Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-10
https://doi.org/10.22049/cco.2024.29624.2091

CCO
Commun. Comb. Optim.

***Research Article***

# On the energy of the line graph of unitary Cayley graphs

**Fateme Movahedi**

Department of Mathematics, Faculty of Sciences, Golestan University, Gorgan, Iran
f.movahedi@gu.ac.ir

*Received: 1 April 2024; Accepted: 9 July 2024*
*Published Online: 17 July 2024*

**Abstract:** The energy of a graph $G$ is the sum of the absolute values of the eigenvalues of its adjacency matrix. The energy of the line graph of graph $G$ is denoted by $E(L(G))$. The unitary Cayley graph $X_n$ is a graph with the vertex set $Z_n = \{0, 1, \ldots, n-1\}$ and the edge set $\{(a, b) : ged(a-b, n) = 1\}$. In this paper, we focus on the line graph of the unitary Cayley graph $X_n$ and compute the spectrum of line graphs of $X_n$ and its complement graph $\overline{X_n}$. We also obtain the energy of the line graph of $X_n$ and $\overline{X_n}$.

**Keywords:** graph energy, unitary Cayley graph, spectrum, complement, line graph

**AMS Subject classification:** 05C50, 05C25

## 1. Introduction

Let $G$ be a simple graph of order $n$ with $m$ edges. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the graph $G$ are the eigenvalues of the adjacency matrix of graph $G$. Assume that $\lambda_1 > \lambda_2 > \cdots > \lambda_t$ are a non-increasing sequence of eigenvalues of $G$ and their multiplicities are $m_1, m_2, \ldots, m_t$, respectively. The spectrum of $G$ is written as

$$Spec(G) = \begin{pmatrix} \lambda_1 & \ldots & \lambda_t \\ m_1 & \ldots & m_t \end{pmatrix}.$$

In [7, 13], the energy of graph $G$ is defined as $E(G) = \sum_{i=1}^{t} m_i|\lambda_i|$. The line graph of $G$, denoted by $L(G)$ is the graph that each vertex of it corresponding to an edge of $G$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges in $G$ have a common vertex [9]. The energy of the line graph of graph $G$ is denoted by $E(L(G))$. That is, $E(L(G))$ is the sum of the absolute values of eigenvalues of $A(L(G))$ [1]. Some results are obtained on the energy of the line graphs that can be found in [6, 8, 14].

Let $n > 1$ be a positive integer. The unitary Cayley graph $X_n = Cay(Z_n, U_n)$ is defined by the additive symmetric group $Z_n = \{0, 1, \ldots, n-1\}$ of integers modulo $n$

and the multiplicative group $U_n$ of its units. The vertex set of $X_n$ is the elements of $Z_n$ and the edge set is as $\{(a,b) : a,b \in Z_n, gcd(a-b,n)=1\}$ [12].

In [16], the energy of graph $X_n$ is computed. The spectrum of $X_n$ for $n=p^\alpha$ where $p$ is a prime and $n \geq 1$ is obtained [16]. Ilić [11] calculated the energy of the complement of the unitary Cayley graph $X_n$. In [3], the authors obtained the minimum edge dominating energy of some Cayley graphs including the unitary Cayley graphs. In [15] by computing the eigenvalues and Laplacian eigenvalues of the unitary addition Cayley graph and its complement, some bounds for energy and Laplacian energy for these graphs were obtained. Chen and Huang [2] computed the formulas for the eigenvalues of the Unitary Cayley graph and using these values presented the energy, the Kirchhoff index, and the number of spanning trees of this graph.

In this paper, we focus on the line graphs of $X_n$ and its complement $\overline{X_n}$. We compute the spectrum of $L(X_n)$ and $L(\overline{X_n})$. We also obtain the energy of the line graph of the unitary Cayley graph $X_n$ and its complement $\overline{X_n}$.

## 2. Main Results

In this section, we compute the eigenvalues of the line graph of the unitary Cayley graph $X_n$ and its complement graph. Then, we calculate the energy of the graphs $L(X_n)$ and $L(\overline{X_n})$ via their eigenvalues. To do this, we recall the following result on the eigenvalues of the line graph of a regular graph.

