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Research Article

On the comaximal graph of a non-quasi-local atomic domain

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Abstract: Let R be an atomic domain such that R has at least two maximal ideals. Let $Irr(R)$ denote the set of all irreducible elements of R and let $J(R)$ denote the Jacobson radical of R. Let $\mathcal{I}(R) = \{R\pi \mid \pi \in Irr(R)\backslash J(R)\}$. In this paper, with R, we associate an undirected graph denoted by $\mathbb{CGI}(R)$ whose vertex set is $\mathcal{I}(R)$ and distinct vertices $R\pi_1$ and $R\pi_2$ are adjacent if and only if $R\pi_1 + R\pi_2 = R$. The aim of this paper is to study the interplay between some graph properties of $\mathbb{CGI}(R)$ and the algebraic properties of R.

Keywords: irreducible element, atomic domain, connected graph, girth of a graph, clique number of a graph

AMS Subject classification: 13F15, 05C25.

1. Introduction

The rings considered in this paper are commutative with identity. Let R be a ring. The work done on the comaximal graph of R is motivated by the research work done by Sharma and Bhatwadekar in [13]. With R, in [13], Sharma and Bhatwadekar associated an undirected graph denoted by $G(R)$ whose vertex set is the set of all elements of R and distinct vertices a and b are adjacent if and only if $Ra + Rb = R$ and they explored mainly on the coloring of $G(R)$. In [8], Maimani et al. called the graph studied in [13] as the *comaximal graph* of R and denoted it by $\Gamma(R)$. Let us denote the group of units of R by $U(R)$, the set of all non-units of R by $NU(R)$, and the Jacobson radical of R by $J(R)$. In [8], some subgraphs of $\Gamma(R)$ were also

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investigated. Interesting among the subgraphs studied in [8] is the subgraph of $\Gamma(R)$ (where R admits at least two maximal ideals) induced by $NU(R)\backslash J(R)$. Several other researchers have done work on the comaximal graphs of rings (see for example, [5, 10, 12]).

Unless otherwise specified, the integral domains considered in this paper are not fields. Let R be an integral domain. Let $a \in R\backslash\{0\}$ and $b \in R$. We say that *a is a divisor of b* in R if $b = ac$ for some $c \in R$. Let $x, y \in R$ with $x \neq 0$. We say that x and y are *associates* in R if $x = uy$ for some $u \in U(R)$. It is not hard to show that x and y are associates in R if and only if $Rx = Ry$. Let π be a non-zero non-unit of R. Recall that π is called an *atom* or *irreducible element* if π cannot be written as the product of two non-units of R. The integral domain R is called an *atomic domain* if each non-zero non-unit of R can be written as a (finite) product of irreducible elements of R [4]. An atomic domain is also referred to as a *factorization domain* [[7], see page 155]. It is well-known that if an integral domain satisfies the ascending chain condition (a.c.c.) on principal ideals, then it is atomic [[7], Proposition 1.1.1, page 156], and so, any Noetherian domain is atomic. In [[4], Section 1], Grams constructed an example of an atomic domain A such that A does not satisfy a.c.c. on principal ideals.

Let R be a ring. We denote the set of all prime ideals of R by $Spec(R)$ and the set of all maximal ideals of R by $Max(R)$. For a set A, we denote the cardinality of A by $|A|$. The ring R is said to be *quasi-local* (respectively, *semi-quasi-local*) if $|Max(R)| = 1$ (respectively, $|Max(R)| < \infty$). Thus R is non-quasi-local if and only if $|Max(R)| \geq 2$. A Noetherian quasi-local (respectively, semi-quasi-local) ring is called a *local* (respectively, *semi-local*) ring. If a set A is a subset of a set B and $A \neq B$, then we denote it by $A \subset B$ or $B \supset A$.

Let R be an atomic domain with $|Max(R)| \geq 2$. Let $Irr(R)$ denote the set of all irreducible elements of R. Let $\mathcal{I}(R) = \{R\pi \mid \pi \in Irr(R)\backslash J(R)\}$. In this paper, with R, we associate an undirected graph denoted by $\mathbb{CGI}(R)$ whose vertex set is $\mathcal{I}(R)$ and distinct vertices $R\pi$ and $R\pi'$ are adjacent if and only if $R\pi + R\pi' = R$. The purpose of this paper is to discuss some results on the connectedness of $\mathbb{CGI}(R)$, the girth of $\mathbb{CGI}(R)$, and the clique number of $\mathbb{CGI}(R)$.

For definitions and notations from commutative ring theory that are used and left undefined in this paper, the reader can refer any of the following books (for example, [1, 3, 6, 9, 11]).

The graphs considered in this paper are undirected and simple. We use only some basic definitions and basic results from graph theory. For definitions and notations from graph theory that are not mentioned in this paper, the reader can refer [2]. Before we give a brief account of results that are proved in this paper, it is desirable to mention the needed notations from graph theory. Let $G = (V, E)$ be a connected graph. For any distinct $u, v \in V$, we denote the distance between u and v in G by $d_G(u, v)$ or $d(u, v)$. The diameter of G is denoted by $diam(G)$. For any $v \in V$, the eccentricity of v in G is denoted by $e_G(v)$ or $e(v)$. The radius of G is denoted by $r(G)$. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. We denote the girth of G by $gr(G)$, the clique number of G by $\omega(G)$, and the chromatic number of G by $\chi(G)$. It is well-known that $\omega(G) \leq \chi(G)$. If G does not contain any cycle, then we define $gr(G) = \infty$.

This paper consists of three sections including the introduction. For an atomic domain R with $|Max(R)| \geq 2$, the following results are proved in Section 2. It is proved that $\mathbb{CGI}(R)$ is connected and $diam(\mathbb{CGI}(R)) \leq 3$ (Proposition 1). A necessary and sufficient condition is determined such that $diam(\mathbb{CGI}(R)) = 1$ (Theorem 1) (respectively, $diam(\mathbb{CGI}(R)) = 2$ (Proposition 2), $diam(\mathbb{CGI}(R)) = 3$ (Proposition 3)). A necessary and sufficient condition is determined such that $r(\mathbb{CGI}(R)) = 1$ (Proposition 4). If $2 \leq |Max(R)| < \infty$, then a necessary and sufficient condition is provided such that $r(\mathbb{CGI}(R)) = 2$ (Proposition 5). In this section, some problems are posed for which we are not aware of their solutions.

Let R be an atomic domain with $|Max(R)| \geq 2$. In Section 3, we discuss some results on $\omega(\mathbb{CGI}(R))$ and $gr(\mathbb{CGI}(R))$. A necessary and sufficient condition is determined such that $\omega(\mathbb{CGI}(R)) < \infty$ (Theorem 2). It is shown that $gr(\mathbb{CGI}(R)) \in \{3, 4, \infty\}$ (Corollaries 3 and 4). If $|\mathcal{I}(R)| < \infty$, then R is characterized such that $|\mathcal{I}(R)| = |Max(R)|$ (Proposition 9).

Several examples are given to illustrate the results proved in this paper.

2. Some results on the connectedness of $\mathbb{CGI}(R)$

Throughout this paper, unless otherwise specified, we use R to denote an atomic domain with $|Max(R)| \geq 2$. The aim of this section is to prove some results regarding the connectedness of $\mathbb{CGI}(R)$. First, we state and prove some lemmas that are needed in the proofs of main results of this section.

