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Research Article

γ-total dominating graphs of lollipop, umbrella, and coconut graphs

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Abstract: A total dominating set of a graph G is a set $D \subseteq V(G)$ such that every vertex of G is adjacent to some vertex in D. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set. The γ-total dominating graph $TD_\gamma(G)$ of G is the graph whose vertices are minimum total dominating sets, and two minimum total dominating sets of $TD_\gamma(G)$ are adjacent if they differ by only one vertex. In this paper, we determine the total domination numbers of lollipop graphs, umbrella graphs, and coconut graphs, and especially their γ-total dominating graphs.

Keywords: total domination number, total dominating graph, gamma graph

AMS Subject classification: 05C69.

1. Introduction

Let G be a graph whose vertex set is $V(G)$ and edge set is $E(G)$. For a vertex $v \in V(G)$, the *open* and *closed neighborhoods* of v are $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. For a set $D \subseteq V(G)$, the *open* and *closed neighborhoods* of D are $N(D) = \bigcup_{v \in D} N(v)$ and $N[D] = N(D) \cup D$, respectively. We write $G[D]$ for the *subgraph of G induced by D*.

A *dominating set* of G is a set $D \subseteq V(G)$ with $N(v) \cap D \neq \emptyset$ for each $v \in V(G) \setminus D$. For a review of domination in graphs, see [12, 13]. The *gamma graph*

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$\gamma \cdot G$ of G, defined by Subramanian and Sridharan [22], is the graph where its vertices are minimum dominating sets, and two vertices D_1 and D_2 of $\gamma \cdot G$ are adjacent if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1$ and $v \notin D_1$. For additional results on $\gamma \cdot G$, see [15, 20, 21]. Fricke *et al.* [9] also defined the *gamma graph* $G(\gamma)$ of G to be the graph where $V(G(\gamma)) = V(\gamma \cdot G)$, and two vertices D_1 and D_2 of $G(\gamma)$ are adjacent if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1, v \notin D_1$, and $uv \in E(G)$. Further results concerning $G(\gamma)$ can be found in [2, 4]. For the graphs using the other types of domination with the same adjacency condition as $\gamma \cdot G$ and $G(\gamma)$, see [5–7, 18, 19, 24] and [17], respectively.

Haas and Seyffarth [10] defined the *k-dominating graph* $D_k(G)$ of G, as the graph whose vertices are dominating sets with cardinality at most k, and two vertices of $D_k(G)$ are adjacent if they differ by either adding or deleting a single vertex. For more details, see [11, 16, 23]. The *k-total dominating graph* [1] and the *k-independent dominating graph* [8] are defined similarly using total dominating sets and independent dominating sets, respectively.

A set $D \subseteq V(G)$ is a *total dominating set* of G if $N(v) \cap D \neq \emptyset$ for each $v \in V(G)$. The minimum cardinality of a total dominating set of G is called the *total domination number* $\gamma_t(G)$. A total dominating set D is a $\gamma_t(G)$*-set* if $|D| = \gamma_t(G)$. The total domination in graphs was introduced by Cockayne *et al.* [3]. The *γ-total dominating graph* $TD_\gamma(G)$ of G, defined by Wongsriya and Trakultraipruk [24], is the graph whose vertices are $\gamma_t(G)$-sets, and two $\gamma_t(G)$-sets D_1 and D_2 of $TD_\gamma(G)$ are adjacent if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1$ and $v \notin D_1$. In this paper, we determine the total domination numbers of lollipop, umbrella, and coconut graphs in Section 3. Then we study their γ-total dominating graphs in Sections 4 and 5.

2. Preliminary Results

In this section, we recall some definitions and results, which are used in our main results.

A *path* and a *complete graph* with k vertices are denoted by P_k and K_k, respectively. If v is adjacent to a vertex of degree one, then v is a *support vertex*. We first provide a straightforward observation.

Observation 1. Each support vertex of a graph G is in every $\gamma_t(G)$-set.

The total domination numbers of paths established by Henning [14] are shown in the following lemma.

Lemma 1 ([14]). *Let $k \geq 2$ be an integer. Then $\gamma_t(P_k) = \lfloor \frac{k+2}{4} \rfloor + \lfloor \frac{k+3}{4} \rfloor$.*

The *Cartesian product* of graphs G and H, denoted by $G \Box H$, is the graph with $V(G \Box H) = V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) of $V(G \Box H)$ are adjacent if either $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

In [24], the authors determined the γ-total dominating graphs of paths as listed below.

