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Research Article

2-rainbow domination number of the subdivision of graphs

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Abstract: Let G be a simple graph and $f : V(G) \to P(\{1, 2\})$ be a function where for each vertex $v \in V(G)$ with $f(v) = \emptyset$ we have $\bigcup_{u \in N_G(v)} f(u) = \{1, 2\}$. Then f is a 2-rainbow dominating function (a $2RDF$) of G. The weight of f is $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight among all of 2−rainbow dominating functions is 2−rainbow domination number and is denoted by $\gamma_{r2}(G)$. In this paper, we provide some bounds for the 2−rainbow domination number of the subdivision graph $S(G)$ of a graph G. Also, among some other interesting results, we determine the exact value of $\gamma_{r2}(S(G))$ when G is a tree, a bipartite graph, $K_{r,s}$, $K_{n_1,n_2,\ldots,n_k}$ and K_n.

Keywords: 2-Rainbow domination number, subdivision, bipartite graph, tree

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1. Introduction

Let $G = (V(G), E(G))$ be a simple and finite graph. The *open neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. The *closed neighborhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$. When $S \subseteq V(G)$, the *induced subgraph* of G on S is obtained by removing all of vertices in $V(G) \setminus S$ (and their incident edges) from G. A subset D of G is a dominating set of G if each vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. The domination number of G, denoted by $\gamma(G)$, is the minimum size of a dominating set of G. In recent years the domination theory (which is an interesting branch in graph theory) attracts the attention of many authors and

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this concept is expanded to other related parameters of domination like distance domination [10], weighted domination [12], Roman domination [9], signed Roman domination [5], outer independent double Roman domination [18], outer independent total double Roman domination [1], rainbow domination [6], independent 2-rainbow domination [14, 19, 22], total 2-rainbow domination [2], outer-independent total 2-rainbow domination [16], identifying code [3], et cetera. The concept of rainbow domination was introduced in [6] and has been studied extensively since then. Note that the power set of $\{1, 2\}$ is $P(\{1, 2\}) = \big\{\emptyset, \{1\}, \{2\}, \{1, 2\}\big\}$. Let $f : V(G) \to P(\{1, 2\})$ be a function where for each vertex $v \in V(G)$ with $f(v) = \emptyset$ we have $\bigcup_{u \in N_G(v)} f(u) = \{1, 2\}$. Then, f is a 2−rainbow dominating function of G (a $2RDF$ for convenient). The weight of f is $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight among all of 2−rainbow dominating functions is 2−rainbow domination number and is denoted by $\gamma_{r2}(G)$. In [24], all graphs with 2−rainbow domination number 1 or 2 are characterized and some sharp bounds for general graphs are provided, see [23] and [21] for more bounds. In [6] it is shown that the concept of 2-rainbow domination of a graph coincides with the ordinary domination of the prism produced by it, and for the path and cycle graphs it is proved that $\gamma_{r2}(P_n) = \lfloor \frac{n}{2} \rfloor + 1$ and $\gamma_{r2}(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$. In [7] it is proved that the problem of deciding if a graph has a 2-rainbow dominating function of a given weight is an NP-complete problem even when restriction to bipartite graphs or chordal graphs is considered, the exact values of 2-rainbow domination numbers of several important classes of graphs are determined, and it is shown that for the generalized Petersen graphs this number is bounded by sharp bounds (see also [8] and [11]). In [23], Wu and Jafari Rad proved that if G is a connected graph of order $n \geq 3$, then $\gamma_{r2}(G) \leq \frac{3n}{4}$ and they characterized all of graphs achieving the equality. A lower bound for 2-rainbow domination number of a tree using its domination number is provided in [23], in which for an arbitrary graph other bounds are obtained in terms of the diameter of graph. Also, 2-rainbow domination number of functigraphs and their complements is considered in [20]. The relation between ordinary domination number and 2-rainbow domination number of a connected graph G is investigated in [4] and it is shown that $\gamma(G) \leq \gamma_{r2}(G) \leq 2\gamma(G)$. In [9], the relation between 2-rainbow domination number and Roman domination number of graphs is investigated and an upper bound for the 2- rainbow domination number for each tree of order at least three in terms of the number of vertices, stems and leaves of the tree is obtained. The subdivision operation of G is an operation that replaces any edge by a path of order at least two. If each edge is replaced by a path of order three (and length two), then the subdivision graph is denoted by $S(G)$. In [17] some (algebraic) properties of the subdivision graph of a graph is investigated and it is shown that except the cycle C_n, when G is a connected graph of order at least three, then the automorphism groups of G and $S(G)$ are isomorphic. Domination number and identifying code number of the subdivision of some famous families of graphs are investigated and determined in [3]. Some upper and lower bounds for the mixed metric dimension of $S(G)$ is provided in [13]. The minimum number of edges that must be subdivided in order to increase the total k-rainbow domination number of a graph is considered in [15]. Here we will determine the 2−rainbow domination number of the subdivision graph of some famous families of graphs.