Lemma 1. *If $\mathfrak{p} \in Spec(R)$ is such that $\mathfrak{p} \not\subseteq J(R)$, then there exists $\pi \in \mathfrak{p}$ such that $R\pi \in \mathcal{I}(R)$.*

Proof. Let $a \in \mathfrak{p} \backslash J(R)$. It is clear that $a \in NU(R) \backslash \{0\}$. Since R is atomic, $a = \prod_{i=1}^{n} \pi_i$, where $\pi_i \in Irr(R)$ for each $i \in \{1, \ldots, n\}$. By the choice of a, it follows that $\pi_i \notin J(R)$ for each $i \in \{1, \ldots, n\}$ and $\pi_j \in \mathfrak{p}$ for some $j \in \{1, \ldots, n\}$. With $\pi = \pi_j$, it follows that $\pi \in \mathfrak{p}$ and $R\pi \in \mathcal{I}(R)$. □

Lemma 2. *If $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ are such that $\pi_1\pi_2 \notin \mathfrak{m}$ for some $\mathfrak{m} \in Max(R)$, then there exists $\pi \in Irr(R) \cap \mathfrak{m}$ with $R\pi \in \mathcal{I}(R)$ and $R\pi_1 + R\pi = R = R\pi_2 + R\pi$.*

Proof. As $\mathfrak{m} \in Max(R)$ and $\pi_1\pi_2 \notin \mathfrak{m}$ by hypothesis, $R\pi_1\pi_2 + \mathfrak{m} = R$. Therefore, there exist $r \in R$ and $m \in \mathfrak{m}$ such that $r\pi_1\pi_2 + m = 1$. Hence, $R\pi_1\pi_2 + Rm = R$. It is clear that $m \in \mathfrak{m} \backslash J(R)$. Since $\mathfrak{m} \in Spec(R)$, the proof of Lemma 1 shows that there exists $\pi \in Irr(R) \cap \mathfrak{m}$ such that π divides m in R and $R\pi \in \mathcal{I}(R)$. It follows from $R\pi_1\pi_2 + Rm = R$ that $R\pi_1\pi_2 + R\pi = R$. Therefore, $R\pi_1 + R\pi = R = R\pi_2 + R\pi$. □

Proposition 1. *$\mathbb{CGI}(R)$ is connected and $diam(\mathbb{CGI}(R)) \leq 3$.*

Proof. We use an argument found in the proof of [[8], Theorem 3.1]. Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be distinct. We can assume that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$. Hence, there exists $\mathfrak{m} \in Max(R)$ such that $R\pi_1 + R\pi_2 \subseteq \mathfrak{m}$ by [[1], Corollary 1.4]. We consider the following cases.

Case 1. $\pi_1\pi_2 \notin J(R)$.
Note that there exists $\mathfrak{n} \in Max(R)$ such that $\pi_1\pi_2 \notin \mathfrak{n}$. Hence, we obtain from Lemma 2 that there exists $\pi \in Irr(R) \cap \mathfrak{n}$ with $R\pi \in \mathcal{I}(R)$ and $R\pi_1 + R\pi = R = R\pi_2 + R\pi$. Therefore, $R\pi_1 - R\pi - R\pi_2$ is a path of length two between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$.

Case 2. $\pi_1\pi_2 \in J(R)$.
Let $i \in \{1, 2\}$. As $\pi_i \notin J(R)$, there exists $\mathfrak{m}_i \in Max(R)$ such that $\pi_i \notin \mathfrak{m}_i$. As $\pi_1\pi_2 \in J(R)$, it follows that $\pi_1 \in \mathfrak{m}_2$ and $\pi_2 \in \mathfrak{m}_1$. Hence, it is clear that $\mathfrak{m}_1 \neq \mathfrak{m}_2$. Observe that $R\pi_1 + \mathfrak{m}_1 = R$ and so, there exist $s \in R$ and $a_1 \in \mathfrak{m}_1$ such that $s\pi_1 + a_1 = 1$. Thus $R\pi_1 + Ra_1 = R$. Note that $\mathfrak{m}_1 \in Spec(R)$ and $a_1 \in \mathfrak{m}_1 \backslash J(R)$. The proof of Lemma 1 shows that there exists $\pi \in Irr(R) \cap \mathfrak{m}_1$ such that π is a divisor of a_1 in R and $R\pi \in \mathcal{I}(R)$. It is clear that $R\pi_1 + R\pi = R$. Hence, $R\pi_1$ and $R\pi$ are adjacent in $\mathbb{CGI}(R)$. As $\pi_1 \in \mathfrak{m}_2$, we get that $\pi \notin \mathfrak{m}_2$. Observe that $R\pi + R\pi_2 \subseteq \mathfrak{m}_1$ and $\pi\pi_2 \notin \mathfrak{m}_2$. Hence, by Lemma 2, there exists $\pi' \in Irr(R) \cap \mathfrak{m}_2$ with $R\pi' \in \mathcal{I}(R)$ such that $R\pi - R\pi' - R\pi_2$ is a path of length two between $R\pi$ and $R\pi_2$ in $\mathbb{CGI}(R)$. Therefore, $R\pi_1 - R\pi - R\pi' - R\pi_2$ is a path of length three between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$.
This proves that $\mathbb{CGI}(R)$ is connected and $diam(\mathbb{CGI}(R)) \leq 3$. □

The following Lemmas 3 and 4 are needed in the proof of Theorem 1.

Lemma 3. *If $|Max(R)| \geq n$ for some $n \in \mathbb{N}\backslash\{1\}$, then $|\mathcal{I}(R)| \geq n$.*

Proof. Let $\{\mathfrak{m}_i \mid i \in \{1, 2, \ldots, n\}\} \subseteq Max(R)$. Let $i \in \{1, 2, \ldots, n\}$. Since distinct maximal ideals of a ring are not comparable under inclusion, it follows from [[1], Proposition 1.11(i)] that $\mathfrak{m}_i \nsubseteq \bigcup_{j \in A_i} \mathfrak{m}_j$, where $A_i = \{1, 2, \ldots, n\}\backslash\{i\}$. Let $a_i \in \mathfrak{m}_i \backslash (\bigcup_{j \in A_i} \mathfrak{m}_j)$. Since R is an atomic domain, a_i can be written as a (finite) product of atoms of R. Note that no irreducible divisor of a_i in R can belong to $\bigcup_{j \in A_i} \mathfrak{m}_j$ and at least one irreducible divisor π_i of a_i in R such that $\pi_i \in \mathfrak{m}_i$. Thus $\pi_i \notin \bigcup_{j \in A_i} \mathfrak{m}_j$ and so, $R\pi_i \in \mathcal{I}(R)$. It is now evident that $R\pi_i \neq R\pi_j$ for all distinct $i, j \in \{1, 2, \ldots, n\}$. From $\{R\pi_i \mid i \in \{1, 2, \ldots, n\}\} \subseteq \mathcal{I}(R)$, it follows that $|\mathcal{I}(R)| \geq n$. □

By Lemma 3, it follows that $|\mathcal{I}(R)| \geq 2$. As $\mathbb{CGI}(R)$ is connected by Proposition 1, we get that $1 \leq diam(\mathbb{CGI}(R))$.

Lemma 4. *If $diam(\mathbb{CGI}(R)) = 1$, then $\mathfrak{m}$ is principal for each $\mathfrak{m} \in Max(R)$.*

Proof. Let $\mathfrak{m} \in Max(R)$. As $|Max(R)| \geq 2$, it follows that $\mathfrak{m} \not\subseteq J(R)$. Since $\mathfrak{m} \in Spec(R)$, it follows that there exists $\pi \in Irr(R) \cap \mathfrak{m}$ with $R\pi \in \mathcal{I}(R)$. Hence, $R\pi \subseteq \mathfrak{m}$. Let $a \in \mathfrak{m} \backslash J(R)$. Observe that there exists an irreducible divisor π' of a in R such that $\pi' \in \mathfrak{m}$ and $R\pi' \in \mathcal{I}(R)$. If $R\pi' \neq R\pi$, then it follows from the assumption $diam(\mathbb{CGI}(R)) = 1$ that $R\pi + R\pi' = R$. This is impossible, since $R\pi + R\pi' \subseteq \mathfrak{m}$. Therefore, $R\pi' = R\pi$. Note that $a \in R\pi' = R\pi$. This shows that $\mathfrak{m} \subseteq R\pi \cup J(R)$ and so, $\mathfrak{m} \subseteq R\pi$. Therefore, $\mathfrak{m} = R\pi$ is principal. □

Theorem 1. *For an atomic domain R with $|Max(R)| \geq 2$, the following statements are equivalent:*

(1) $diam(\mathbb{CGI}(R)) = 1$.