Theorem 2 ([24]). *Let $k \geq 1$ be an integer. Then $TD_\gamma(P_{4k}) \cong P_1$.*

Theorem 3 ([24]). *Let $k \geq 0$ be an integer. Then $TD_\gamma(P_{4k+3}) \cong P_{k+2}$.*

Theorem 4 ([24]). *Let $k \geq 0$ be an integer. Then $TD_\gamma(P_{4k+2}) \cong P_{k+1} \Box P_{k+1}$.*

Theorem 5 ([24]). *Let $k \geq 1$ be an integer. Then $TD_\gamma(P_{4k+1}) \cong P_k$.*

We denote $P_k : v_1v_2v_3 \cdots v_k$ to be the path. From the proofs of Theorems 3, 4, and 5, we can get Lemmas 2, 3, and 4 shown below, respectively.

Lemma 2. *Let $k \geq 0$ be an integer and $TD_\gamma(P_{4k+3}) \cong P_{k+2} \cong D_1D_2 \cdots D_{k+2}$, where D_x is a $\gamma_t(P_{4k+3})$-set for all $x \in \{1, 2, \ldots, k+2\}$.*

(1) *If $v_{4k+3} \in D_x$, then either $x = 1$ or $x = k+2$.*

(2) *If D_{k+2} contains the vertex v_{4k+3}, then $D_{k+2} = (D_{k+1} \setminus \{v_{4k+1}\}) \cup \{v_{4k+3}\}$.*

We consider the $\gamma_t(P_{4k+2})$-sets of $TD_\gamma(P_{4k+2}) \cong P_{k+1} \Box P_{k+1}$ as the entries in a matrix.

Lemma 3. *Let $k \geq 0$ be an integer and $D_{x,y}$ the $\gamma_t(P_{4k+2})$-set at the position (x, y) (row x and column y) of $TD_\gamma(P_{4k+2}) \cong P_{k+1} \Box P_{k+1}$ for all $x, y \in \{1, 2, \ldots, k+1\}$.*

(1) *If $v_{4k+2} \in D_{x,y}$, then either $x = 1$, $x = k+1$, $y = 1$, or $y = k+1$.*

(2) *If $D_{x,k+1}$ contains the vertex v_{4k+2}, then*

  (2.1) *$D_{x,k+1} = (D_{x,k} \setminus \{v_{4k}\}) \cup \{v_{4k+2}\}$ for each $x \in \{1, 2, \ldots, k+1\}$,*

  (2.2) *$D_{k+1,k+1} = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\} \cup \{v_{4k+1}, v_{4k+2}\}$, and*

  (2.3) *$D_{k+1,1}, D_{k+1,2}, \ldots, D_{k+1,k+1}$ are the only $\gamma_t(P_{4k+2})$-sets containing the vertex v_{4k-1}.*

Lemma 4. *Let $k \geq 1$ be an integer. Then each $\gamma_t(P_{4k+1})$-set does not contain the vertex v_{4k+1}.*

3. Total Domination Numbers of Lollipop, Umbrella, and Coconut Graphs

The definitions of a lollipop graph, a umbrella graph, and a coconut graph are appeared in this section. In particular, the total domination numbers of those graphs are determined.

Let p and q be positive integers. A *lollipop graph* $L_{p,q}$ is obtained by affixing an endpoint of a path P_p to a vertex of a complete graph K_q. Throughout this paper, we let the vertices of $L_{p,q}$ be as shown in Figure 1.

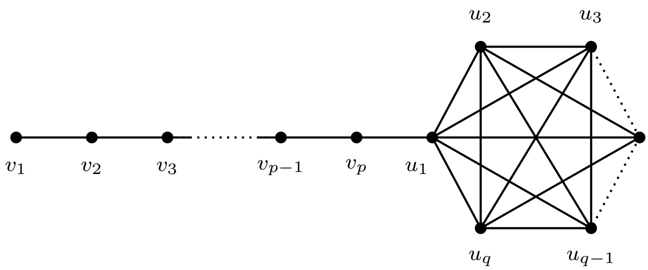

Figure 1. **The lollipop graph** $L_{p,q}$

An *umbrella graph* $U_{p,q}$ is obtained by appending an endpoint of a path P_p to the central vertex of a fan graph $K_1 \vee P_{q-1}$. A *coconut graph* $C_{p,q}$ is obtained by appending an endpoint of a path P_p to the support vertex of a complete bipartite graph $K_{1,q-1}$. We let the vertices of $U_{p,q}$ and $C_{p,q}$ be as shown in Figures 2 and 3, respectively.

