm: scheme and discussion

The proposed algorithm (Algorithm $\mathcal{A}$) computes a feasible solution $G_F = (V, F)$ of a given strongly laminar Min-k-SCCP instance $\mathcal{I}$, where G_F is a spanning submultigraph of the graph G, whose each connected component D intersects the set S, i.e. $V(D) \cap S \neq \varnothing$. As outer parameters, Algorithm $\mathcal{A}$ takes the (κ, η)-algorithm $\mathcal{A}_{\kappa,\eta}$ for vertebrate pairs (see Def. 4 and Lemma 3) and $(3\kappa + \eta + 2)$-approximation algorithm $\mathcal{A}^*_{ATSP}$ for the ATSP [25].
At step 1, we construct the subfamily $\mathcal{L}_{\bar{S}} = \mathcal{L}$ defined by the following equation

$$\mathcal{L}_{\bar{S}} = \big\{L \in \mathcal{L}_{\geqslant 2} \colon (S \cap L = \varnothing) \wedge (\forall U \in \mathcal{L} \colon L \subset U)(S \cap U \neq \varnothing)\big\}.$$

By construction, all of them do not intersect the set S. Therefore, the instance $\mathcal{I}'$ obtained at step 2 is a singleton instance. Then, at step 2b, we obtain a laminar family $\mathcal{L}'$ by the formula

$$\mathcal{L}' = \mathcal{L} \setminus \big\{L \in \mathcal{L} \colon (\exists U \in \mathcal{L}_{\bar{S}})\,(L \subseteq U)\big\} \cup \bigcup_{L \in \mathcal{L}_{\bar{S}}} \{v_L\}.$$

Further, the S-backbone B can be computed at step 3 from the cycle cover provided by $(2,0)$-light algorithm [23, Th. 4.1] or by sampling. By applying algorithm $\mathcal{A}_{\kappa,\eta}$ at

Algorithm $\mathcal{A}$

Input: strongly laminar Min-k-SCCP$_S$ instance $\mathcal{I} = (G, \mathcal{L}, k, S, x, y)$
Parameters: (κ, η)-algorithm $\mathcal{A}_{\kappa,\eta}$ for vertebrate pairs and algorithm $\mathcal{A}^*_{ATSP}$
Output: a feasible solution $G_F = (V, F)$ of the instance $\mathcal{I}$.

1: set up $\mathcal{L}_{\bar{S}}$ the sub-family of maximal non-singleton elements of the laminar family $\mathcal{L}$, which do not intersect with the given set S;
2: construct Min-k-SCCP$_S$ instance $\mathcal{I}' = (G', \mathcal{L}', k, S, x', y')$, where
 a: multigraph $G' = (V', E')$ is obtained by contracting each $L \in \mathcal{L}_{\bar{S}}$ into the corresponding node v_L,
 b: laminar family $\mathcal{L}'$ is obtained from the family $\mathcal{L}$ by replacement any any $L \in \mathcal{L}_{\bar{S}}$ by the singleton $\{v_L\}$ and exclusion all the subsets of L,
 c: vector x' is a restriction of x on E' and $y' \colon \mathcal{L}' \to \mathbb{R}_+$ is defined by:

$$y'_U = \begin{cases} y_U, & \text{if } U \in \mathcal{L} \cap \mathcal{L}' \\ y_U + D_U/2, & \text{otherwise;} \end{cases} \tag{5.6}$$

3: construct an S-backbone B;
4: apply algorithm $\mathcal{A}_{\kappa,\eta}$ to the vertebrate pair $(\mathcal{I}', B)$ and compute the Eulerian multiset of arcs F' (in the graph G');
5: **for each** $L \in \mathcal{L}_{\bar{S}}$ **do**
6: obtain a traveling salesman tour F_L in L by applying $\mathcal{A}^*_{ATSP}$ to $(\mathcal{I}, L)$,
7: extend F_L by a nice path P_L in $G[L]$ to ensure that the resulting solution remains a Eulerian graph
8: **end for**
9: set $F = \left(\bigcup_{L \in \mathcal{L}_{\bar{S}}} (F_L \cup P_L)\right) \dot{\cup} E(B) \dot{\cup} F'$;
10: **return** (V, F).

step 4, we extend B to a solution for the contracted graph G', which is augmented with ATSP tours for each $L \in \mathcal{L}_{\bar{S}}$ at steps 5–8. Finally, at step 9, we combine all the parts of the resulting solution.

Lemma 4. *For any $k > 1$ and $\kappa, \eta \geqslant 0$, existence of a polynomial time (κ, η)-algorithm for vertebrate pairs (Def. 5) implies the existence of an algorithm which, for an arbitrary strongly laminar Min-k-SCCP$_S$ instance $\mathcal{I}$, computes in polynomial time a feasible solution (V, F) of cost*

$$\mathit{cost}(F) \leqslant (3\kappa + \eta + 4)\,\mathrm{LP}(\mathcal{I}). \tag{5.7}$$