(2) R is a PID.

Proof. (1) ⇒ (2). Assume that $diam(\mathbb{CGI}(R)) = 1$. Let $\mathfrak{p} \in Spec(R)$, $\mathfrak{p} \neq (0)$. We claim that $\mathfrak{p}$ is principal. Let $\mathfrak{m} \in Max(R)$ be such that $\mathfrak{p} \subseteq \mathfrak{m}$. Observe that there exists $\pi \in Irr(R)$ such that $\mathfrak{m} = R\pi$ by the proof of Lemma 4. Let $a \in \mathfrak{p} \backslash \{0\}$. Since $a \in NU(R)$ and R is atomic, there exists an irreducible element π' of R such that π' is a divisor of a in R and $\pi' \in \mathfrak{p}$. From $\mathfrak{p} \subseteq \mathfrak{m} = R\pi$, we obtain that $\pi' = u\pi$ for some $u \in U(R)$. Hence, $\pi = u^{-1}\pi' \in \mathfrak{p}$. Therefore, $R\pi = \mathfrak{m} \subseteq \mathfrak{p}$ and so, $\mathfrak{p} = \mathfrak{m} = R\pi$ is principal. This proves that any prime ideal of R is principal and hence, we obtain from [[6], Exercise 10, page 8] that R is a PID.

(2) ⇒ (1). Assume that R is a PID. Let $\pi \in Irr(R)$. Note that $\mathfrak{p} = R\pi \in Spec(R)$ and $\mathfrak{p} \neq (0)$ and so, we obtain from [[1], Example 3, page 5] that $R\pi = \mathfrak{p} \in Max(R)$. As $|Max(R)| \geq 2$, $|\mathcal{I}(R)| \geq 2$ by Lemma 3. Let $R\pi, R\pi' \in \mathcal{I}(R)$ be distinct. Since $R\pi, R\pi'$ are distinct members of $Max(R)$, we obtain that $R\pi + R\pi' = R$. This shows that $R\pi$ and $R\pi'$ are adjacent in $\mathbb{CGI}(R)$ and so, $diam(\mathbb{CGI}(R)) = 1$. □

In Proposition 2, we determine a necessary and sufficient condition in order that $diam(\mathbb{CGI}(R)) = 2$. We use the following lemma in its proof.

Lemma 5. *If $\pi_1\pi_2 \in J(R)$ for some distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$, then there does not exist any $R\pi \in \mathcal{I}(R)$ such that $R\pi$ is adjacent to both $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$.*

Proof. Let $R\pi \in \mathcal{I}(R)$ be such that $R\pi_1$ and $R\pi$ are adjacent in $\mathbb{CGI}(R)$. Then $R\pi_1 + R\pi = R$. Let $\mathfrak{m} \in Max(R)$ be such that $\pi \in \mathfrak{m}$. Hence, $\pi_1 \notin \mathfrak{m}$. Note that $\pi_1\pi_2 \in J(R) \subset \mathfrak{m}$. Therefore, $\pi_2 \in \mathfrak{m}$ and so, $R\pi_2 + R\pi \subseteq \mathfrak{m}$. Hence, $R\pi_2$ and $R\pi$ are not adjacent in $\mathbb{CGI}(R)$. □

Proposition 2. *With R as in the statement of Theorem 1, the following statements are equivalent:*

(1) $diam(\mathbb{CGI}(R)) = 2$.

(2) R is not a PID and $\pi_1\pi_2 \notin J(R)$ for any distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$.

Proof. (1) $\Rightarrow$ (2). Assume that $diam(\mathbb{CGI}(R)) = 2$. By (2) $\Rightarrow$ (1) of Theorem 1, we get that R is not a PID. Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be distinct. If they are not adjacent in $\mathbb{CGI}(R)$, then there exists $R\pi \in \mathcal{I}(R)$ such that $R\pi$ is adjacent to both $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$. Hence, we obtain from Lemma 5 that $\pi_1\pi_2 \notin J(R)$.
(2) $\Rightarrow$ (1). Assume that R is not a PID and $\pi_1\pi_2 \notin J(R)$ for any distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that they are not adjacent in $\mathbb{CGI}(R)$. It follows from (1) $\Rightarrow$ (2) of Theorem 1 that $diam(\mathbb{CGI}(R)) \geq 2$. Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be distinct. If they are not adjacent in $\mathbb{CGI}(R)$, then $\pi_1\pi_2 \notin J(R)$ by assumption. It follows from the proof of Case(1) of Proposition 1 that there exists a path of length two between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$. This proves that $diam(\mathbb{CGI}(R)) \leq 2$ and so, $diam(\mathbb{CGI}(R)) = 2$. $\square$

If R is not a PID, then in the following corollary, we provide a condition on $J(R)$ such that the statement (2) of Proposition 2 holds.

Corollary 1. *Let R be as in the statement of Theorem 1. If R is not a PID and $J(R) \in Spec(R)$, then $diam(\mathbb{CGI}(R)) = 2$.*

Proof. Assume that R is not a PID and $J(R) \in Spec(R)$. Then for any $a, b \in R\backslash J(R)$, $ab \notin J(R)$. As $\pi \notin J(R)$ for any $R\pi \in \mathcal{I}(R)$, we obtain from (2) $\Rightarrow$ (1) of Proposition 2 that $diam(\mathbb{CGI}(R)) = 2$. $\square$

We provide Example 1 to illustrate $J(R) \in Spec(R)$ is not necessary to ensure that $diam(\mathbb{CGI}(R)) = 2$. We use the following corollary in its verification.

Corollary 2. *With R as in the statement of Theorem 1, if R is not a PID and $|Max(R)| = 2$, then $diam(\mathbb{CGI}(R)) = 2$.*

Proof. Let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1, 2\}\}$. It is clear that $J(R) = \bigcap_{i=1}^{2} \mathfrak{m}_i$. Note that $V(\mathbb{CGI}(R)) = \bigcup_{i=1}^{2} V_i$, where $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1\backslash\mathfrak{m}_2\}$ and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in \mathfrak{m}_2\backslash\mathfrak{m}_1\}$. It follows from the proof of Lemma 3 that $V_i \neq \emptyset$ for each $i \in \{1, 2\}$. For any $R\pi \in V_1$ and $R\pi' \in V_2$, it is evident that $R\pi + R\pi' = R$ and so, $R\pi$ and $R\pi'$ are adjacent in $\mathbb{CGI}(R)$. Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be distinct. If $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$, then both $R\pi_1$ and $R\pi_2$ must be in either V_1 or V_2. Without loss of generality, we can assume that both $R\pi_1$ and $R\pi_2$ are in V_1. Observe that $\pi_1\pi_2 \notin \mathfrak{m}_2$ and so, $\pi_1\pi_2 \notin J(R)$. As R is not a PID by assumption, it follows from (2) $\Rightarrow$ (1) of Proposition 2 that $diam(\mathbb{CGI}(R)) = 2$. $\square$

For any ring R, we denote the polynomial ring in one variable X over R by $R[X]$.

Example 1. If $T = \mathbb{Z}[X]$, $\mathfrak{m}_1 = T2 + TX$, $\mathfrak{m}_2 = T3 + TX$, and $S = T\backslash(\bigcup_{i=1}^{2} \mathfrak{m}_i)$, then $R = S^{-1}T$ is a unique factorization domain (UFD), $|Max(R)| = 2$ and $diam(\mathbb{CGI}(R)) = 2$.