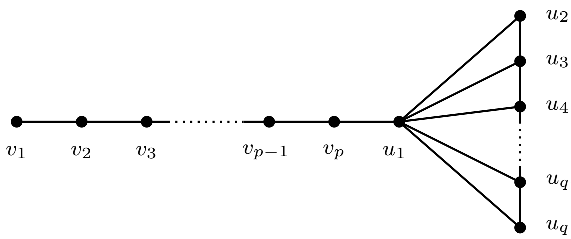

Figure 2. **The umbrella graph** $U_{p,q}$

Note that $L_{p,1} \cong U_{p,1} \cong C_{p,1} \cong P_{p+1}$. By Lemma 1, $\gamma_t(L_{p,1}) = \gamma_t(U_{p,1}) = \gamma_t(C_{p,1}) = \lfloor \frac{p+3}{4} \rfloor + \lfloor \frac{p+4}{4} \rfloor$. For $q \geq 2$, we obtain the following theorem.

Theorem 6. *Let $p \geq 1$ and $q \geq 2$ be integers. Then $\gamma_t(L_{p,q}) = \gamma_t(U_{p,q}) = \gamma_t(C_{p,q}) = \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor$.*

Proof. If $q = 2$, then $L_{p,q} \cong P_{p+2}$, so $\gamma_t(L_{p,2}) = \gamma_t(P_{p+2}) = \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor$ by Lemma 1. Let $q \geq 3$ and P' be the graph obtained from $L_{p,q}$ by deleting the vertices $u_3, u_4, \ldots, u_q$. Clearly, $P' \cong P_{p+2}$ and then $\gamma_t(P') = \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor$. Let D be a

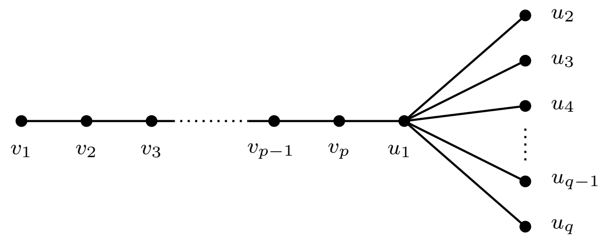

Figure 3. **The coconut graph** $C_{p,q}$

$\gamma_t(L_{p,q})$-set. We show that $|D| \geq \gamma_t(P')$. If $u_1 \in D$, then to dominate u_1, D contains either v_p or, without loss of generality, u_2. In both cases, D is a total dominating set of P', and thus $|D| \geq \gamma_t(P')$. On the other hand, we assume that $u_1 \notin D$. Since D is a $\gamma_t(L_{p,q})$-set, without loss of generality, D contain exactly two vertices u_2 and u_3 from $\{u_2, u_3, \ldots, u_q\}$. Then $D' = (D \setminus \{u_3\}) \cup \{u_1\}$ is a total dominating set of P', and hence $|D| = |D'| \geq \gamma_t(P')$. Therefore, $\gamma_t(L_{p,q}) = |D| \geq \gamma_t(P')$. Note that $U_{p,q}$ and $C_{p,q}$ are spanning subgraphs of $L_{p,q}$, so $\gamma_t(U_{p,q}) \geq \gamma_t(L_{p,q})$ and $\gamma_t(C_{p,q}) \geq \gamma_t(L_{p,q})$.

We next determine the upper bounds of $\gamma_t(L_{p,q}), \gamma_t(U_{p,q})$, and $\gamma_t(C_{p,q})$. If $p \equiv 0, 1, 2 \pmod 4$, let $D = \{v_i, v_{i+1} : i \equiv 2 \pmod 4, i < p\} \cup \{v_p, u_1\}$; otherwise, let $D = \{v_i, v_{i+1} : i \equiv 2 \pmod 4\} \cup \{u_1\}$. Then D is a total dominating set of $L_{p,q}$ with $|D| = \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor$, and hence $\gamma_t(L_{p,q}) \leq \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor$. Likewise, $\gamma_t(U_{p,q}) \leq \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor$ and $\gamma_t(C_{p,q}) \leq \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor$. The theorem follows. □

4. γ-Total Dominating Graphs of Lollipop Graphs

In this section, we study the γ-total dominating graph of a lollipop graph $L_{p,q}$. If $q = 1$, then $L_{p,q} \cong P_{p+1}$. Theorems 2 - 5 provide the results on $TD_\gamma(L_{p,1}) \cong TD_\gamma(P_{p+1})$. For $q \geq 2$, we divide the value of p into four cases. If $p = 4k + 2$, then we get the following theorem.