2. Main Results

First of all, we provide some bounds for the 2-rainbow domination number of the subdivision $S(G)$ of an arbitrary graph G.

Theorem 1. *Let $t \in \mathbb{N}$ be an integer and $H = tK_2$ be an induced subgraph of an n-vertex graph G. Then we have $\gamma_{r2}(S(G)) \leq 2(n - t)$.*

Proof. Assume that $V(G) = \{x_1, \dots, x_n\}$, $V(S(G)) = V(G) \cup \{z_{i,j} \mid x_i x_j \in E(G)\}$ and

$$E(H) = \{x_{i_1}x_{j_1}, x_{i_2}x_{j_2}, \dots, x_{i_t}x_{j_t}\} \subseteq E(G).$$

Define the function $f : V(S(G)) \to P(\{1, 2\})$ as

$$f(v) = \begin{cases} \{1\} & \text{if } v \in \{x_{i_1}, x_{i_2}, \dots, x_{i_t}\} \\ \{2\} & \text{if } v \in \{x_{j_1}, x_{j_2}, \dots, x_{j_t}\} \\ \{1,2\} & \text{if } v \in V(G) \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_t}, x_{j_1}, x_{j_2}, \dots, x_{j_t}\} \\ \emptyset & \text{if } v \notin V(G). \end{cases}$$

Since H is an induced subgraph of G, it is easy to check that f is a $2RDF$ for $S(G)$ and hence, $\gamma_{r2}(S(G)) \leq w(f) = t \times 1 + t \times 1 + (n - 2t) \times 2 = 2n - 2t$. □

Corollary 1. *Let G be a graph of order $n \geq 2$. Then $\gamma_{r2}(S(G)) \leq 2n - 2$.*

Proof. If $E(G) = \emptyset$, then $S(G) = G$ and the function $f : V(S(G)) \to P(\{1, 2\})$ defined by $f(v) = \{1\}$ for each $v \in V(S(G)) = V(G)$, is a $2RDF$ and hence

$$\gamma_{r2}(S(G)) = \gamma_{r2}(G) \leq w(f) = n \leq 2n - 2.$$

Thus assume that $E(G) \neq \emptyset$ and hence, K_2 is an induced subgraph of G. Now the result follows directly from Theorem 1. □

Corollary 2. *Let G be a graph of order $n \geq 2$ with s isolated vertices and t connected components of order at least two. Then $\gamma_{r2}(S(G)) \leq 2(n - t) - s$.*

Proof. Choose one edge from each connected component of order at least two to produce an induce tK_2 in G and consider the function $f : V(S(G)) \to P(\{1, 2\})$ as defined in the proof of Theorem 1 by modifying it for each isolated vertex $v \in V(G)$ as $f(v) = \{1\}$. □

Theorem 2. *If $t \geq 2$ and the path P_t is an induced subgraph in an n-vertex graph G, then $\gamma_{r2}(S(G)) \leq 2n - t$.*

Proof. Assume that $V(G) = \{x_1, \ldots, x_n\}$, $V(S(G)) = V(G) \cup \{z_{i,j} \mid x_i x_j \in E(G)\}$ and $P_t = x_{i_1} x_{i_2} \ldots x_{i_t}$. Now define the function $f : V(S(G)) \to P(\{1,2\})$ as

$$f(v) = \begin{cases} \{1\} & \text{if } v = x_{i_k} \in V(P_t) \text{ with } \ k \equiv 1 \pmod 2 \\ \{2\} & \text{if } v = x_{i_k} \in V(P_t) \text{ with } \ k \equiv 0 \pmod 2 \\ \{1,2\} & \text{if } v \in V(G) \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\} \\ \emptyset & \text{if } v \notin V(G). \end{cases}$$

Since P_t is an induced subgraph of G, f is a $2RDF$ for $S(G)$ and hence,

$$\gamma_{r2}(S(G)) \leq w(f) = t \times 1 + (n - t) \times 2 = 2n - t.$$

□

Corollary 3. *For each n-vertex graph G with diameter d, we have $\gamma_{r2}(S(G)) \leq 2n - d$.*

Proof. By Theorem 2, the proof is straightforward. □

Now in the following result we determine the $2-$rainbow domination number of the subdivision of each bipartite graph. This result leads to the determination of the $2-$rainbow domination number of the subdivision of each complete bipartite graph and each tree.