Proof. It is well-known that T is a UFD. Observe that S is a multiplicatively closed subset (m.c. subset) of T. It follows from [[1], Proposition 3.11(iv)] and [[6], Theorem 5] that $R = S^{-1}T$ is a UFD and so, R is an atomic domain. Observe that $\mathfrak{m}_1, \mathfrak{m}_2 \in Max(T)$ and are distinct. As $\{\mathfrak{m}_i \mid i \in \{1,2\}\}$ is the set of prime ideals of T maximal with respect to not meeting S, it follows from [[1], Proposition 3.11(iv)] that $Max(R) = \{S^{-1}\mathfrak{m}_i \mid i \in \{1,2\}\}$. Hence, $|Max(R)| = 2$. Observe that $T2 \in Spec(T)$ and $S^{-1}T2 \in Spec(R)$ with $(0) \subset S^{-1}T2 \subset S^{-1}\mathfrak{m}_1$. Hence, $\mathfrak{p} = S^{-1}T2 \in Spec(R)\backslash Max(R)$. As $\mathfrak{p} \neq (0)$, it follows from [[1], Example (3), page 5] that R is not a PID. Therefore, it follows from Corollary 2 that $diam(\mathbb{CGI}(R)) = 2$. Note that $(2+X)(3+X) \in \bigcap_{i=1}^{2} S^{-1}\mathfrak{m}_i = J(R)$ but neither $2+X$ nor $3+X$ belongs to $J(R)$. Hence, $J(R) \notin Spec(R)$. □

In Proposition 3, we determine a necessary and sufficient condition in order that $diam(\mathbb{CGI}(R)) = 3$. We use the following lemma in its proof.

Lemma 6. *Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be distinct. Then $d(R\pi_1, R\pi_2) = 3$ in $\mathbb{CGI}(R)$ if and only if $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$ and $\pi_1\pi_2 \in J(R)$.*

Proof. Assume that $d(R\pi_1, R\pi_2) = 3$ in $\mathbb{CGI}(R)$. It is clear that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$. If $\pi_1\pi_2 \notin J(R)$, then we know from the proof of Case(1) of Proposition 1 that $d(R\pi_1, R\pi_2) = 2$ in $\mathbb{CGI}(R)$. This is a contradiction and so, $\pi_1\pi_2 \in J(R)$.

Conversely, assume that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$ and $\pi_1\pi_2 \in J(R)$. It follows from Lemma 5 that $d(R\pi_1, R\pi_2) \geq 3$ in $\mathbb{CGI}(R)$. Since $diam(\mathbb{CGI}(R)) \leq 3$ by Proposition 1, we obtain that $d(R\pi_1, R\pi_2) = 3$ in $\mathbb{CGI}(R)$. □

Proposition 3. *Let R be as in the statement of Theorem 1. Then $diam(\mathbb{CGI}(R)) = 3$ if and only if there exist distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$ and $\pi_1\pi_2 \in J(R)$.*

Proof. Note that $diam(\mathbb{CGI}(R)) \leq 3$ by Proposition 1. Hence, it is clear that $diam(\mathbb{CGI}(R)) = 3$ if and only if there exist distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $d(R\pi_1, R\pi_2) = 3$ in $\mathbb{CGI}(R)$. Therefore, the proof of this proposition follows immediately from Lemma 6. □

The following example illustrates Proposition 3.

Example 2. Let $n \geq 3$ and let $p_3, \ldots, p_n$ be distinct odd prime numbers. If $T = \mathbb{Z}[X]$, $\mathfrak{m}_1 = T2 + T(X-1)$, $\mathfrak{m}_2 = T2 + TX$, $\mathfrak{m}_j = Tp_j + TX$ for each $j \in \{3, \ldots, n\}$, and $S = T\backslash(\bigcup_{i=1}^{n} \mathfrak{m}_i)$, then $R = S^{-1}T$ is a UFD and $diam(\mathbb{CGI}(R)) = 3$.

Proof. It is clear that $\mathfrak{m}_i \in Max(T)$ for each $i \in \{1, 2, 3, \ldots, n\}$ and $\mathfrak{m}_i \neq \mathfrak{m}_j$ for all distinct $i, j \in \{1, 2, 3, \ldots, n\}$. Observe that S is a m.c. subset of T. Since T is a

UFD, it follows as in Example 1 that $R = S^{-1}T$ is a UFD. Therefore, R is an atomic domain. As distinct maximal ideals of a ring are not comparable under inclusion, it follows from [[1], Proposition 1.11(i)] that $\{\mathfrak{m}_i \mid i \in \{1, 2, 3, \ldots, n\}\}$ is the set of prime ideals of T maximal with respect to not meeting S. Hence, it follows from [[1], Proposition 3.11(iv)] that $Max(R) = \{S^{-1}\mathfrak{m}_i \mid i \in \{1, 2, 3, \ldots, n\}\}$. Note that $2, X$ are non-associate prime elements of T and $T2 \cap S = TX \cap S = \emptyset$. Hence, $2, X$ are non-associate prime elements of R. Observe that $2 \notin S^{-1}\mathfrak{m}_3$ and $X \notin S^{-1}\mathfrak{m}_1$ and so, $2, X \notin J(R)$. Hence, $R2, RX \in \mathcal{I}(R)$. It is clear that $R2 + RX = S^{-1}\mathfrak{m}_2$ and so, $R2$ and RX are not adjacent in $\mathbb{CGI}(R)$. Note that $2X \in J(R)$. Therefore, $diam(\mathbb{CGI}(R)) = 3$ by Proposition 3. □

We do not know whether or not there exists an atomic domain R with an infinite number of maximal ideals such that $diam(\mathbb{CGI}(R)) = 3$.

We next discuss some results on $r(\mathbb{CGI}(R))$. In the following proposition, we determine a necessary and sufficient condition in order that $r(\mathbb{CGI}(R)) = 1$.

Proposition 4. *Let R be as in the statement of Theorem 1. Then $r(\mathbb{CGI}(R)) = 1$ if and only if there exists $\mathfrak{m} \in Max(R)$ such that $\mathfrak{m}$ is principal.*

Proof. Assume that $r(\mathbb{CGI}(R)) = 1$. Then there exists $R\pi \in \mathcal{I}(R)$ such that $e(R\pi) = 1$ in $\mathbb{CGI}(R)$. Thus implies that $R\pi$ is adjacent to all $R\pi' \in \mathcal{I}(R)\backslash\{R\pi\}$ in $\mathbb{CGI}(R)$. Let $\mathfrak{m} \in Max(R)$ be such that $\pi \in \mathfrak{m}$. Since $R\pi$ is adjacent to all the other vertices in $\mathbb{CGI}(R)$, it follows as shown in the proof of Lemma 4 that $\mathfrak{m} = R\pi$ is principal.

Conversely, if there exists $\mathfrak{m} \in Max(R)$ such that $\mathfrak{m}$ is principal, then there exists $R\pi \in \mathcal{I}(R)$ such that $\mathfrak{m} = R\pi$. Let $R\pi' \in \mathcal{I}(R)\backslash\{R\pi\}$. As π and π' are non-associates in R, it follows that $\pi' \notin R\pi = \mathfrak{m}$. Hence, $R\pi + R\pi' = R$. Therefore, $R\pi$ is adjacent to all $R\pi' \in \mathcal{I}(R)\backslash\{R\pi\}$ in $\mathbb{CGI}(R)$. This proves that $e(R\pi) = 1$ in $\mathbb{CGI}(R)$ and so, $r(\mathbb{CGI}(R)) = 1$. □

For a ring T, we denote $T\backslash\{0\}$ by T^*. For any $f(X) \in T[X]\backslash\{0\}$, we denote the degree of $f(X)$ by $deg(f(X))$. The following example illustrates Proposition 4.

Example 3. Let $T = \mathbb{Q}[X]$. If $R = \mathbb{Q}[X^2, X^3]$, then $diam(\mathbb{CGI}(R)) = 2$ and $r(\mathbb{CGI}(R)) = 1$.