Theorem 7. *Let $k \geq 0$ and $q \geq 2$ be integers. Then $TD_\gamma(L_{4k+2,q}) \cong P_1$.*

Proof. By Theorem 6, we get $\gamma_t(L_{4k+2,q}) = 2k + 2$. Then there is exactly one $\gamma_t(L_{4k+2,q})$-set, which is $D = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\} \cup \{v_{4k+2}, u_1\}$. □

Lemma 5. *Let $k \geq 0$ and $q \geq 2$ be integers. Then each $\gamma_t(L_{4k+1,q})$-set contains the vertex u_1.*

Proof. For $q = 2$, the vertex u_1 is a support vertex of $L_{4k+1,2} \cong P_{4k+3}$, and hence the lemma follows by Observation 1. Let $q \geq 3$ and suppose, contrary to the statement, that there exists a $\gamma_t(L_{4k+1,q})$-set D that does not contain u_1. Thus, D contains exactly two vertices u_i and u_j from $\{u_2, u_3, \ldots, u_q\}$. Let $S = \{v : v \notin N(\{u_i, u_j\})\}$, and then the induced subgraph $L_{4k+1,q}[S]$ is P_{4k+1}. By Theorem 6, $|D| = 2k + 2$ and thus the $2k$ remaining vertices of D must dominate all vertices in $L_{4k+1,q}[S]$, which is impossible. □

Theorem 8. *Let $k \geq 0$ and $q \geq 2$ be integers. Then $TD_\gamma(L_{4k+1,q}) \cong L_{k,q}$.*

Proof. For each $i \in \{2, 3, \ldots, q\}$, let P^i be the subgraph of $L_{4k+1,q}$ induced by $\{v_1, v_2, \ldots, v_{4k+1}, u_1, u_i\}$, and then $P^i \cong P_{4k+3}$. By Theorem 3, for each $i \in \{2, 3, \ldots, q\}$, $TD_\gamma(P^i) \cong P_{k+2} \cong D_1^i D_2^i \cdots D_{k+2}^i$, where D_x^i is a $\gamma_t(P^i)$-set for each $x \in \{1, 2, \ldots, k+2\}$, so by Observation 1, $u_1 \in D_x^i$. By Lemma 2(1), without loss of generality, we may assume that D_{k+2}^i contains u_i. If $x \neq k+2$, then $D_x^i = D_x^j$ for all $i, j \in \{2, 3, \ldots, q\}$, so we let $D_x = D_x^i$. Next, we claim that D_{k+2}^i and D_{k+2}^j are adjacent for all $i \neq j$. By Lemma 2(2), we get $D_{k+2}^i = (D_{k+1} \setminus \{v_{4k+1}\}) \cup \{u_i\} = [(D_{k+1} \setminus \{v_{4k+1}\}) \cup \{u_j\}] \setminus \{u_j\} \cup \{u_i\} = (D_{k+2}^j \setminus \{u_j\}) \cup \{u_i\}$, so the claim holds.

Note that $\gamma_t(P^i) = 2k + 2 = \gamma_t(L_{4k+1,q})$, and every $\gamma_t(P^i)$-set is also a $\gamma_t(L_{4k+1,q})$-set for each $i \in \{2, 3, \ldots, q\}$. Hence, $D_1, \ldots, D_{k+1}, D_{k+2}^2, \ldots, D_{k+2}^q$ are $\gamma_t(L_{4k+1,q})$-sets containing u_1. By Lemma 5, each $\gamma_t(L_{4k+1,q})$-set contains u_1, so it is a $\gamma_t(P^i)$-set for some $i \in \{2, 3, \ldots, q\}$. Therefore, $D_1, \ldots, D_{k+1}, D_{k+2}^2, \ldots, D_{k+2}^q$ are the only $\gamma_t(L_{4k+1,q})$-sets, and in addition they form the lollipop graph $L_{k,q}$ (see Figure 4). □