Theorem 3. *For each bipartite graph G we have $\gamma_{r2}(S(G)) = |V(G)|$.*

Proof. Let X, Y be two partite sets of the bipartite graph G and assume that $|X| = r, |Y| = s$. The function $f : V(S(G)) \to P(\{1,2\})$ defined by

$$f(v) = \begin{cases} \{1\} & \text{if } v \in X \\ \{2\} & \text{if } v \in Y \\ \emptyset & \text{otherwise,} \end{cases}$$

is a $2RDF$ for $S(G)$ and hence, $\gamma(S(G)) \leq w(f) = |X| + |Y| = |V(G)|$. If $E(G) = \emptyset$, then $S(G) = G$ is an empty graph on $r + s$ vertices and the function f is a $2RDF$ of minimum weight $r + s$, hence $\gamma_{r2}(S(G)) = r + s = |V(G)|$. Thus, assume that $E(G) \neq \emptyset$ and let $g : V(S(G)) \to P(\{1,2\})$ be a $2RDF$ of minimum weight, i.e. $\gamma_{r2}(S(G)) = w(g)$. Note that $w(g) \leq w(f) = |V(G)|$. If for each $v \in X \cup Y$ we have $g(v) \neq \emptyset$, then

$$\gamma_{r2}(S(G)) = w(g) \geq \sum_{v \in X \cup Y} |g(v)| \ \geq |X \cup Y| = |V(G)|,$$

which implies that $w(g) = |V(G)|$ and the proof is complete. Thus, assume that there exists $v \in X \cup Y$ such that $g(v) = \emptyset$. Let

$$X_1 = \{x \mid x \in X,\ g(x) = \emptyset\},\ Y_1 = \{y \mid y \in Y,\ g(y) = \emptyset\},\ r_1 = |X_1|,\ s_1 = |Y_1|.$$

Note that $X_1 \cup Y_1 \neq \emptyset$ and hence, $r_1 + s_1 \geq 1$. Without loss of generality, we can assume that $r_1 \geq s_1$ and hence, $r_1 \geq 1$. Since $|g(u)| \geq 1$ for each $u \in (X \setminus X_1) \cup (Y \setminus Y_1)$ we obtain

$$\sum_{x \in X} |g(x)| = \sum_{x \in X \setminus X_1} |g(x)| \geq (r - r_1), \quad \sum_{y \in Y} |g(y)| = \sum_{y \in Y \setminus Y_1} |g(y)| \geq (s - s_1).$$

Since g is a $2RDF$, for each $x \in X_1$ we have $\cup_{z \in N_{S(G)}(x)} g(z) = \{1, 2\}$. This implies that

$$\sum_{z \in V(S(G)) \setminus V(G)} |g(z)| \geq \sum_{x \in X_1} \sum_{z \in N_{S(G)}(x)} |g(z)| \geq \sum_{x \in X_1} 2 = 2r_1.$$

Note that for each $v \in X \cup Y$ we have $N_{S(G)}(v) \cap (X \cup Y) = \emptyset$. Therefore,

$$\begin{aligned}
\gamma_{r2}(S(G)) &= w(g) \\
&= \sum_{x \in X} |g(x)| + \sum_{y \in Y} |g(y)| + \sum_{z \in V(S(G)) \setminus V(G)} |g(z)| \\
&\geq (r - r_1) + (s - s_1) + 2r_1 \\
&= (r + s) + (r_1 - s_1) \\
&\geq (r + s) + 0 \\
&= |V(G)|.
\end{aligned}$$

Hence, $\gamma_{r2}(S(G)) = |V(G)|$ and the proof is complete. □

Corollary 4. *For each complete bipartite graph $K_{r,s}$ we have $\gamma_{r2}(S(K_{r,s})) = r + s$.*

Proof. By Theorem 3, the proof is straightforward. □

Since each tree is a bipartite graph, the following result follows directly.

Corollary 5. *Let T be a tree of order n. then $\gamma_{r2}(S(T)) = n$.*

For the 2-rainbow domination number of the subdivision of complete multipartite graphs we have the following interesting result.

Theorem 4. *Let $k \geq 3$ be an integer and $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph of order $n = n_1 + n_2 + \cdots + n_k$ in which $n_1 \geq n_2 \geq \cdots \geq n_k$. Then, we have $\gamma_{r2}(S(G)) = 2n - n_1 - n_2$.*

Proof. Assume that $V(G) = X^1 \cup X^2 \cup \cdots \cup X^k$, in which X^i, $1 \leq i \leq k$, is the i-th part of the vertices of complete k-partite graph G, $|X^i| = n_i$, $X^i = \{x_1^i, x_2^i, \ldots, x_{n_i}^i\}$ and $V(S(G)) = V(G) \cup B$ where

$$B = \left\{x_{rs}^{ij} \mid 1 \leq i < j \leq k, 1 \leq r \leq n_i, 1 \leq s \leq n_j, N_{S(G)}(x_{rs}^{ij}) = \{x_r^i, x_s^j\}\right\}.$$

Define the function $f : V(S(G)) \to P(\{1,2\})$ as

$$f(v) = \begin{cases} \{1\} & \text{if } v \in X^1 \\ \{2\} & \text{if } v \in X^2 \\ \{1,2\} & \text{if } v \in V(G) \setminus \{X^1, X^2\} \\ \emptyset & \text{otherwise.} \end{cases}$$

It can be easily checked that f is $2RDF$ for $S(G)$ and hence,

$$\gamma_{r2}(S(G)) \leq w(f) = n_1 + n_2 + 2(n - n_1 - n_2) = 2n - n_1 - n_2.$$

Now let g be $2RDF$ for $S(G)$ with the minimum weight. We consider three following cases.