Proof. It follows from [[1], Corollary 7.7] that R is Noetherian. Therefore, R is an atomic domain. Observe that $R = \mathbb{Q} + X^2\mathbb{Q}[X]$. It follows from [[1], Exercise 2(i), page 11] that $U(T) = \mathbb{Q}^*$ and so, $U(R) = \mathbb{Q}^*$. Let $r \in J(R)$. Then $1 - r \in U(R)$ by [[1], Proposition 1.9]. Hence, $1 - r = \alpha$ for some $\alpha \in \mathbb{Q}^*$. Therefore, $r = 1 - \alpha \in \mathbb{Q} \cap J(R) = (0)$ and so, $J(R) = (0)$. We claim that X^2, X^3 are irreducible elements of R. Let $k \in \{2, 3\}$. Let $X^k = r_1r_2$ for some $r_1, r_2 \in R$. Then $k = deg(r_1) + deg(r_2)$. Since $k > 0$, it follows that $deg(r_i) > 0$ for some $i \in \{1, 2\}$. Without loss of generality, we can assume that $deg(r_1) > 0$. Since R does not contain any $g(X) \in T$ with

$deg(g(X)) = 1$, it follows that $deg(r_1) \geq 2$. As $k \in \{2, 3\}$, we get that $deg(r_1) = k$ and so, $deg(r_2) = 0$. Hence, $r_2 \in U(R)$. This shows that X^2, X^3 are irreducible elements of R and it is clear that they are non-associates in R. Therefore, $RX^2 + RX^3$ is not a principal ideal of R. Hence, R is not a PID. From $J(R) = (0)$, it follows that $Max(R)$ is infinite. As R is not a PID and $J(R) \in Spec(R)$, we obtain from Corollary 1 that $diam(\mathbb{CGI}(R)) = 2$.

We next show that $r(\mathbb{CGI}(R)) = 1$. Let $i \in \mathbb{C}$ be such that $i^2 = -1$. Note that the mapping $\phi : T \to \mathbb{Q}(i)$ defined by $\phi(f(X)) = f(i)$ for any $f(X) \in T$ is an onto ring homomorphism with $T(X^2 + 1)$ as its kernal. It is not hard to verify that $T(X^2+1) \cap R = R(X^2+1)$. The restriction of ϕ to R maps R onto $\mathbb{Q}$ with its kernel equals $R(X^2 + 1)$. Therefore, $\frac{R}{R(X^2+1)} \cong \mathbb{Q}$ as rings and so, $R(X^2 + 1) \in Max(R)$. Thus R admits a principal maximal ideal and so, we obtain from Proposition 4 that $r(\mathbb{CGI}(R)) = 1$. □

In the following example, we provide a UFD R with $|Max(R)| = 2$ and $diam(\mathbb{CGI}(R)) = 2 = r(\mathbb{CGI}(R))$.

Example 4. Let T, R be as in the statement of Example 1. Then R is a UFD, $|Max(R)| = 2$ and $diam(\mathbb{CGI}(R)) = 2 = r(\mathbb{CGI}(R))$.

Proof. In the notation of the statement of Example 1, $T = \mathbb{Z}[X]$, $R = S^{-1}T$, where $S = T \backslash (\bigcup_{i=1}^{2} \mathfrak{m}_i)$ with $\mathfrak{m}_1 = T2 + TX$ and $\mathfrak{m}_2 = T3 + TX$. It is already noted in the proof of Example 1 that R is a UFD and $Max(R) = \{S^{-1}\mathfrak{m}_i \mid i \in \{1, 2\}\}$. Thus $|Max(R)| = 2$. It is verified in the proof of Example 1 that $diam(\mathbb{CGI}(R)) = 2$ and so, $r(\mathbb{CGI}(R)) \leq 2$. Note that $S^{-1}\mathfrak{m}_1 = R2 + RX$. As $R2, RX$ are distinct prime ideals of R, it follows that $S^{-1}\mathfrak{m}_1$ is not principal. Similarly, since $S^{-1}\mathfrak{m}_2 = R3+RX$, $R3, RX$ are distinct prime ideals of R, we get that $S^{-1}\mathfrak{m}_2$ is not principal. Hence by Proposition 4, $r(\mathbb{CGI}(R)) \geq 2$ and so, $r(\mathbb{CGI}(R)) = 2$. □

We do not know any necessary and sufficient condition in order that $r(\mathbb{CGI}(R)) = 2$. For an atomic domain R with $2 \leq |Max(R)| < \infty$, in the following proposition, we provide a necessary and sufficient condition in order that $r(\mathbb{CGI}(R)) = 2$.

Proposition 5. *For an atomic domain R with $2 \leq |Max(R)| < \infty$, $r(\mathbb{CGI}(R)) = 2$ if and only if $\mathfrak{m}$ is not principal for each $\mathfrak{m} \in Max(R)$.*

Proof. Assume that $r(\mathbb{CGI}(R)) = 2$. Then $\mathfrak{m}$ is not principal for each $\mathfrak{m} \in Max(R)$ by Proposition 4. (For this part of the proof, we do not need the assumption that $|Max(R)| < \infty$.)

Conversely, assume that $\mathfrak{m}$ is not principal for each $\mathfrak{m} \in Max(R)$. Let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1, 2, \ldots, n\}\}$. It follows from Proposition 4 that $r(\mathbb{CGI}(R)) \geq 2$. We know from the proof of Lemma 3 that there exists $R\pi_1 \in \mathcal{I}(R)$ such that $\pi_1 \in \mathfrak{m}_1 \backslash (\bigcup_{j=2}^{n} \mathfrak{m}_j)$. Let $R\pi \in \mathcal{I}(R)$ be such that $R\pi \neq R\pi_1$. We claim that $d(R\pi_1, R\pi) \leq 2$ in $\mathbb{CGI}(R)$. We can assume that $R\pi_1$ and $R\pi$ are not adjacent in $\mathbb{CGI}(R)$. It follows from the choice of π_1 that $\pi \in \mathfrak{m}_1$. As $\pi \notin J(R)$, we get that $\pi \notin \mathfrak{m}_j$ for some $j \in \{2, \ldots, n\}$. Hence, $\pi_1\pi \notin \mathfrak{m}_j$ and so, we obtain from Lemma 2 that there exists $R\pi' \in \mathcal{I}(R)$ such that $R\pi_1 - R\pi' - R\pi$ is a path of length two between $R\pi_1$ and $R\pi$ in $\mathbb{CGI}(R)$. This shows that $e(R\pi_1) \leq 2$ in $\mathbb{CGI}(R)$ and so, $r(\mathbb{CGI}(R)) \leq 2$. Therefore, $r(\mathbb{CGI}(R)) = 2$. □

If $diam(\mathbb{CGI}(R)) = 2$ and if no maximal ideal of R is principal, then it follows from Proposition 4 that $r(\mathbb{CGI}(R)) \geq 2$ and so, $r(\mathbb{CGI}(R)) = 2$. In the following example, we provide such an atomic domain.

Example 5. If $T = \mathbb{Z}[X]$, then $diam(\mathbb{CGI}(T)) = 2 = r(\mathbb{CGI}(T))$.

Proof. It is well-known that T is a UFD and so, T is an atomic domain. As $T2+TX$ is not principal, it follows that T is not a PID. It follows from [[1], Exercise 4, page 11] that $J(T) = (0) \in Spec(T)$. Hence, $Max(T)$ is infinite and it follows from Corollary 1 that $diam(\mathbb{CGI}(T)) = 2$. It is well-known that any maximal ideal of T is of the form $Tp+Tf(X)$, where p is a prime number and $f(X) \in T$ is such that $f(X)+pT$ is irreducible modulo $\frac{\mathbb{Z}}{p\mathbb{Z}}$. Since $Tp \in Spec(T)$ and $f(X) \notin Tp$, we get that $Tp+Tf(X)$ is not principal. Thus no maximal ideal of T is principal and so, $r(\mathbb{CGI}(T)) \geq 2$ and therefore, $diam(\mathbb{CGI}(T)) = 2 = r(\mathbb{CGI}(T))$. □

In the following example, we mention a UFD R with $diam(\mathbb{CGI}(R)) = 3$ and $r(\mathbb{CGI}(R)) = 2$.

Example 6. If T, S, R are as in the statement of Example 2, then $diam(\mathbb{CGI}(R)) = 3$ and $r(\mathbb{CGI}(R)) = 2$.