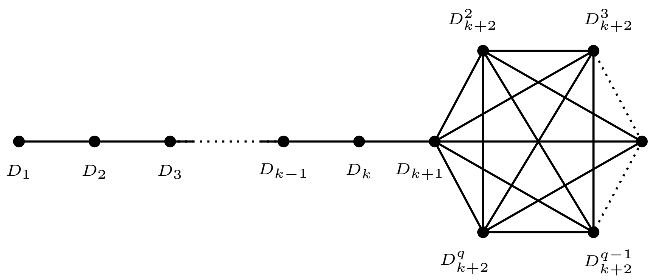

Figure 4. **The γ-total dominating graph of $L_{4k+1,q}$**

The *Johnson graph* $J_{p,q}$ is the graph whose vertices correspond to the q-element subsets of $\{1, 2, \ldots, p\}$, where two vertices are adjacent when they meet in a $(q-1)$-element set. Clearly, $J_{p,q}$ has $\binom{p}{q}$ vertices. In Figure 5, we show the Johnson graph $J_{4,2}$.

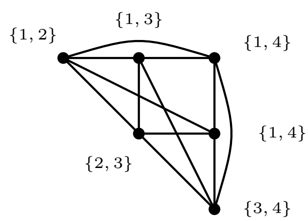

Figure 5. **The Johnson graph $J_{4,2}$**

Note that $\gamma_t(K_p) = 2$. It follows from the definition that the γ-total domination graph of K_p is precisely the Johnson graph $J_{p,2}$, as stated the following theorem.

Theorem 9. *Let $p \geq 2$ be an integer. Then $TD_\gamma(K_p) \cong J_{p,2}$.*

Let $L^r_{p,q} = L_{p,q} \square P_r$, where the vertices of $L^r_{p,q}$ are labeled as shown in Figure 6. For convenience, we write $q-1$ vertices $v_{r,p+2}, v_{r,p+3}, \ldots, v_{r,p+q}$ of $L^r_{p,q}$ for $u_1, u_2, \ldots, u_{q-1}$, respectively. Let $JL^r_{p,q}$ be the graph obtained from $L^r_{p,q}$ by adding the vertices $u_q, u_{q+1}, \ldots, u_{\binom{q}{2}}$ such that $u_1, u_2, \ldots, u_{q-1}, u_q, u_{q+1}, \ldots, u_{\binom{q}{2}}$ form the Johnson graph $J_{q,2}$. We illustrate the graph $JL^4_{5,4}$ in Figure 7.

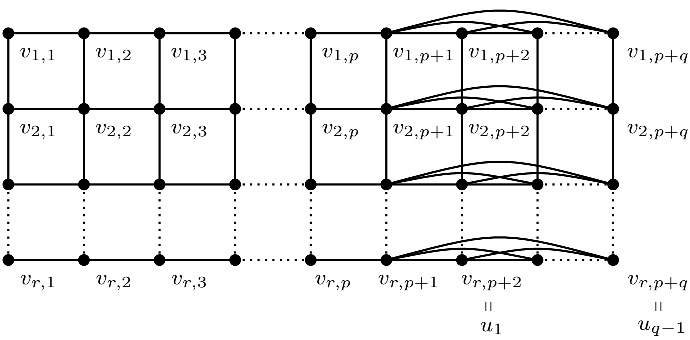

Figure 6. **The graph $L^r_{p,q}$**

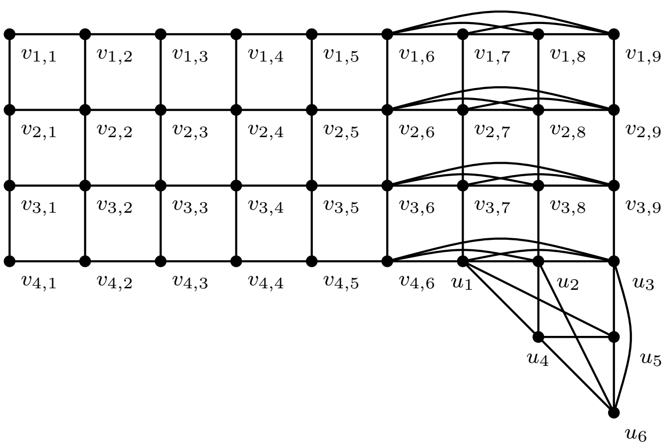

Figure 7. **The graph $JL^4_{5,4}$**

Theorem 10. *Let $k \geq 1$ and $q \geq 2$ be integers. Then $TD_\gamma(L_{4k,q}) \cong JL^{k+1}_{k-1,q}$.*