Case 1. $g(v) = \emptyset$ for each $v \in V(G)$.

Since g is a $2RDF$, this implies that $g(x_{rs}^{ij}) \neq \emptyset$ (i.e. $|g(x_{rs}^{ij})| \geq 1$) for each $x_{rs}^{ij} \in B$. Thus, $w(g) \geq |E(G)|$ which using Handshaking Lemma means that $w(g) \geq \frac{1}{2}\sum_{i=1}^{k} n_i(n - n_i)$. Since $w(g) = \gamma_{r2}(S(G)) \leq 2n - n_1 - n_2$, we obtain

$$\frac{1}{2}\sum_{i=1}^{k} n_i(n - n_i) \leq 2n - n_1 - n_2 = (n - n_1) + (n - n_2).$$

Hence,

$$\sum_{i=1}^{k} n_i(n - n_i) \leq 2(n - n_1) + 2(n - n_2). \tag{1}$$

Since $k \geq 3$ and $n_3 \geq 1$, inequality (1) implies that

$$n_1(n - n_1) + n_2(n - n_2) < 2(n - n_1) + 2(n - n_2).$$

If $n_2 \geq 2$, then $n_1 \geq n_2 \geq 2$ and this implies that

$$n_1(n-n_1)+n_2(n-n_2) \geq 2(n-n_1)+2(n-n_2),$$

which is a contradiction. Thus, $n_2 = 1$ and since $n_2 \geq n_3 \geq \cdots \geq n_k \geq 1$, we have

$$1 = n_2 = n_3 = \cdots = n_k, \quad n_1 = n-(n_2+n_3+\cdots+n_k) = n-(k-1).$$

Now from inequality (1) we obtain

$$(n-k+1)(k-1)+(k-1)(n-1) \leq 2(k-1)+2(n-1).$$

This implies that $n \leq \frac{k(k-1)}{2(k-2)}+1$. Since $k \leq n$, we have $k \leq \frac{k(k-1)}{2(k-2)}+1$ which leads to the inequality $k^2-5k+4 \leq 0$. Since k is an integer, $k \in \{1,2,3,4\}$ and since $3 \leq k$, we have $k=3$ or $k=4$. If $k=3$, then the inequality $n \leq \frac{k(k-1)}{2(k-2)}+1$ implies that $n \leq 4$ and since $3 = k \leq n$ we have $n=3$ or $n=4$. Therefore, $G = K_{1,1,1}$ or $G = K_{2,1,1}$. If $k=4$ then two inequalities $n \leq \frac{k(k-1)}{2(k-2)}+1$ and $k \leq n$ imply that $n=4$ and hence, $G = K_{1,1,1,1}$. By investigation we see that $\gamma_{r2}(S(G)) = 2n-n_1-n_2$ when $G \in \{K_{1,1,1}, K_{2,1,1}, K_{1,1,1,1}\}$ and Figure 1 provides an optimal $2RDF$ for each of these graphs.

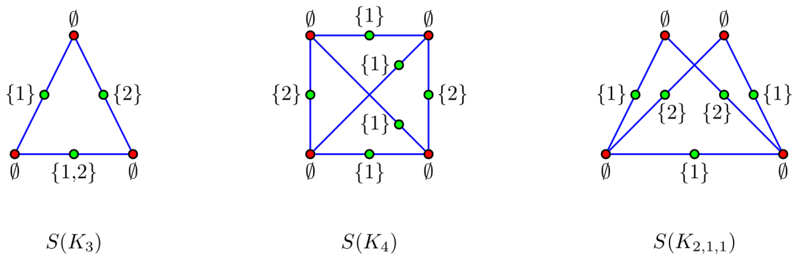

Figure 1: Optimal 2−rainbow dominating functions for $S(K_3)$, $S(K_4)$ and $S(K_{2,1,1})$

Case 2. There exists a vertex $x_r^i \in V(G)$ such that $|g(x_r^i)| = 1$.
Without loss of generality, assume that $g(x_r^i) = \{1\}$. Now we use the following algorithm to modify the 2-Rainbow dominating function g. If there exist j, s such that $g(x_{rs}^{ij}) \neq \emptyset$, then we define the function $g_1 : V(S(G)) \to P(\{1,2\})$ as

$$g_1(u) = \begin{cases} \emptyset & \text{if } u = x_{rs}^{ij} \\ g(x_s^j) \cup \{2\} & \text{if } u = x_s^j \\ g(u) & \text{if } u \notin \{x_s^j, x_{rs}^{ij}\}. \end{cases}$$