Proof. In the notation of Example 2, $T = \mathbb{Z}[X]$, $S = T\backslash(\bigcup_{i=1}^{n} \mathfrak{m}_i)$ with $\mathfrak{m}_1 = T2 + T(X-1)$, $\mathfrak{m}_2 = T2 + TX$, $\mathfrak{m}_j = Tp_j + TX$ ($j \in \{3, \ldots, n\}$), where $p_3, \ldots, p_n$ are distinct odd prime numbers, and $R = S^{-1}T$. It is noted in the proof of Example 2 that R is a UFD and $Max(R) = \{S^{-1}\mathfrak{m}_i \mid i \in \{1, 2, 3, \ldots, n\}\}$. Hence, $3 \leq |Max(R)| = n < \infty$. It is shown in the proof of Example 2 that $diam(\mathbb{CGI}(R)) = 3$. Note that $2, p_3, \ldots, p_n, X, X-1$ are pairwise non-associate prime elements of T with $T2 \cap S = TX \cap S = T(X-1) \cap S = Tp_j \cap S = \emptyset$ for each $j \in \{3, \ldots, n\}$ and so, $2, X, X-1, p_j$ ($j \in \{3, \ldots, n\}$) are pairwise non-associate prime elements of R. From $2, X-1 \in S^{-1}\mathfrak{m}_1, 2, X \in S^{-1}\mathfrak{m}_2$, and $p_j, X \in S^{-1}\mathfrak{m}_j (j \in \{3, \ldots, n\})$, we obtain that $S^{-1}\mathfrak{m}_i$ is not principal for each $i \in \{1, 2, 3, \ldots, n\}$. It now follows from Proposition 5 that $r(\mathbb{CGI}(R)) = 2$. □

We do not know whether there exists an atomic domain R with $Max(R)$ is infinite such that $diam(\mathbb{CGI}(R)) = 3$ and $r(\mathbb{CGI}(R)) = 2$ (respectively, $diam(\mathbb{CGI}(R)) = 3 = r(\mathbb{CGI}(R))$).

3. Some results on $gr(\mathbb{CGI}(R))$ and $\omega(\mathbb{CGI}(R))$

As in Section 2, unless otherwise specified, we use R to denote an atomic domain with $|Max(R)| \geq 2$. In this section, we discuss some results on $gr(\mathbb{CGI}(R))$ and $\omega(\mathbb{CGI}(R))$. In Theorem 2, we determine a necessary and sufficient condition in order that $\omega(\mathbb{CGI}(R)) < \infty$. We use the following proposition in its proof.

Proposition 6. *Let $n \in \mathbb{N}\backslash\{1\}$. If $\{\mathfrak{m}_i \mid i \in \{1, 2, \ldots, n\}\} \subseteq Max(R)$, then there exists $R\pi_i \in \mathcal{I}(R)$ with $\pi_i \in \mathfrak{m}_i$ for each $i \in \{1, 2, \ldots, n\}$ such that the subgraph of $\mathbb{CGI}(R)$ induced by $\{R\pi_i \mid i \in \{1, 2, \ldots, n\}\}$ is a clique.*

Proof. We prove this proposition using induction on n. Suppose that $n = 2$. As $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, there exist $a_1 \in \mathfrak{m}_1$ and $a_2 \in \mathfrak{m}_2$ such that $Ra_1 + Ra_2 = R$. Let $i \in \{1, 2\}$ and let $\pi_i \in Irr(R) \cap \mathfrak{m}_i$ be a divisor of a_i in R. Then it is clear that $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ and $R\pi_1 + R\pi_2 = R$ and so, the subgraph of $\mathbb{CGI}(R)$ induced by $\{R\pi_1, R\pi_2\}$ is a clique. Let $n \geq 3$ and assume by induction that this proposition is true for $n - 1$. Hence, by induction hypothesis, there exists $R\pi_i \in \mathcal{I}(R)$ with $\pi_i \in \mathfrak{m}_i$ for each $i \in \{1, 2, \ldots, n-1\}$ such that the subgraph of $\mathbb{CGI}(R)$ induced by $\{R\pi_i \mid i \in \{1, 2, \ldots, n-1\}\}$ is a clique. We need to consider the following cases.

Case 1. $\prod_{i=1}^{n-1} \pi_i \notin \mathfrak{m}_n$.
Since $\mathfrak{m}_n \in Max(R)$, it follows that $R(\prod_{i=1}^{n-1} \pi_i) + \mathfrak{m}_n = R$. Hence, there exist $r \in R$ and $m_n \in \mathfrak{m}_n$ such that $r(\prod_{i=1}^{n-1} \pi_i) + m_n = 1$. Therefore, $R(\prod_{i=1}^{n-1} \pi_i) + Rm_n = R$. As R is atomic and $\mathfrak{m}_n \in Spec(R)$, it follows that there exists $\pi_n \in Irr(R) \cap \mathfrak{m}_n$ such that π_n is a divisor of m_n in R and $R(\prod_{i=1}^{n-1} \pi_i) + R\pi_n = R$. Note that $R\pi_n \in \mathcal{I}(R)$. Thus for each $i \in \{1, 2, \ldots, n\}$, there exists $R\pi_i \in \mathcal{I}(R)$ with $\pi_i \in \mathfrak{m}_i$ such that the subgraph of $\mathbb{CGI}(R)$ induced by $\{R\pi_i \mid i \in \{1, 2, \ldots, n\}\}$ is a clique.

Case 2. $\prod_{i=1}^{n-1} \pi_i \in \mathfrak{m}_n$.
Since $R\pi_s + R\pi_t = R$ for all distinct $s, t \in \{1, 2, \ldots, n-1\}$, it follows that there exists a unique $s \in \{1, 2, \ldots, n-1\}$ such that $\pi_s \in \mathfrak{m}_n$. Let us denote $\{1, 2, \ldots, n-1\}\backslash\{s\}$ by A and the element $\prod_{i \in A} \pi_i$ by a. Observe that $a \notin \mathfrak{m}_n$, $\mathfrak{m}_s \neq \mathfrak{m}_n$, and so, it follows that $\mathfrak{m}_s a \not\subseteq \mathfrak{m}_n$. Therefore, $\mathfrak{m}_s a + \mathfrak{m}_n = R$ and so, there exist elements $y_s \in \mathfrak{m}_s$ and $y_n \in \mathfrak{m}_n$ such that $y_s a + y_n = 1$. Hence, $R(y_s a) + Ry_n = R$. Since R is atomic and $\mathfrak{m}_s, \mathfrak{m}_n \in Spec(R)$, it follows that there exist $\pi'_s \in Irr(R) \cap \mathfrak{m}_s$ and $\pi_n \in Irr(R) \cap \mathfrak{m}_n$ such that π'_s is a divisor of y_s in R and π_n is a divisor of y_n in R and $R(\pi'_s a) + R\pi_n = R$. Observe that $R\pi'_s, R\pi_n \in \mathcal{I}(R)$. Note that $a\pi_n \notin \mathfrak{m}_s$. Therefore, $R(a\pi_n) + \mathfrak{m}_s = R$. Hence, there exist $r \in R$ and $m_s \in \mathfrak{m}_s$ such that $ra\pi_n + m_s = 1$. Since R is atomic and $\mathfrak{m}_s \in Spec(R)$, there exists $\pi''_s \in Irr(R) \cap \mathfrak{m}_s$ such that π''_s divides m_s in R and $R(a\pi_n) + R\pi''_s = R$. It is clear that $R\pi''_s \in \mathcal{I}(R)$. Now, the elements $\{\pi_i \mid i \in \{1, 2, \ldots, n-1\}\backslash\{s\}\} \cup \{\pi''_s\} \cup \{\pi_n\} \subseteq Irr(R)$ are such that $\pi_i \in \mathfrak{m}_i$ for each $i \in \{1, 2, \ldots, n-1\}\backslash\{s\}$, $\pi''_s \in \mathfrak{m}_s$, $\pi_n \in \mathfrak{m}_n$, and the subgraph of $\mathbb{CGI}(R)$ induced by $\{R\pi_i \mid i \in A\} \cup \{R\pi''_s, R\pi_n\}$ is a clique, where $A = \{1, 2, \ldots, n-1\}\backslash\{s\}$. This completes the proof. □

Theorem 2. *For an atomic domain R with $|Max(R)| \geq 2$, $\omega(\mathbb{CGI}(R)) < \infty$ if and only if $|Max(R)| < \infty$. Moreover, if $|Max(R)| < \infty$, then $\omega(\mathbb{CGI}(R)) = \chi(\mathbb{CGI}(R)) = |Max(R)|$.*

Proof. Assume that $\omega(\mathbb{CGI}(R)) < \infty$. Let $\omega(\mathbb{CGI}(R)) = t$. We assert that $|Max(R)| \leq t$. Suppose that $|Max(R)| \geq t+1$. Then Proposition 6 implies that $\omega(\mathbb{CGI}(R)) \geq t+1$. This is a contradiction. Therefore, $|Max(R)| \leq t$ and so, $|Max(R)| < \infty$.