Proof. For each $i \in \{2, 3, \ldots, q\}$, let P^i be the subgraph of $L_{4k,q}$ induced by $\{v_1, v_2, \ldots, v_{4k}, u_1, u_i\}$, so $TD_\gamma(P^i) \cong TD_\gamma(P_{4k+2}) \cong P_{k+1} \Box P_{k+1}$ by Theorem 2. For each $i \in \{2, 3, \ldots, q\}$ and $x, y \in \{1, 2, \ldots, k+1\}$, let $D^i_{x,y}$ be the $\gamma_t(P^i)$-set at the position (x, y) of $TD_\gamma(P^i)$. By Lemma 3(1), without loss of generality, we may assume that $D^i_{x,k+1}$ contains u_i. If $y \neq k+1$, then $D^i_{x,y} = D^j_{x,y}$ for all $i, j \in \{2, 3, \ldots, q\}$. Hence, for all $x \in \{1, 2, \ldots, k+1\}$, we let $D_{x,y} = D^i_{x,y}$ if $y \neq k+1$; otherwise, let $D^i_{x,k+1} = D_{x,k+i-1}$ for all $i \in \{2, 3, \ldots, q\}$. Note that $D_{x,k}$ is adjacent to $D_{x,k+i-1}$ for all $i \in \{2, 3, \ldots, q\}$. We next show that $D_{x,k+i-1}$ and $D_{x,k+j-1}$ are adjacent for all $i \neq j$. By Lemma 3(2.1), for each $x \in \{1, 2, \ldots, k+1\}$, we get $D_{x,k+i-1} = D^i_{x,k+1} = (D_{x,k} \setminus \{v_{4k}\}) \cup \{u_i\} = [(D_{x,k} \setminus \{v_{4k}\}) \cup \{u_j\}] \setminus \{u_j\} \cup \{u_i\} = (D^j_{x,k+1} \setminus \{u_j\}) \cup \{u_i\} = (D_{x,k+j-1} \setminus \{u_j\}) \cup \{u_i\}$, as desired.

Note that $\gamma_t(P^i) = 2k+2 = \gamma_t(L_{4k,q})$, and a $\gamma_t(P^i)$-set is a $\gamma_t(L_{4k,q})$-set containing u_1 and vice versa. Thus, all $D_{x,y}$'s with $1 \leq x \leq k+1$ and $1 \leq y \leq k+q-1$ are the only $\gamma_t(L_{4k,q})$-sets containing u_1, and they form the graph $L^{k+1}_{k-1,q}$ in $TD_\gamma(L_{4k,q})$ (see Figure 8).