**Lemma 1.** [5] *Let $G$ be a regular graph of degree $r \geq 2$ with $n$ vertices and $m$ edges. Then the following relations hold.*

*(i) For $1 \leq i \leq n$, $\lambda_i(L(G)) = \lambda_i(G) + r - 2$,*

*(ii) for $n+1 \leq i \leq m$, $\lambda_i(L(G)) = -2$.*

In the following theorem, we obtain the line graph spectrum of the unitary Cayley graph $X_n$ for $n = p^\alpha$ where $p$ is prime.

**Theorem 1.** *Let $X_n$ be the unitary Cayley graph where $n = p^\alpha$ and $p$ be prime. If $L(X_n)$ is the line graph of the unitary Cayley graph $X_n$, then the spectrum of $L(X_n)$ is given as follows*

$$Spec(L(X_n)) = \begin{pmatrix} 2\phi(n)-2 & 2\big(\phi(n)-1\big)-n & \phi(n)-2 & -2 \\ 1 & p-1 & n-p & \frac{n\phi(n)}{2}-n \end{pmatrix}.$$

*where $\phi(n)$ is the Euiler function.*

*Proof.* Assume that $X_n$ is the unitary Cayley graph of order $n = p^\alpha$ and size $m = \frac{n\phi(n)}{2}$ whose the degree of all vertices is $\phi(n)$.

If $\alpha = 1$, then the unitary Cayley graph $X_p$ is the complete graph $K_p$. Note that the spectrum of graph $K_p$ is $p-1$ with multiplicity $p-1$. Since $n=p$ and $\phi(p)=p-1$, we can consider the following from $Spec(K_n)$.

$$Spec(K_n)=\begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix}=\begin{pmatrix} \phi(n) & \phi(n)-n \\ 1 & n-1 \end{pmatrix}.$$

Since $K_n$ is $(n-1)$-regular graph, then using Lemma 1, we obtain the eigenvalues of the line graph $L(K_n)$ for $1\leq i\leq \frac{n(n-1)}{2}=m$.

**i)** $\lambda_1(L(K_n))=\lambda_1(K_n)+\phi(n)-2=2\phi(n)-2$.

**ii)** For $2\leq i\leq p-1$, we have $\lambda_i(L(K_n))=\lambda_i(K_n)+\phi(n)-2=\phi(n)-n+\phi(n)-2=2\phi(n)-n-2$.

**iii)** For $n\leq i\leq \frac{n(n-1)}{2}=m$, we have $\lambda_i(L(K_n))=-2$.

Therefore, the $Spec(L(K_n))$ is as follows

$$Spec(L(K_n))=\begin{pmatrix} 2\phi(n)-2 & 2\big(\phi(n)-1\big)-n & -2 \\ 1 & p-1 & \frac{n\phi(n)}{2}-n \end{pmatrix}.$$

Now, we assume $\alpha\geq 2$. In the proof of theorem 3.1 in [16], the spectrum of the unitary Cayley graph $X_n$ for $n=p^\alpha$ is obtained as follows

$$Spec(X_{p^\alpha})=\begin{pmatrix} p^\alpha-p^{\alpha-1} & -p^{\alpha-1} & 0 \\ 1 & p-1 & p^\alpha-p \end{pmatrix}.$$

Since $n=p^\alpha$ and $\phi(n)=p^\alpha-p^{\alpha-1}=n-p^{\alpha-1}$, we can consider the following $Spec(X_n)$ where $n=p^\alpha$.

$$Spec(X_{p^\alpha})=\begin{pmatrix} \phi(n) & \phi(n)-n & 0 \\ 1 & p-1 & n-p \end{pmatrix}.$$

Using Lemma 1, $X_n$ is a $\phi(n)$-regular graph. Thus we can obtain the eigenvalues of the line graph $L(X_n)$.

**i)** $\lambda_1(L(X_n))=\lambda_1(X_n)+\phi(n)-2=2\phi(n)-2$.

**ii)** For $2\leq i\leq p$, we have $\lambda_i(L(X_n))=\lambda_i(X_n)+\phi(n)-2=\phi(n)-n+\phi(n)-2=2\big(\phi(n)-1\big)-n$.