Since g is a $2RDF$, the function g_1 is a $2RDF$ and $w(g_1) \leq w(g)$. On the other hand, since $\gamma_{r2}(S(G)) = w(g)$, we have $w(g_1) = \gamma_{r2}(S(G))$ i.e. g_1 is also a $2RDF$ with the minimum weight. Thus, we can replace g by g_1. By repeating this algorithm if it's necessary, we can suppose that $g(x^{ip}_{rq}) = \emptyset$ for each $x^{ip}_{rq} \in N_{S(G)}(x^i_r)$ and hence, $\{2\} \subseteq g(x^p_q)$ for each $x^p_q \in V(S(G)) \setminus X^i$ (because g is a $2RDF$). If $|g(x^p_q)| = 2$ for each $x^p_q \in V(S(G)) \setminus X^i$, then

$$\begin{aligned} w(g) &\geq \sum_{v\in X^i} \sum_{u\in N_{S(G)}[v]} |g(u)| + \sum_{x^j_s \in V(S(G))\setminus X^i} |g(x^j_s)| \\ &\geq n_i \times 1 + (n - n_i) \times 2 \\ &= 2n - n_i \\ &> 2n - n_1 - n_2, \end{aligned}$$

which is a contradiction. Thus, there exists $x^j_s \in V(S(G)) \setminus X^i$ such that $|g(x^j_s)| = 1$. Since $g(x^{ij}_{rs}) = \emptyset$, $N_{S(G)}(x^{ij}_{rs}) = \{x^i_r, x^j_s\}$ and $g(x^i_r) = \{1\}$, we must have $g(x^j_s) = \{2\}$. Now since $|g(x^j_s)| = 1$, we can use the previous algorithm to modify g and thus we can assume that $g(x^{jp}_{sq}) = \emptyset$ for each $x^{jp}_{sq} \in N_{S(G)}(x^j_s)$, and hence $\{1\} \subseteq g(x^p_q)$. Let x^p_q be an arbitrary vertex in $V(G) \setminus (X^i \cap X^j)$. since

$$g(x^{ip}_{rq}) = \emptyset,\ N_{S(G)}(x^{ip}_{rq}) = \{x^i_r, x^p_q\},\ g(x^i_r) = \{1\}$$

we must have $\{2\} \subseteq g(x^p_q)$ and since

$$g(x^{jp}_{sq}) = \emptyset,\ N_{S(G)}(x^{jp}_{sq}) = \{x^j_s, x^p_q\},\ g(x^j_s) = \{2\}$$

we must have $\{1\} \subseteq g(x^p_q)$. Hence, $g(x^p_q) = \{1, 2\}$ for each $x^p_q \in V(G) \setminus (X^i \cap X^j)$. Thus,

$$2n-n_1-n_2 \geq w(g) \geq \sum_{v\in V(G)} |g(v)| \geq n_i \times 1 + n_j \times 1 + (n - n_i - n_j) \times 2 = 2n - n_i - n_j$$

which using the inequality $n_i + n_j \leq n_1 + n_2$ implies that

$$n_i + n_j = n_1 + n_2, \quad w(g) = 2n - n_1 - n_2 = \sum_{v\in V(G)} |g(v)|.$$

Therefore, $\gamma_{r2}(S(G)) = w(g) = 2n - n_1 - n_2$ which completes the proof in this case (note that in this case two functions g and f are defined on $V(S(G))$ almost similarly but they may be unequal).

Case 3. For each $v \in V(G)$ we have $|g(v)| \in \{0, 2\}$ and there exists a vertex $x^i_r \in V(G)$ such that $|g(x^i_r)| = 2$.

Since $w(g) \leq 2n - n_1 - n_2 < 2n$, there exist some vertices in $V(G)$ whose assigned weights by the function g are 0 and hence, some of their neighbors (which are vertices in $V(S(G)) \setminus V(G))$ have non-zero weights. If there exists a vertex $x_{sq}^{jp} \in V(S(G)) \setminus V(G)$ with $g(x_{sq}^{jp}) = \{1, 2\}$, then define the function $g_1 : V(S(G)) \to P(\{1, 2\})$ as

$$g_1(u) = \begin{cases} \emptyset & \text{if } u = x_{sq}^{jp} \\ g(x_s^j) \cup \{1\} & \text{if } u = x_s^j \\ g(x_q^p) \cup \{2\} & \text{if } u = x_q^p \\ g(u) & \text{otherwise.} \end{cases}$$