Proof. Since $\mathcal{A}_{\kappa,\eta}$ and $\mathcal{A}^*_{ATSP}$ are polynomial time algorithms, all the steps of Algorithm $\mathcal{A}$ can be carried out in polynomial time as well.
To prove (5.7), obtain upper bounds for the costs $C(F')$, $c(E(B))$ and $c(F_L)$ sepa-

rately. By (5.4) and (5.6), for $c(F')$ we have

$$
\begin{aligned}
c(F') &\leqslant \kappa\cdot \mathrm{LP}(\mathcal{I}') + \eta\cdot\Big(\sum_{L\in\mathcal{L}_{\bar S}} 2y'_{\{v_L\}} + \sum_{v\notin V(B)\colon \{v\}\in\mathcal{L}\cap\mathcal{L}'} 2y'_{\{v\}}\Big)\\
&= \kappa\cdot\Big(\sum_{L\in\mathcal{L}\cap\mathcal{L}'} 2y_L + \sum_{L\in\mathcal{L}_{\bar S}}(2y_L + D_L)\Big)\\
&\quad + \eta\cdot\Big(\sum_{v\notin V(B)\colon \{v\}\in\mathcal{L}\cap\mathcal{L}'} 2y_{\{v\}} + \sum_{L\in\mathcal{L}_{\bar S}}(2y_L + D_L)\Big)\\
&= \kappa\cdot\sum_{L\in\mathcal{L}\cap\mathcal{L}'} 2y_L + (\kappa+\eta)\cdot\sum_{L\in\mathcal{L}_{\bar S}}(2y_L + D_L) + \eta\cdot\sum_{v\notin V(B)\colon \{v\}\in\mathcal{L}\cap\mathcal{L}'} 2y_{\{v\}}.
\end{aligned}
$$

Next, for each F_L due to [25, Lemma 12],

$$
\begin{aligned}
c(F_L) + c(P_L) &\leqslant (2\kappa+2)\,\mathrm{LP}(\mathcal{I}_L) + (\kappa+\eta)(\mathrm{LP}(\mathcal{I}_L) - D_L)\\
&= (3\kappa+\eta+2)\,\mathrm{LP}(\mathcal{I}_L) - (\kappa+\eta)D_L,
\end{aligned}
$$

where

$$
\mathrm{LP}(\mathcal{I}_L) = \sum_{U\in\mathcal{L}\colon U\subset L} 2y_U.
$$

Further, by [23, Th. 4.1],

$$
c(E(B)) \leqslant 2\,\mathrm{LP}(\mathcal{I}') = 2\cdot\Big(\sum_{L\in\mathcal{L}\cap\mathcal{L}'} 2y_L + \sum_{L\in\mathcal{L}_{\bar S}}(2y_L + D_L)\Big).
$$

Finally, taking into account that $D_L \leqslant \mathrm{LP}(\mathcal{I})$ and

$$
\sum_{L\in\mathcal{L}_{\bar S}} 2y_L + \sum_{v\notin V(B)\colon \{v\}\in\mathcal{L}\cap\mathcal{L}'} 2y_{\{v\}} + \sum_{L\in\mathcal{L}_{\bar S}} \mathrm{LP}(\mathcal{I}_L) \leqslant \mathrm{LP}(\mathcal{I}),
$$

we obtain

$$
\begin{aligned}
c(E(B)) + c(F') + \sum_{L\in\mathcal{L}_{\bar S}}\big(c(F_L) + c(P_L)\big) &\leqslant (\kappa+2)\sum_{L\in\mathcal{L}\cap\mathcal{L}'} 2y_L + (\kappa+\eta+2)\sum_{L\in\mathcal{L}_{\bar S}}(2y_L + D_L)\\
&\quad + \eta\sum_{v\notin V(B)\colon \{v\}\in\mathcal{L}\cap\mathcal{L}'} 2y_{\{v\}} + \sum_{L\in\mathcal{L}_{\bar S}}\big((3\kappa+\eta+2)\,\mathrm{LP}(\mathcal{I}_L)\\
&\quad - (\kappa+\eta)D_L\big)\\
&\leqslant (3\kappa+\eta+2)\,\mathrm{LP}(\mathcal{I}) + 2\sum_{L\in\mathcal{L}_{\bar S}} D_L\\
&\leqslant (3\kappa+\eta+4)\,\mathrm{LP}(\mathcal{I}).
\end{aligned}
$$

Lemma is proved. □

Now, we are ready to formulate our main results.

Theorem 1. *For an arbitrary $k \geqslant 1$ and $\varepsilon > 0$, there exists a polynomial time algorithm that assigns to an arbitrary instance (G, c, S) of the Min-k-SCCP$_S$ an approximate solution (V, F_S) of cost $c(F_S) \leqslant (24 + \varepsilon) \cdot \mathcal{P}^*$, where $\mathcal{P}$ is an LP-relaxation of the MILP-model (4.1)–(4.4).*

Proof of Theorem 1 easily follows from Lemma 3 and Lemma 4.

Theorem 2. *For an arbitrary $k \geqslant 1$ and $\varepsilon > 0$, the Min-k-SCCP can be approximated within a ratio $24 + \varepsilon$ by a polynomial time algorithm.*

Proof of Theorem 2 can be obtained by successive application of Lemma 1, Lemma 2, and Theorem 1.

6. Conclusion

In this paper, we proposed the first non-trivial extension of the breakthrough framework designed Svensson, Tarnawski, Végh, Traub, and Vygen for the polynomial time approximation of the ATSP within constant ratio.
As a consequence, we showed that the Min-k-SCCP can be approximated in polynomial time within the ratio $24 + \varepsilon$ for any fixed $k > 1$ and $\varepsilon > 0$. We believe that our approach admits further extension to wider class of asymmetric routing problems and consider such an extension as a possible topic for the future research. In addition, we plan to perform an implementation and numerical evaluation of the proposed algorithm and report on the obtained experimental results in one of the forthcoming papers.

Acknowledgements: This research was supported by Russian Science Foundation, grant no. 22-21-00672, https://rscf.ru/project/22-21-00672/.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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