**iii)** For $p+1\leq i\leq n$, we have $\lambda_i(L(X_n))=\lambda_i(X_n)+\phi(n)-2=0+\phi(n)-2=\phi(n)-n$.

**iv)** For $n+1\leq i\leq m=\frac{n\phi(n)}{2}$, the eigenvalues of $L(X_n)$ are $\lambda_i(L(X_n))=-2$.

Therefore, according to the above eigenvalues, the result is completed. □

**Theorem 2.** *The energy of the line graph of the unitary Cayley graph $X_n$ for $n=p^\alpha$ where $p$ is prime, is given as follows*

$$E(L(X_n))=2n\big(\phi(n)-2\big).$$

*Proof.* Let $X_n$ be the $\phi(n)$-regular graph with $n = p^\alpha$ vertices and $m = \frac{n\phi(n)}{2}$ edges. According to the definition, the energy of the line graph of a graph is equal to the sum of the absolute values of the eigenvalues of the adjacency matrix of its line graph. Using Theorem 1, we obtain the energy of $L(X_n)$, where $n = p^\alpha$.
For $\alpha = 1$ where $n = p$, we obtained the spectrum of $L(X_n)$ as follows

$$Spec(L(X_n)) = \begin{pmatrix} 2\phi(n)-2 & 2\big(\phi(n)-1\big)-n & -2 \\ 1 & p-1 & \frac{p\phi(n)}{2}-p \end{pmatrix}.$$

Therefore, we get

$$\begin{aligned} E(L(X_p)) &= \sum_{i=1}^{m} \big|\lambda_i(X_p)\big| \\ &= 2\phi(n)-2+(p-1)\Big(2\big(\phi(n)-1\big)-2\Big) \\ &+ \Big(\frac{p(\phi(n)}{2}-p\Big)\big(|-2|\big) \\ &= 2p\phi(n)-np+p\phi(n)-3p \\ &= p\big(\phi(n)-n\big)+p\big(2\phi(n)-3\big). \end{aligned} \tag{2.1}$$

With considering $n = p$ and $\phi(n) = p-1$, we have from (2.1),

$$\begin{aligned} E(L(X_p)) &= p\big(\phi(n)-n\big)+p\big(2\phi(n)-3\big) \\ &= p(p-1-p)+p\big(2\phi(n)-3\big) \\ &= -p+p\big(2\phi(n)-3\big) \\ &= p\big(2\phi(n)-4\big) \\ &= 2n\big(\phi(n)-2\big). \end{aligned}$$

For $\alpha \geq 2$, using Theorem 1, the spectrum of line graph $L(X_n)$ is as follows

$$Spec(L(X_n)) = \begin{pmatrix} 2\phi(n)-2 & 2\big(\phi(n)-1\big)-n & \phi(n)-2 & -2 \\ 1 & p-1 & n-p & \frac{n\phi(n)}{2}-n \end{pmatrix}.$$

Therefore, we get

$$\begin{aligned} E(L(X_{p^\alpha})) &= \sum_{i=1}^{m} \big|\lambda_i(X_{p^\alpha})\big| \\ &= 2\phi(n)-2+(p-1)\Big(2\phi(n)-2-n\Big)+(n-p)\big(\phi(n)-2\big) \\ &+ \Big(\frac{n(\phi(n)}{2}-n\Big)\big(|-2|\big) \\ &= p\phi(n)-pn-3n+2n\phi(n) \end{aligned}$$

$$= p\big(\phi(n) - n\big) + n\big(2\phi(n) - 3\big). \tag{2.2}$$

With putting $n = p^\alpha$ and $\phi(n) = p^\alpha - p^{\alpha-1}$ in (2.2), we have

$$\begin{aligned}
E(L(X_{p^\alpha})) &= p\big(\phi(n) - n\big) + n\big(2\phi(n) - 3\big) \\
&= p\big(p^\alpha - p^{\alpha-1} - p^\alpha\big) + n\big(2\phi(n) - 3\big) \\
&= p(-p^{\alpha-1}) + n\big(2\phi(n) - 3\big) \\
&= -n + n\big(2\phi(n) - 3\big) \\
&= n\big(2\phi(n) - 4\big) \\
&= 2n\big(\phi(n) - 2\big).
\end{aligned}$$

□

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The direct product of $G_1$ and $G_2$ is the graph $G = (V, E)$ denoted by $G_1 \otimes G_2$ where $V = V_1 \times V_2$, the direct product of $V_1$ and $V_2$ with $(v_1, v_2)$ and $(u_1, u_2)$ are adjacent in $G$ if and only if $v_1$, $u_1$ are adjacent in $G_1$ and $v_2$, $u_2$ are adjacent in $G_2$. The following result is obtained about the direct product of the unitary Cayley graphs.