Since g is a $2RDF$, g_1 is a $2RDF$ and the fact $w(g_1) \leq w(g)$ using the optimality of $w(g)$ implies that $w(g_1) = w(g)$. Thus, we can replace g by g_1 and we can repeat this method if it is necessary. Hence, we can assume that each vertex in $V(S(G)) \setminus V(G)$ has assigned $\emptyset$, $\{1\}$ or $\{2\}$ under g (i.e. the weight is either 0 or 1). Now if there exists a vertex $v \in V(G)$ such that $|g(v)| = 1$, then we can apply Case 2 to complete the proof, otherwise we continue the following proof. Note that $g(x_r^i) = \{1, 2\}$ and if $u \in N_{S(G)}(x_r^i)$, then $u = x_{rs}^{ij}$ for some $j \neq i$ and $s \in \{1, 2, \ldots, n_j\}$. Consider the function $g_2 : V(S(G)) \to P(\{1, 2\})$ defined by

$$g_2(u) = \begin{cases} \emptyset & \text{if } u \in N_{S(G)}(x_r^i) \\ g(x_s^j) \cup g(x_{rs}^{ij}) & \text{if } u = x_s^j,\ j \neq i \\ g(u) & \text{otherwise.} \end{cases}$$

Since g is a $2RDF$, g_2 is a $2RDF$ and the fact $w(g_2) \leq w(g)$ using the optimality of $w(g)$ implies that $w(g_2) = w(g)$. Thus, we can replace g by g_2. Similarly (and if it is necessary), we can repeat this method and we can use it for each vertex in $V(G)$ whose weight is 2 in such a way that its neighbors (which are vertices in $V(S(G)) \setminus V(G))$ have weight 0. Let g be the final (optimal) $2RDF$ for $S(G)$ after repeating this algorithm and these function replacements. Again, if there exists a vertex $v \in V(G)$ such that $|g(v)| = 1$, then we can apply Case 2 to complete the proof. Let $V' = \{v \ : \ v \in V(G),\ g(v) \neq \emptyset\}$ and $n' = |V'|$. Note that $g(v') = \{1, 2\}$ for each $v' \in V'$ and $n' < n$. We have $\sum_{v \in V(G)} |g(v)| = 2n' \leq w(g) \leq 2n - n_1 - n_2$.

Clearly, if $w(g) = 2n'$, then the weight of each vertex in $V(S(G)) \setminus V(G)$ is 0. But then non of the vertices in $V(G)$ has weight 0, otherwise g is not $2RDF$. In graph $S(G)$, the neighborhood of each vertex in $V(G)$ is a subset of $V(S(G)) \setminus V(G)$. Therefore, if there is a vertex in $V(G)$ assigned the empty set, then at least one vertex in $V(S(G)) \setminus V(G)$ must have non-zero weight. It follows that if $w(g) = 2n'$, then $n' = n$ which is a contradiction and the proof is complete in this case. Thus, assume that $2n' < w(g)$. Since $2n' < 2n - n_1 - n_2$ and $n_2 \leq n_1$ we have $n_2 \leq \frac{n_1 + n_2}{2} < (n - n')$ which implies that $n_2 < (n - n') = \big|\{v \ : \ v \in V(G),\ g(v) = \emptyset\}\big|$.

If there exists $j \in \{1, 2, \ldots, k\}$ such that $\{v \mid \in V(G),\ g(v) = \emptyset\} \subseteq X^j$, then we must have $j = 1$ and $n_1 = (n - n')$. This means that for each $v \in (X^2 \cup X^3 \cup \cdots \cup X^k)$ we have $g(v) = \{1, 2\}$ and hence, the weight of each neighbor of v is 0. Thus, the weight of each neighbor of each vertex in X^1 is 0, which is a contradiction. Therefore, $n - n'$

vertices of weight 0 in the set $V(G)$ are distributed among at least two partite sets, i.e. there exist $i_1 \neq i_2$ such that

$$\{v \ : \ v \in V(G),\ g(v) = \emptyset\} \cap X^{i_1} \neq \emptyset, \quad \{v \ : \ v \in V(G),\ g(v) = \emptyset\} \cap X^{i_2} \neq \emptyset.$$

Let $V'' = V' \cup \big(\bigcup_{v \in V'} N_{S(G)}(v)\big)$. Note that $g(v) = \{1,2\}$ for each $v \in V'$, and $g(z) = \emptyset$ for each $z \in N_{S(G)}(v)$. Hence, $\sum_{u \in V''} |g(u)| = 2|V'| = 2n'$. Let $H = S(G) - V''$, i.e. let H be the induced subgraph of $S(G)$ on the vertices in $V(G) \setminus V'$ and their common neighbors in $S(G)$. Thus

$$V(H) = V(S(G)) \setminus V'' = (V(G) \setminus V') \cup \{x^{ij}_{rs} \ : \ x^i_r \in V(G) \setminus V',\ x^j_s \in V(G) \setminus V'\}.$$

Note that H is (isomorphic to) the subdivision of a complete (bipartite or) multipartite graph, say H^*, the restriction of g to $V(H)$, say $g_{|_{V(H)}}$, is a $2RDF$ for H (and has the minimum weight, otherwise by using the weight of vertices in V'' we obtain a $2RDF$ for G with smaller weight which is a contradiction), $g(v) = \emptyset$ for each $v \in (V(G) \setminus V')$ and $|g(x^{ij}_{rs})| = 1$ for each $x^{ij}_{rs} \in V(H) \setminus (V(G) \setminus V')$. Now we consider two following subcases.