Conversely, assume that $|Max(R)| < \infty$. Let $|Max(R)| = n$ and let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1, 2, \ldots, n\}\}$. It is clear that $n \geq 2$. Let $i \in \{1, 2, \ldots, n\}$. We know from the proof of Lemma 3 that there exists $\pi_i \in Irr(R) \cap (\mathfrak{m}_i \backslash (\bigcup_{j \in A_i} \mathfrak{m}_j))$, where $A_i = \{1, 2, \ldots, n\} \backslash \{i\}$. It is clear from the choice of the elements $\pi_1, \pi_2, \ldots, \pi_n$ that $R\pi_i + R\pi_j = R$ for all distinct $i, j \in \{1, 2, \ldots, n\}$. Hence, the subgraph of $\mathbb{CGI}(R)$ induced by $\{R\pi_i \mid i \in \{1, 2, \ldots, n\}\}$ is a clique. Therefore, $\omega(\mathbb{CGI}(R)) \geq n$. We next verify that $\chi(\mathbb{CGI}(R)) \leq n$. Let $\{c_1, c_2, \ldots, c_n\}$ be a set of n distinct colors. We now color the vertices of $\mathbb{CGI}(R)$ as follows: Let $R\pi \in V(\mathbb{CGI}(R)) = \mathcal{I}(R)$. It is clear that $NU(R) = \bigcup_{i=1}^{n} \mathfrak{m}_i$. If $i \in \{1, 2, \ldots, n\}$ is the least positive integer such that $\pi \in \mathfrak{m}_i$, then color $R\pi$ using the color c_i. Let $R\pi, R\pi' \in \mathcal{I}(R)$ be such that they are adjacent in $\mathbb{CGI}(R)$. Then $R\pi + R\pi' = R$. Let $i \in \{1, 2, \ldots, n\}$ be least such that $\pi \in \mathfrak{m}_i$ and let $j \in \{1, 2, \ldots, n\}$ be least such that $\pi' \in \mathfrak{m}_j$. As $R\pi + R\pi' = R$, we obtain that $\mathfrak{m}_i \neq \mathfrak{m}_j$ and so, $i \neq j$. In the above assignment of colors to the vertices of $\mathbb{CGI}(R)$, $R\pi$ receives the color c_i and $R\pi'$ receives the color c_j. As $i \neq j$, it follows that $c_i \neq c_j$. This shows that the above assignment of colors to the vertices of $\mathbb{CGI}(R)$ is a proper vertex coloring of $\mathbb{CGI}(R)$. Thus the vertices of $\mathbb{CGI}(R)$ can be properly colored using a set of n distinct colors. So, we obtain that $n \leq \omega(\mathbb{CGI}(R)) \leq \chi(\mathbb{CGI}(R)) \leq n$. Hence, we get that $\omega(\mathbb{CGI}(R)) = \chi(\mathbb{CGI}(R)) = n = |Max(R)|$.
If $|Max(R)| < \infty$, then it is already verified in the previous paragraph that $\omega(\mathbb{CGI}(R)) = \chi(\mathbb{CGI}(R)) = |Max(R)|$. □

If $T = \mathbb{Z}[X]$, then it is already noted in the proof of Example 5 that $Max(T)$ is infinite. Hence, $\omega(\mathbb{CGI}(T))$ is not finite by Theorem 2. Let n, R be as in the statement of Example 2. It is shown in the proof of Example 2 that R is a UFD and $|Max(R)| = n$. Therefore, $\omega(\mathbb{CGI}(R)) = \chi(\mathbb{CGI}(R)) = n$ by the moreover part of Theorem 2.

If $|Max(R)| \not< \infty$, then we do not know whether $\mathbb{CGI}(R)$ contains an infinite clique. In the following proposition, we provide a condition under which $\mathbb{CGI}(R)$ contains an infinite clique.

Proposition 7. *If $J(R) \in Spec(R)$, then $\mathbb{CGI}(R)$ admits an infinite clique.*

Proof. Assume that $J(R) \in Spec(R)$. It is clear that $Max(R)$ is infinite. Let $n \geq 2$. Let $\{\mathfrak{m}_i \mid i \in \{1, 2, \ldots, n\}\} \subset Max(R)$. Then we know from Proposition 6 that for each $i \in \{1, 2, \ldots, n\}$, there exists $R\pi_i \in \mathcal{I}(R)$ with $\pi_i \in \mathfrak{m}_i$ such that the subgraph of $\mathbb{CGI}(R)$ induced by $\{R\pi_i \mid i \in \{1, 2, \ldots, n\}\}$ is a clique. (For this part of the proof, we do not need the assumption that $J(R) \in Spec(R)$.) Since $J(R) \in Spec(R)$ by assumption, it follows that $\prod_{i=1}^{n} \pi_i \notin J(R)$. Hence, there exists $\mathfrak{m}_{n+1} \in Max(R)$ such that $\prod_{i=1}^{n} \pi_i \notin \mathfrak{m}_{n+1}$. It is clear that $\mathfrak{m}_{n+1} \notin \{\mathfrak{m}_i \mid i \in \{1, 2, \ldots, n\}\}$. Observe that $R(\prod_{i=1}^{n} \pi_i) + \mathfrak{m}_{n+1} = R$. Hence, there exist $r \in R$ and $a_{n+1} \in \mathfrak{m}_{n+1}$ such that $r(\prod_{i=1}^{n} \pi_i) + a_{n+1} = 1$. Since R is atomic, there exists $\pi_{n+1} \in Irr(R) \cap \mathfrak{m}_{n+1}$ such that π_{n+1} is a divisor of a_{n+1} in R. It is clear that $R(\prod_{i=1}^{n} \pi_i) + R\pi_{n+1} = R$ and so, $R\pi_{n+1} \in \mathcal{I}(R)$. Note that $R\pi_i + R\pi_{n+1} = R$ for each $i \in \{1, 2, \ldots, n\}$. Therefore, the subgraph of $\mathbb{CGI}(R)$ induced by $\{R\pi_i \mid i \in \{1, 2, \ldots, n+1\}\}$ is a clique. The above procedure can be repeated thereby yielding for each $j \in \mathbb{N}$, $\mathfrak{m}_j \in Max(R)$ and $R\pi_j \in \mathcal{I}(R)$ with $\pi_j \in \mathfrak{m}_j$ such that the subgraph of $\mathbb{CGI}(R)$ induced by $\{R\pi_j \mid j \in \mathbb{N}\}$ is an infinite clique. □

We next discuss some results on $gr(\mathbb{CGI}(R))$.

Corollary 3. *If $|Max(R)| \geq 3$, then $gr(\mathbb{CGI}(R)) = 3$.*

Proof. Assume that $|Max(R)| \geq 3$. Then Proposition 6 implies that $\omega(\mathbb{CGI}(R)) \geq 3$. Therefore, $gr(\mathbb{CGI}(R)) = 3$. □

If $|Max(R)| = 2$, then with the help of the following proposition, we show in Corollary 4 that $gr(\mathbb{CGI}(R)) \in \{4, \infty\}$. In the following proposition, we determine a necessary and sufficient condition in order that $\mathbb{CGI}(R)$ is a bipartite graph.