**Lemma 2.** [16] *Assume that $(m, n) = 1$. If $X_m$ and $X_n$ are the unitary Cayley graphs, then $X_m \otimes X_n \simeq X_{mn}$.*

**Theorem 3.** *Let $L(X_m)$ and $L(X_n)$ be the line graph of the unitary Cayley graphs $X_m$ and $X_n$, respectively. If $(m, n) = 1$, then*

$$L(X_m) \otimes L(X_n) \simeq L(X_{mn}).$$

*Proof.* Since for two graphs $G_1$ and $G_2$, if $G_1 \simeq G_2$ then $L(G_1) \simeq L(G_2)$. Thus by applying Lemma 2.2, it is sufficient to prove $L(X_m) \otimes L(X_n) \simeq L(X_m \otimes X_n)$. Therefore, we show that there is a one-to-one correspondence between the vertices and edges of two graphs $L(X_m) \otimes L(X_n)$ and $L(X_m \otimes X_n)$.

Assume that $e_1 = a_1 b_1 \in V(L(Z_m)) = E(Z_m)$ and $e_2 = a_2 b_2 \in V(L(Z_n)) = E(Z_n)$. Thus, $(a_1 - b_1, m) = 1 = (a_2 - b_2, n)$. Since $a_1 \sim b_1$ in $Z_m$ and $a_2 \sim b_2$ in $Z_n$ thus $(a_1, a_2) \sim (b_1, b_2)$ in $Z_m \otimes Z_n$.
Therefore, $e = \big((a_1, a_2), (b_1, b_2)\big) \in E(Z_m \otimes Z_n) = V(L(Z_m \otimes Z_n))$. Consequently, any vertex $(e_1, e_2) = (a_1 b_1, a_2 b_2)$ in graph $L(Z_m) \otimes L(Z_n)$ corresponding to the vertex $e = \big((a_1, a_2), (b_1, b_2)\big)$ in $L(Z_m \otimes Z_n)$. Therefore, there is the bijective correspondence between vertex sets of $L(Z_m) \otimes L(Z_n)$ and $L(Z_m \otimes Z_n)$.

We show that there is a bijective correspondence between edges in the line graph $(Z_m \otimes Z_n)$ and the direct product of $L(Z_m)$ and $L(Z_n)$. Let $k \in E(L(Z_m \otimes Z_n))$. Thus, there are two vertices $e$ and $e'$ in $L(Z_m \otimes Z_n)$ such that $k = ee'$ where $e =$

$\big((a_1,b_1),(a_2,b_2)\big)$ and $e'=\big((a'_1,b'_1),(a'_2,b'_2)\big)$ for $a_1,a_2,a'_1,a'_2\in Z_m$ and $b_1,b_2,b'_1,b'_2\in Z_n$. According to the line graph of a graph and without loss of generality assume that $(a_1,b_1)=(a'_1,b'_1)$. Therefore, $a_1=a'_1$ and $b_1=b'_1$.

On the other hand, the corresponding with vertices $e$ and $e'$ in $L(Z_m\otimes Z_n)$, we have $k_1=(a_1a_2,b_1b_2)\in L(Z_m)\otimes L(Z_n)$ and $k_2=(a'_1a'_2,b'_1b'_2)\in L(Z_m)\otimes L(Z_n)$. Therefore, it is sufficient to show that $k_1$ and $k_2$ are adjacent in $L(Z_m)\otimes L(Z_n)$. That is, we show $a_1a_2\sim a'_1a'_2$ in $L(Z_m)$ and $b_1b_2\sim b'_1b'_2$ in $L(Z_n)$. Using the definition of the line graph $L(Z_m)$ and since $a_1=a'_1$ and $b_1=b'_1$, then it is clear to have $k_1$ and $k_2$ are adjacent in $L(Z_m)\otimes L(Z_n)$. This completes the proof. □

**Corollary 1.** *If $n=p_1^{\alpha_1}\dots p_k^{\alpha_k}$, then the direct product of unitary Cayley graph $L(X_{P_1^{\alpha_1}})\otimes\cdots\otimes L(X_{P_k^{\alpha_k}})\simeq L(X_n)$.*

The tensor product $A\otimes B$ of the $r\times s$ matrix $A=(a_{ij})$ and the $t\times u$ matrix $B=(b_{ij})$ is defined as $rt\times su$ matrix got by replacing each entry $a_{ij}$ of $A$ with the double array $a_{ij}B$. For two graphs $G_1$ and $G_2$, $A(G_1\otimes G_2)=A(G_1)\otimes A(G_2)$.