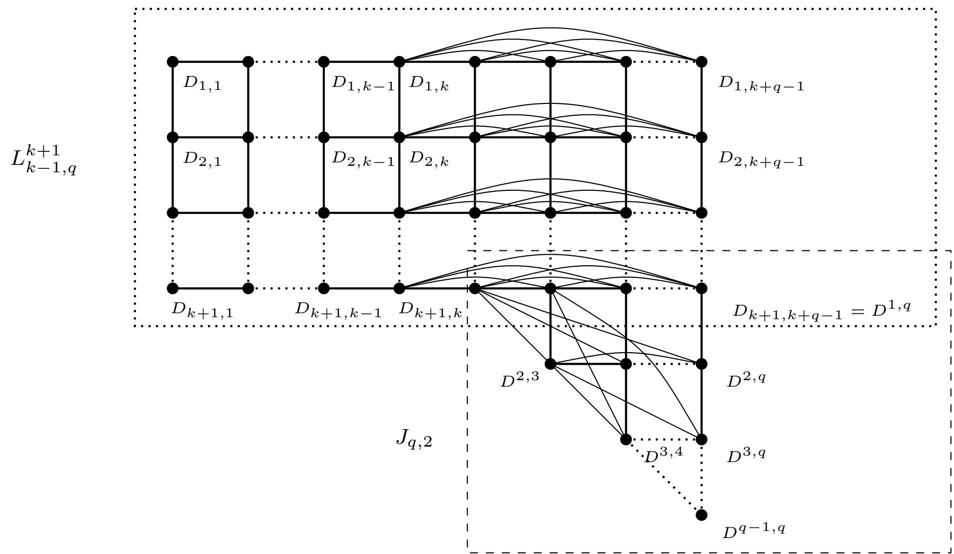

Figure 8. **The γ-total dominating graph of $L_{4k,q}$**

Finally, we find all $\gamma_t(L_{4k,q})$-sets that do not contain u_1. Then such a set contains $2k$ vertices from $\{v_1, v_2, \ldots, v_{4k}\}$ and two vertices from $\{u_2, u_3, \ldots, u_q\}$. Thus, it is the union of $D = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\}$ and $\{u_i, u_j\}$ for some distinct $i, j \in \{2, 3, \ldots, q\}$. By Lemma 3(2.2), for each $i \in \{2, 3, \ldots, q\}$, $D_{k+1,k+i-1} = D^i_{k+1,k+1} = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\} \cup \{u_1, u_i\} = D \cup \{u_1, u_i\}$. For all $1 \leq i < j \leq q$, let $D^{i,j} = D \cup \{u_i, u_j\}$. Theorem 10 implies that all $D^{i,j}$'s form the Johnson graph $J_{q,2}$ in $TD_\gamma(L_{4k,q})$ (see Figure 8). Moreover, for all $2 \leq i < j \leq q$, $D^{i,j}$ is not adjacent to

$D_{x,y}$ for all $y \leq k$, which does not contain $u_2, u_3, \ldots, u_q$. By Lemma 3(2.3), for each $x \neq k+1$ and $y \in \{2, 3, \ldots, q\}$, $D_{x,k+y-1} = D^y_{x,k+1}$ contains u_1 and u_y but not v_{4k-1}, so $D_{x,k+y-1} \setminus \{u_1\} \cup \{u_j\}$ is not a total dominating set for all $j \notin \{1, y\}$ since v_{4k} is not dominated. This means that $D_{x,k+y-1}$ with $x \neq k+1$ is not adjacent to $D^{i,j}$ for all $2 \leq i < j \leq q$. This completes the proof. □

Lemma 6. *Let $k \geq 1$ and $q \geq 2$ be integers. Then each $\gamma_t(L_{4k-1,q})$-set does not contain the vertex u_i for all $i \in \{2, 3, \ldots, q\}$.*

Proof. Assume on contrary that there exists a $\gamma_t(L_{4k-1,q})$-set D containing u_i for some $i \in \{2, 3, \ldots, q\}$. To dominate u_i, we need at least one vertex $u_j \in D$ for some $j \in \{1, 2, \ldots, q\}$ with $j \neq i$. Let $S = \{v : v \notin N(\{u_i, u_j\})\}$. If $j = 1$, then the induced subgraph $L_{4k-1,q}[S] \cong P_{4k-2}$; otherwise, $L_{4k-1,q}[S] \cong P_{4k-1}$. Note that $|D| = 2k+1$, so Lemma 1 implies that the $2k - 1$ remaining vertices of D cannot dominate all vertices in $L_{4k-1,q}[S]$, a contradiction. □

Theorem 11. *Let $k \geq 1$ and $q \geq 2$ be integers. Then $TD_\gamma(L_{4k-1,q}) \cong P_k$.*

Proof. For each $i \in \{2, 3, \ldots, q\}$, let P^i be the subgraph of $L_{4k-1,q}$ induced by $\{v_1, v_2, \ldots, v_{4k-1}, u_1, u_i\}$, and then by Theorem 5, $TD_\gamma(P^i) \cong P_k \cong D^i_1 D^i_2 \cdots D^i_k$, where D^i_x is a $\gamma_t(P^i)$-set for all $x \in \{1, 2, \ldots, k\}$. By Lemma 4, $D^i_1, D^i_2, \ldots, D^i_k$ do not contain u_i for each $i \in \{2, 3, \ldots, q\}$, so without loss of generality, we may assume that $D^i_x = D^j_x$ for all $i, j \in \{2, 3, \ldots, q\}$, and we let $D_x = D^i_x$. Since $\gamma_t(P^i) = 2k+1 = \gamma_t(L_{4k-1,q})$ and every $\gamma_t(P^i)$-set is a $\gamma_t(L_{4k-1,q})$-set for all $i \in \{2, 3, \ldots, q\}$, we get $D_1, D_2, \ldots, D_k$ are $\gamma_t(L_{4k-1,q})$-sets. Lemma 6 implies that each $\gamma_t(L_{4k-1,q})$-set is also a $\gamma_t(P^i)$-set for some $i \in \{2, 3, \ldots, q\}$. Therefore, $D_1, D_2, \ldots, D_k$ are the only $\gamma_t(L_{4k-1,q})$-sets, and they form the path with k vertices in $TD_\gamma(L_{4k-1,q})$. □

5. γ-Total Dominating Graphs of Umbrella and Coconut Graphs

Let p and q be positive integers. If $q = 1$, then we immediately get $TD_\gamma(U_{p,1}) \cong TD_\gamma(P_{p+1}) \cong TD_\gamma(C_{p,1})$ by Theorems 2 - 5. For $q = 2$, we determine $TD_\gamma(U_{p,q})$ and $TD_\gamma(C_{p,q})$ in Theorem 12 (below) by the following discussions.