Subcase I. H^* is a bipartite graph.
By Theorem 3 we have $\gamma_{r2}(H) = |V(H^*)|$ and hence,

$$\begin{aligned} 2n - n_1 - n_2 &\geq \gamma_{r2}(S(G)) \\ &= w(g) \\ &= 2n' + w(g_{|_{V(H)}}) \\ &\geq 2n' + \gamma_{r2}(H) \\ &= 2n' + |V(H^*)| \\ &= 2n' + (n - n') \\ &= n + n'. \end{aligned}$$

Thus, $n_1 + n_2 \leq (n - n')$. Since $(n - n')$ vertices of weight 0 in the set $V(G) \setminus V'$ are distributed among exactly two partite sets (Note that H^* is bipartite and a subgraph of G) and $n_1 \geq n_2 \geq \cdots \geq n_k$, we must have $n_1 + n_2 = (n - n')$ and (without loss of generality) we can assume that $V(H^*) = X^1 \cup X^2$. Therefore, $n' = n - n_1 - n_2$ and since $\big|V(H) \setminus (X^1 \cup X^2)\big| = n_1 n_2$ we have

$$\begin{aligned} w(g) &= 2n' + w(g_{|_{V(H)}}) \\ &= 2(n - n_1 - n_2) + \big((n_1 + n_2) \times 0 + (n_1 n_2) \times 1\big) \\ &= (2n - n_1 - n_2) + (n_1 n_2 - n_1 - n_2). \end{aligned}$$

Note that we have $n_1 n_2 - n_1 - n_2 \leq 0$ if and only if $n_1(n_2 - 1) \leq n_2$. Since $n_1 \geq n_2$, this happens just when $n_2 = 1$ or $n_1 = n_2 = 2$. If $n_2 = 1$, then each vertex of H

in X^1 (whose weight is 0) has just one neighbor (whose weight is 1, i.e. is assigned either $\{1\}$ or $\{2\}$ under g), and this is a contradiction because $g_{|_{V(H)}}$ is a $2RDF$. If $n_1 = n_2 = 2$, then $(n_1n_2 - n_1 - n_2) = 0$. Thus, $(n_1n_2 - n_1 - n_2) \geq 0$ and hence,

$$\gamma_{r2}(S(G)) = w(g) = (2n - n_1 - n_2) + (n_1n_2 - n_1 - n_2) \geq (2n - n_1 - n_2)$$

which completes the proof.

Subcase II. H^* is a multipartite graph.
Let H^* be a complete k'-partite graph with partite sets of size $n'_1 \geq n'_2 \geq \cdots \geq n'_{k'}$. Since $g_{|_{V(H)}}(v) = \emptyset$ for each $v \in V(H^*)$, and by considering the Case 1 and its proof, we have

$$\begin{aligned}
\gamma_{r2}(S(G)) &= w(g) \\
&= 2n' + w(g_{|_{V(H)}}) \\
&\geq 2n' + \big(2(n - n') - n'_1 - n'_2\big) \\
&= 2n - n'_1 - n'_2 \\
&\geq 2n - n_1 - n_2
\end{aligned}$$

and this completes the proof. □

Note that K_2 is a tree and it is easy to see that $\gamma_{r2}(S(K_2)) = 2 = 2 \times 2 - 2$. Also, for each $n \geq 3$ the complete graph K_n can be regarded as the complete n-partite graph $K_{1,1,\ldots,1}$ and by Theorem 4 we have $\gamma_{r2}(S(K_{1,1,\ldots,1})) = 2n - 2$. Hence, the following result directly follows.

Corollary 6. *For each $n \geq 2$ we have $\gamma_{r2}(S(K_n)) = 2n - 2$.*

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References

[1] H. Abdollahzadeh Ahangar, M. Chellali, S.M. Sheikholeslami, and J.C. Valenzuela-Tripodoro, *On the outer independent total double Roman domination in graphs*, Mediterr. J. Math. **20** (2023), no. 3, Article number 171. https://doi.org/10.1007/s00009-023-02317-1.

[2] H. Abdollahzadeh Ahangar, M. Khaibari, N. Jafari Rad, and S.M. Sheikholeslami, *Graphs with large total 2-rainbow domination number*, Iran. J. Sci. Technol. Trans. A Sci. **42** (2018), 841–846.
https://doi.org/10.1007/s40995-017-0465-9.

[3] S. Ahmadi, E. Vatandoost, and A. Behtoei, *Domination number and identifying code number of the subdivision graphs*, J. Algebraic Sys., to appear.

[4] M. Ali, M. T. Rahim, M. Zeb, and G. Ali, *On* 2*-rainbow domination of some families of graph*, Int. J. Math. Soft Comput. **1** (2011), no. 1, 47–53.