**Lemma 3.** [16] *If $G_1$ and $G_2$ are any two graphs, then*

$$E(G_1\otimes G_2)=E(G_1)E(G_2).$$

**Theorem 4.** *If $n>1$ and $n=p_1^{\alpha_1}\dots p_k^{\alpha_k}$ where $p_i$ are distinct primes and $\alpha_i$ are positive integers for $1\le i\le k$, then*

$$E(X_n)=2^kn\big(\phi(n)-2^k\big).$$

*Proof.* Let $X_n$ be the unitary Cayley graph. Using Lemma 3, Corollary 1 and Theorem 2 we get

$$\begin{aligned}E(L(X_n))&=E\big(L(X_{P_1^{\alpha_1}})\otimes\cdots\otimes L(X_{P_k^{\alpha_k}})\big)\\&=E\big(L(X_{P_1^{\alpha_1}})\big)\cdots E\big(L(X_{P_k^{\alpha_k}})\big)\\&=\Big(2P_1^{\alpha_1}\big(\phi(P_1^{\alpha_1})-2\big)\Big)\cdots\Big(2P_k^{\alpha_k}\big(\phi(P_k^{\alpha_k})-2\big)\Big)\\&=2^k\big(p_1^{\alpha_1}\dots p_k^{\alpha_k}\big)\Big(\phi(p_1^{\alpha_1})\dots\phi(p_k^{\alpha_k})-2^k\Big)\\&=2^kn\Big(\phi(p_1^{\alpha_1}\dots p_k^{\alpha_k})-2^k\Big).\end{aligned}$$

Therefore, the result holds. □

Graph $G$ is said to be hyperenergetic if its energy exceeds the energy of the complete graph $K_n$ equivalently $E(G)>2n-2$ [10]. Using the following lemma, we obtain Theorem 5 about hyperenergetic from the unitary Cayley graph $X_n$.

**Lemma 4.** [10] *Let $G$ be a graph of order $n \geq 5$ and of size $m$. If $m \geq 2n$, then $L(G)$ is hyperenergetic.*

**Theorem 5.** *Let $X_n$ be the unitary Cayley graph where $n \geq 5$. If $\phi(n) \geq 4$, then $L(X_n)$ is hyperenergetic.*

*Proof.* $X_n$ is a graph of the order $n \geq 5$ and size $m = \frac{n\phi(n)}{2} \geq 2n$, therefore using Lemma 4, the result is completed. □

**Lemma 5.** [4] *If $G$ is a $r$-regular with $n$ vertices then*

$$P_{\overline{G}}(x) = (-1)^n \frac{x-n+r+1}{x+r+1} P_G(-x-1),$$

*where $P_{\overline{G}}$ is the characteristic polynomial of the complement of the graph $G$.*

**Theorem 6.** *Let $X_n$ be the unitary Cayley graph with $n = p^\alpha$ where $\alpha \geq 2$ and $p \geq 2$. Then the spectrum of the line graph of $\overline{X_n}$ is as follows*

$$Spec(L(\overline{X_n})) = \begin{pmatrix} 2(n-\phi(n)-2) & n-\phi(n)-4 & -2 \\ p & n-p & \frac{n\phi(n)}{2}-n \end{pmatrix}.$$

*Proof.* Let $X_n$ be the $\phi(n)$-regular graph with $n$ vertices. Using Lemma 5 and Theorem 3.1 in [16], the characteristic polynomial of $\overline{X_n}$ is as follows

$$\begin{aligned} P_{\overline{X_n}}(\lambda) &= (-1)^n \Big(\frac{\lambda-n+\phi(n)+1}{\lambda+\phi(n)+1}\Big) P_{X_n}(-\lambda-1) \\ &= (-1)^n \Big(\frac{\lambda-n+\phi(n)+1}{\lambda+\phi(n)+1}\Big)\big(-\lambda-\phi(n)-1\big) \\ &\quad \big(-\lambda-\phi(n)+n-1\big)^{p-1}\big(-\lambda-1\big)^{n-p} \\ &= (-1)^{2n}\big(\lambda-n+\phi(n)+1\big)\big(\lambda+\phi(n)-n+1\big)^{p-1}(\lambda+1)^{n-p} \\ &= \big(\lambda+\phi(n)-n+1\big)^p(\lambda+1)^{n-p}. \end{aligned}$$