Proposition 8. *With R as in the statement of Theorem 1, the following statements are equivalent:*

(1) $\mathbb{CGI}(R)$ is a bipartite graph.

(2) $|Max(R)| = 2$.

Moreover, if $|Max(R)| = 2$, then $\mathbb{CGI}(R)$ is a complete bipartite graph.

Proof. (1) ⇒ (2). Assume that $\mathbb{CGI}(R)$ is a bipartite graph. If $|Max(R)| \geq 3$, then we know from Corollary 3 that $\mathbb{CGI}(R)$ contains a cycle of length three. This is impossible by [[2], Theorem 1.5.10]. Therefore, $|Max(R)| = 2$.

(2) ⇒ (1). Assume that $|Max(R)| = 2$. Let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1, 2\}\}$. It is already noted in the proof of Corollary 2 that $V(\mathbb{CGI}(R)) = \bigcup_{i=1}^{2} V_i$, where $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1 \backslash \mathfrak{m}_2\}$ and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in \mathfrak{m}_2 \backslash \mathfrak{m}_1\}$ and $V_i \neq \emptyset$ for each $i \in \{1, 2\}$. It is clear that $V_1 \cap V_2 = \emptyset$. Moreover, it is noted in the proof of Corollary 2 that no two vertices of V_i are adjacent in $\mathbb{CGI}(R)$ for each $i \in \{1, 2\}$ and for any $R\pi \in V_1, R\pi' \in V_2$, $R\pi$ and $R\pi'$ are adjacent in $\mathbb{CGI}(R)$. This proves that $\mathbb{CGI}(R)$ is a complete bipartite graph with vertex partition V_1 and V_2.

If $|Max(R)| = 2$, then it is already verified in the previous paragraph that $\mathbb{CGI}(R)$ is a complete bipartite graph. □

Corollary 4. *If $|Max(R)| = 2$, then $gr(\mathbb{CGI}(R)) \in \{4, \infty\}$. Moreover, $gr(\mathbb{CGI}(R)) = \infty$ if and only if at least one maximal ideal of R is principal.*

Proof. Assume that $|Max(R)| = 2$. We know from the moreover part of Proposition 8 that $\mathbb{CGI}(R)$ is a complete bipartite graph. Hence, $gr(\mathbb{CGI}(R)) \in \{4, \infty\}$.
For any complete bipartite graph G, $gr(G) = \infty$ if and only if $r(G) = 1$. Hence, it follows that $gr(\mathbb{CGI}(R)) = \infty$ if and only if $r(\mathbb{CGI}(R)) = 1$. Therefore, by Proposition 4, we obtain that $gr(\mathbb{CGI}(R)) = \infty$ if and only if at least one maximal ideal of R is principal. □

The following example illustrates Proposition 8 and Corollary 4.

Example 7. (1) Let T, R be as in Example 1. Then $\mathbb{CGI}(R)$ is a complete bipartite graph and $gr(\mathbb{CGI}(R)) = 4$.

(2) Let T, R be as in the statement of Example 3. Let $S = R \setminus (\bigcup_{i=1}^{2} \mathfrak{m}_i)$, where $\mathfrak{m}_1 = RX^2 + RX^3$ and $\mathfrak{m}_2 = R(X^2 + 1)$. Let $R_1 = S^{-1}R$. Then R_1 is a Noetherian domain and $gr(\mathbb{CGI}(R_1)) = \infty$.

Proof. (1) It is already noted in the proof of Example 1 that R is a UFD and $|Max(R)| = 2$. Hence, $\mathbb{CGI}(R)$ is a complete bipartite graph by the moreover part of Proposition 8. It is shown in the proof of Example 4 that both the maximal ideals of R are not principal. Hence, we obtain from Corollary 4 that $gr(\mathbb{CGI}(R)) = 4$.

(2) In the notation of the statement of Example 3, $T = \mathbb{Q}[X]$ and $R = \mathbb{Q}[X^2, X^3]$. Note that $R = \mathbb{Q} + X^2\mathbb{Q}[X]$ and it is not hard to verify that $X^2\mathbb{Q}[X] = RX^2 + RX^3$. Since $\frac{R}{X^2\mathbb{Q}[X]} \cong \mathbb{Q}$ as rings, it follows that $\mathfrak{m}_1 = RX^2 + RX^3 \in Max(R)$. It is already shown in the proof of Example 3 that $\mathfrak{m}_2 = R(X^2 + 1) \in Max(R)$. Hence, S is a m.c. subset of R. As R is Noetherian by [[1], Corollary 7.7], it follows from [[1], Proposition 7.3] that $R_1 = S^{-1}R$ is Noetherian. Hence, R_1 is an atomic domain. It is clear that $\{\mathfrak{m}_i \mid i \in \{1, 2\}\}$ is the set of prime ideals of R maximal with respect to not meeting S. Therefore, it follows from [[1], Proposition 3.11(iv)] that $Max(R_1) = \{S^{-1}\mathfrak{m}_i \mid i \in \{1, 2\}\}$. Thus $|Max(R_1)| = 2$. Since $\mathfrak{m}_2$ is a principal ideal of R, it follows that $S^{-1}\mathfrak{m}_2$ is a principal ideal of R_1. Hence, we obtain from Corollary 4 that $gr(\mathbb{CGI}(R_1)) = \infty$.

□

If $\mathbb{CGI}(R)$ is a finite graph, then $\mathcal{I}(R)$ is finite. It follows from Lemma 3 that $|Max(R)| \leq |\mathcal{I}(R)|$. With the assumption $|\mathcal{I}(R)| < \infty$, in the following proposition, we characterize R such that $|Max(R)| = |\mathcal{I}(R)|$.

Proposition 9. *For an atomic domain R with $|Max(R)| \geq 2$ and $|\mathcal{I}(R)| < \infty$, $|Max(R)| = |\mathcal{I}(R)|$ if and only if R is a semi-local PID.*

Proof. Assume that $|Max(R)| = |\mathcal{I}(R)|$. Suppose that $|\mathcal{I}(R)| = n$. Let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1, 2, \ldots, n\}\}$. For each $i \in \{1, 2, \ldots, n\}$, we know from the proof of Lemma 3 that there exists $\pi_i \in Irr(R) \cap (\mathfrak{m}_i \setminus (\bigcup_{j \in A_i} \mathfrak{m}_j))$, where $A_i = \{1, 2, \ldots, n\} \setminus \{i\}$. Note that for all distinct $t, s \in \{1, 2, \ldots, n\}$, $R\pi_t + R\pi_s = R$. As $|\mathcal{I}(R)| = n$ and $R\pi_1, R\pi_2 \ldots, R\pi_n \in \mathcal{I}(R)$ are distinct, it follows that $\mathcal{I}(R) = \{R\pi_i \mid i \in \{1, 2, \ldots, n\}\}$. From $R\pi_i + R\pi_j = R$ for all distinct $i, j \in \{1, 2, \ldots, n\}$, it follows that $diam(\mathbb{CGI}(R)) = 1$. Therefore, we obtain from (1) $\Rightarrow$ (2) of Theorem 1 that R is a PID. Since $|Max(R)| = n$, it is clear that R is semi-local.

Conversely, assume that R is a semi-local PID. Let $|Max(R)| = n$ and let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1, 2, \ldots, n\}\}$. We know from (2) $\Rightarrow$ (1) of Theorem 1 that $diam(\mathbb{CGI}(R)) = 1$. Note that $\omega(\mathbb{CGI}(R)) = |Max(R)| = n$ by the moreover part of Theorem 2. Therefore, $\mathbb{CGI}(R)$ is a complete graph with n vertices. As $V(\mathbb{CGI}(R)) = \mathcal{I}(R)$, it follows that $|Max(R)| = |\mathcal{I}(R)|$. □

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