[5] A. Behtoei, E. Vatandoost, and F. Azizi Rajol Abad, *Signed Roman domination number and join of graphs*, J. Algebraic Sys. **4** (2016), no. 1, 65–77.
https://doi.org/10.22044/jas.2016.729.

[6] B. Brešar, M.A. Henning, and D. F. Rall, *Rainbow domination in graphs*, Taiwanese J. Math. **12** (2008), no. 1, 213–225.
http://doi.org/10.11650/twjm/1500602498.

[7] B. Brešar and T. K. Šumenjak, *On the* 2*-rainbow domination in graphs*, Discrete Appl. Math. **155** (2007), no. 17, 2394–2400.
http://doi.org/10.1016/j.dam.2007.07.018.

[8] S. Brezovnik, D.R. Poklukar, and J. Žerovnik, *On 2-rainbow domination of generalized Petersen graphs* $P(ck, k)$, Mathematics **11** (2023), no. 10, Article ID: 2271.
https://doi.org/10.3390/math11102271.

[9] M. Chellali and N. Jafari Rad, *On 2-rainbow domination and Roman domination in graphs*, Australas. J. Comb. **56** (2013), 85–93.

[10] P. Dankelmann and D. J. Erwin, *Distance domination and generalized eccentricity in graphs with given minimum degree*, J. Graph Theory **94** (2020), no. 1, 5–19.
https://doi.org/10.1002/jgt.22503.

[11] R. Erveš and J. Žerovnik, *On* 2*-rainbow domination number of generalized Petersen graphs* $P(5k, k)$, Symmetry **13** (2021), no. 5, Article ID: 809.
https://doi.org/10.3390/sym13050809.

[12] A. Ghaffari-Hadigheh, *Uncertain weighted dominating set: a prototype application on natural disaster relief managemen*, Soft Comput. **22** (2018), 1003–1012.
https://doi.org/10.1007/s00500-016-2404-7.

[13] A. Ghalavand, S. Klavžar, M. Tavakoli, and I.G. Yero, *On mixed metric dimension in subdivision, middle, and total graphs*, Quaest. Math. **46** (2023), no. 12, 2517–2527.
https://doi.org/10.2989/16073606.2023.2169206.

[14] R. Khoeilar, M. Kheibari, M. Chellali, and S.M. Sheikholeslami, *A sharp upper bound on the independent 2-rainbow domination in graphs with minimum degree at least two*, Comput. Sci. J. Moldova **28** (2020), no. 3, 373–388.

[15] R. Khoeilar, M. Kheibari, Z. Shao, and S.M. Sheikholeslami, *Total* k*-rainbow domination subdivision number in graphs*, Comput. Sci. J. Moldova **28** (2020), no. 2, 152–169.

[16] A. Mahmoodi and L. Volkmann, *Outer-independent total* 2*-rainbow dominating functions in graphs*, Commun. Comb. Optim. **8** (2023), no. 2, 431–444.

https://doi.org/10.22049/cco.2022.27753.1344.

[17] S. M. Mirafzal, *Some algebraic properties of the subdivision graph of a graph*, Commun. Comb. Optim. **9** (2024), no. 2, 297–307.
https://doi.org/10.22049/cco.2023.28270.1494.

[18] D.A. Mojdeh, B. Samadi, Z. Shao, and I.G. Yero, *On the outer independent double Roman domination number*, Bull. Iranian Math. Soc. **48** (2022), no. 4, 1789–1803.
https://doi.org/10.1007/s41980-021-00606-7.

[19] N. Jafari Rad, E. Gholami, A. Tehranian, and H. Rasouli, *A new upper bound on the independent 2-rainbow domination number in trees*, Commun. Comb. Optim. **8** (2023), no. 1, 261–270.
https://doi.org/10.22049/cco.2022.27641.1305.

[20] A. Shaminezhad and E. Vatandoost, *On* 2*-rainbow domination number of functigraph and its complement*, Opuscula Math. **40** (2020), no. 5, 617–627.
http://doi.org/10.7494/OpMath.2020.40.5.617.

[21] Z. Shao, M. Liang, C. Yin, X. Xu, P. Pavlie, and J. Žerovnik, *On rainbow domination number of graphs*, Inform. Sci. **254** (2014), 225–230.
https://doi.org/10.1016/j.ins.2013.07.020.

[22] X. Shi, P. Wu, Z. Shao, V. Samodivkin, S.M. Sheikholeslami, M. Soroudi, and S. Wang, *Changing and unchanging 2-rainbow independent domination*, IEEE Access **7** (2019), 72604–72612.

[23] Y. Wu and N. Jafari Rad, *Bounds on the* 2*-rainbow domination of graph*, Graphs Combin. **29** (2013), no. 4, 1125–1133.
https://doi.org/10.1007/s00373-012-1158-y.

[24] Y. Wu and H. Xing, *Note on the* 2*-rainbow domination and Roman domination in graphs*, Appl. Math. Lett. **23** (2010), no. 6, 706–709.
https://doi.org/10.1016/j.aml.2010.02.012.