Therefore, the eigenvalues of $\overline{X_n}$ are $n-\phi(n)-1$ with multiplicity $p$ and $-1$ with multiplicity $n-p$. Since $\overline{X_n}$ is an $(n-1)-\phi(n)$-regular graph, then the eigenvalues of $L(\overline{X_n})$ are as follows

**i)** For $1 \leq i \leq p$, we get

$$\begin{aligned} \lambda_i(L(\overline{X_n})) &= \lambda_i(\overline{X_n}) + (n-1) - \phi(n) - 2 \\ &= n-\phi(n)-1+(n-1)-\phi(n)-2 \\ &= 2n-2\phi(n)-4 = 2(n-\phi(n)-2). \end{aligned}$$

**ii)** For $p+1 \leq i \leq n$, we get

$$\begin{aligned}\lambda_i(L(\overline{X_n})) &= \lambda_i(\overline{X_n}) + (n-1) - \phi(n) - 2 \\ &= -1 + (n-1) - \phi(n) - 2 \\ &= n - \phi(n) - 4.\end{aligned}$$

**iii)** For $n+1 \leq i \leq \frac{n\phi(n)}{2}$, we have $\lambda_i(L(\overline{X_n})) = -2$.
Therefore, the result holds. □

**Theorem 7.** *Let $X_n$ be the unitary Cayley graph with $n = p^\alpha$ where $\alpha \geq 2$ and $p \geq 2$. If $\overline{X_n}$ is the complement of $X_n$, then the energy of $L(\overline{X_n})$ is given as follows*

*i) If $n = 4$, then $E(L(\overline{X_4})) = 6$.*

*ii) If $n = 9$, then $E(L(\overline{X_9})) = 58$.*

*iii) If $\alpha \geq 2$ and $p \geq 5$, then $E(L(\overline{X_n})) = n(n-5)$.*

*Proof.* Using Theorem 6, we get

$$\begin{aligned}E(L(\overline{X_{p^\alpha}})) &= \sum_{i=1}^{m} \left|\lambda_i\big(L(\overline{X_{p^\alpha}})\big)\right| \\ &= 2p\big|n - \phi(n) - 2\big| + (n-p)\big|n - \phi(n) - 4\big| + \big(\frac{n\phi(n)}{2} - n\big)| - 2|.\end{aligned}$$

We consider the following cases.
**Case 1.** If $\alpha = 2$ and $p = 2, 3$, then

$$E(L(\overline{X_{p^2}})) = 2p\big(n - \phi(n) - 2\big) + (n-p)\big(4 - n + \phi(n)\big) + n\phi(n) + 2n.$$

With putting $\alpha = 2$ and $p = 2, 3$, we obtain $E(L(\overline{X_{2^2}})) = 6$ and $E(L(\overline{X_{3^2}})) = 58$.
**Case 2.** Let $\alpha \geq 2$ and $p \geq 5$. Then

$$\begin{aligned}E(L(\overline{X_{p^\alpha}})) &= 2p\big(n - \phi(n) - 2\big) + (n-p)\big(n - \phi(n) - 4\big) + n\phi(n) - 2n \\ &= pn - p\phi(n) + n^2 - 6n \\ &= p\big(n - \phi(n)\big) + (n^2 - 6n) \\ &= p\big(p^\alpha - p^\alpha + p^{\alpha-1}\big) + (n^2 - 6n) \\ &= p(p^{\alpha-1}) + n^2 - 6n \\ &= p^\alpha + n^2 - 6n \\ &= n + n^2 - 6n.\end{aligned}$$