If $p = 4k + 2$ for some $k \geq 0$, then we can verify that $\{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k - 1\} \cup \{v_{4k+2}, u_1\}$ is the only $\gamma_t(U_{p,q})$-set and the only $\gamma_t(C_{p,q})$-set, so $TD_\gamma(U_{4k+2,q}) \cong P_1 \cong TD_\gamma(C_{4k+2,q})$. Theorem 6 shows that $\gamma_t(U_{p,q}) = \gamma_t(L_{p,q}) = \gamma_t(C_{p,q})$. For $p = 4k + 1$, the similar proof of Lemma 5 provide that u_1 is in every $\gamma_t(U_{4k+1,q})$-set. Observation 1 also give that u_1 is in every $\gamma_t(C_{4k+1,q})$-set. Then we follow the steps in the proof of Theorem 8, so $TD_\gamma(U_{4k+1,q}) \cong L_{k,q} \cong TD_\gamma(C_{4k+1,q})$.

If $q \in \{2, 3\}$, then $U_{4k,q} \cong L_{4k,q}$, so by Theorem 10, $TD_\gamma(U_{4k,q}) \cong JL^{k+1}_{k-1,q}$. We observe that every $\gamma_t(U_{4k,q})$-set is a $\gamma_t(L_{4k,q})$-set, but the converse is not necessarily true. From the proof of Theorem 10, we know that $D^{i,j} = \{v_{4l+2}, v_{4l+3} : 0 \leq l \leq k-1\} \cup \{u_i, u_j\}$ is a $\gamma_t(L_{4k,q})$-set for $2 \leq i < j \leq q$. If $q = 4$, then $D^{2,4}$ is the only $\gamma_t(L_{4k,4})$-set that is not a $\gamma_t(U_{4k,4})$-set, and thus $TD_\gamma(U_{4k,4}) \cong TD_\gamma(L_{4k,4}) - \{D^{2,4}\}$. Similarly, for $q = 5$, $TD_\gamma(U_{4k,5}) \cong TD_\gamma(L_{4k,5}) - \{D^{2,3}, D^{2,4}, D^{2,5}, D^{3,5}, D^{4,5}\}$. Note that u_1 is in every $\gamma_t(U_{4k,q})$-set for all $q \geq 6$ and in every $\gamma_t(C_{4k,q})$-set for all $q \geq 2$, so $TD_\gamma(U_{4k,q}) \cong L^{k+1}_{k-1,q}$ for all $q \geq 6$, and $TD_\gamma(C_{4k,q}) \cong L^{k+1}_{k-1,q}$ for all $q \geq 2$ by following the first two paragraphs in the proof of Theorem 10.

Similar to Lemma 6, we can easily prove that each $\gamma_t(U_{4k-1,q})$-set (and $\gamma_t(C_{4k-1,q})$-set) does not contain u_i for all $i \in \{2, 3, \ldots, q\}$. Then we follow the steps in the proof of Theorem 11, so $TD_\gamma(U_{4k-1,q}) \cong P_k \cong TD_\gamma(C_{4k-1,q})$.

Theorem 12. *Let p and q be positive integers. Then*

$$TD_\gamma(U_{p,q}) \cong \begin{cases} P_1 & \text{if } p = 4k+2, q \geq 2; \\ L_{k,q} & \text{if } p = 4k+1, q \geq 2; \\ L^{k+1}_{k-1,q} & \text{if } p = 4k, q \geq 6; \\ P_k & \text{if } p = 4k-1, q \geq 2; \end{cases}$$

and

$$TD_\gamma(C_{p,q}) \cong \begin{cases} P_1 & \text{if } p = 4k+2, q \geq 2; \\ L_{k,q} & \text{if } p = 4k+1, q \geq 2; \\ L^{k+1}_{k-1,q} & \text{if } p = 4k, q \geq 2; \\ P_k & \text{if } p = 4k-1, q \geq 2. \end{cases}$$

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