Therefore, the result holds. □

**Theorem 8.** *Let $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where $\alpha \geq 2$ and $p \geq 5$. Then*

$$E(L(\overline{X_n})) = n^2 - 5^k n.$$

*Proof.* Similar to the proof from Theorem 3 and using Corollary 1 and Lemma 3, we get

$$E(L(\overline{X_n})) = E(L(\overline{X_{p_1^{\alpha_1}}}))E(L(\overline{X_{p_2^{\alpha_2}}})) \dots E(L(\overline{X_{p_k^{\alpha_k}}})).$$

By applying Theorem 7 in the above relation, we have

$$\begin{aligned} E(L(\overline{X_n})) &= \Big((p_1^{\alpha_1})^2 - 5p_1^{\alpha_1}\Big)\Big((p_2^{\alpha_2})^2 - 5p_2^{\alpha_2}\Big) \cdots \Big((p_k^{\alpha_k})^2 - 5p_k^{\alpha_k}\Big) \\ &= p_1^{\alpha_1} \dots p_k^{\alpha_k} \Big(\prod_{i=1}^{k} (p_i^{\alpha_i} - 5)\Big) \\ &= n\Big(\prod_{i=1}^{k} p_i^{\alpha_i} - \prod_{i=1}^{k} 5\Big) \\ &= n\big(n - 5^k\big). \end{aligned}$$

□

**Acknowledgements:** The present study was supported by Golestan University, Gorgan, Iran (research number: 1648). The author truly appreciates Golestan University for this support. The author is grateful to the referees for their valuable comments and suggestions to improve the quality of the paper.

**Conflict of Interest:** The authors declare that they have no conflict of interest.

**Data Availability:** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

## References

[1] Ş.B. Bozkurt and D. Bozkurt, *On incidence energy*, MATCH Commun. Math. Comput. Chem **72** (2014), 215–225.

[2] B. Chen and J. Huang, *On unitary Cayley graphs of matrix rings*, Discrete Math. **345** (2022), no. 1, Article ID: 112671.
https://doi.org/10.1016/j.disc.2021.112671.

[3] S. Chokani, F. Movahedi, and S.M. Taheri, *The minimum edge dominating energy of the Cayley graphs on some symmetric groups*, Algebr. Struct. their Appl. **10** (2023), no. 2, 15–30.
https://doi.org/10.22034/as.2023.3001.

[4] D. Cvetković, P. Rowlinson, and S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, New York, 2009.

[5] D.M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs: Theory and Application*, Academic Press, New York, 1980.

[6] K.C. Das, S.A. Mojallal, and I. Gutman, *On energy of line graphs*, Linear Algebra Appl. **499** (2016), 79–89.
https://doi.org/10.1016/j.laa.2016.03.003.

[7] I. Gutman, *The energy of a graph*, Ber. Math. Statist. Sekt. Forsch. Graz **103** (1978), 1–22.

[8] I. Gutman, M. Robbiano, E.A. Martins, D.M. Cardoso, L. Medina, and O. Rojo, *Energy of line graphs*, Linear Algebra Appl. **433** (2010), no. 7, 1312–1323.
https://doi.org/10.1016/j.laa.2010.05.009.

[9] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, 1969.

[10] Y. Hou and I. Gutman, *Hyperenergetic line graphs*, MATCH Commun. Math. Comput. Chem **43** (2001), 29–39.

[11] A. Ilić, *The energy of unitary Cayley graphs*, Linear Algebra Appl. **431** (2009), no. 10, 1881–1889.
https://doi.org/10.1016/j.laa.2009.06.025.

[12] W. Klotz and T. Sander, *Some properties of unitary Cayley graphs*, Electron. J. Comb. **14** (2007), Article Number: R45
https://doi.org/10.37236/963.

[13] X. Li, Y. Shi, and I. Gutman, *Graph Energy*, Springer, New York, 2012.

[14] F. Movahedi, *The energy and edge energy of some Cayley graphs on the abelian group* $\mathbb{Z}_n^4$, Commun. Comb. Optim. **9** (2024), no. 1, 119–130.
https://doi.org/10.22049/cco.2023.28642.1647.

[15] N. Palanivel and A.V. Chithra, *Energy and Laplacian energy of unitary addition Cayley graphs*, Filomat **33** (2019), no. 11, 3599–3613.
https://doi.org/10.2298/FIL1911599P.

[16] H.N. Ramaswamy and C.R. Veena, *On the energy of unitary Cayley graphs*, Electron. J. Comb. **16** (2009), no. 1, Article Number: N24
https://doi.org/10.37236/262.