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Research Article

Maximal outerplanar graphs with semipaired domination number double the domination number

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Abstract: A subset S of vertices in a graph G is a dominating set if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. If the graph G has no isolated vertex, then a paired dominating set S of G is a dominating set of G such that $G[S]$ has a perfect matching. Further, a semipaired dominating set of G is a dominating set of G with the additional property that the set S can be partitioned into two element subsets such that the vertices in each subset are at most distance two apart. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. Similarly, the paired (semipaired) domination number $\gamma_{\rm pr}(G)$ ($\gamma_{\rm pr2}(G)$) is the minimum cardinality of a paired (semipaired) dominating set of G. It is known that for a graph G, $\gamma(G) \le \gamma_{\rm pr2}(G) \le \gamma_{\rm pr}(G) \le 2\gamma(G)$. In this paper, we characterize maximal outerplanar graphs G satisfying $\gamma_{\rm pr2}(G) = 2\gamma(G)$. Hence, our result yields the characterization of maximal outerplanar graphs G satisfying $\gamma_{\rm pr}(G) = 2\gamma(G)$.

Keywords: paired-domination, semipaired domination number, maximal outerplanar graphs

AMS Subject classification: 05C69

1. Introduction

A *dominating set* of a graph G is a set S of vertices of G such that every vertex not in S has a neighbor in S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G. A γ-*set of* G is a dominating set of G of

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minimum cardinality $\gamma(G)$. For recent books on domination in graphs, we refer the reader to [4–6].

An *isolate-free graph* is a graph that contains no isolated vertex. Paired domination was introduced in [13, 14] as a model for security applications involving backups for police officers. To model a backup, each vertex in the paired dominating set must be partnered with an adjacent vertex in the set. Formally, a *paired dominating set*, abbreviated PD-set, of an isolate-free graph G is a dominating set S of G with the additional property that the subgraph induced by S, denoted $G[S]$, contains a perfect matching. The *paired domination number*, denoted by $\gamma_{\mathrm{pr}}(G)$, of G is the minimum cardinality of a PD-set of G. A γ_{pr}-*set of* G is a PD-set of G of minimum cardinality $\gamma_{\mathrm{pr}}(G)$. For a state of the art on paired domination in graphs we refer the reader to the survey paper [3] and the book chapter [2].

A relaxed version of paired domination, called semipaired domination, was introduced by Haynes and Henning [8] and studied, for example, in [7, 9–12, 15, 16] and elsewhere. Following the notation introduced in [8], a set S of vertices in an isolate-free graph G is a *semipaired dominating set*, abbreviated semi-PD-set, of G if S is a dominating set of G and every vertex in S is paired with exactly one other vertex in S that is within at most distance 2 from it. Thus, the vertices in the dominating set S can be partitioned into 2-sets such that if $\{u, v\}$ is a 2-set, then $uv \in E(G)$ or the distance between u and v is 2. As defined in [8], we say that u and v are S-*paired* (or simply *paired* if the set S is clear from the context), and that u and v are S-*partners* (or simply *partners*), and we call such a pairing of the vertices of S a *semi-matching* in G. The *semipaired domination number*, denoted by $\gamma_{\mathrm{pr2}}(G)$, is the minimum cardinality of a semi-PD-set of G. A semi-PD-set of G of cardinality $\gamma_{\mathrm{pr2}}(G)$ is a γ_{pr2}-*set of* G.

Every semi-PD-set is a dominating set and every PD-set is a semi-PD-set. Hence, we have the following observation, where it is observed in [14] that $\gamma_{\mathrm{pr}}(G) \le 2\gamma(G)$ for every isolate-free graph G.

Observation 1. ([8]) If G is an isolate-free graph, then $\gamma(G) \le \gamma_{\mathrm{pr2}}(G) \le \gamma_{\mathrm{pr}}(G) \le 2\gamma(G)$.

A *triangulated disc* is a (simple) planar graph all of whose inner faces are triangles. A *maximal outerplanar graph*, abbreviated *mop* in the literature, is a triangulated disc where the outer face contains all vertices. Thus, a maximal outerplanar graph can be embedded in the plane in such a way that all vertices are on the boundary of its outer face (the unbounded face) and all interior faces are triangles. We note that the addition of a single edge in a maximal outerplanar graph results in a graph that is not outerplanar.

In this paper, we characterize maximal outerplanar graphs G satisfying $\gamma_{\mathrm{pr2}}(G) = 2\gamma(G)$. This yields the characterization of maximal outerplanar graphs G satisfying $\gamma_{\mathrm{pr2}}(G) = \gamma_{\mathrm{pr}}(G) = 2\gamma(G)$.

1.1. Notation

For notation and graph theory terminology, we in general follow [6]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. Two vertices in G are *neighbors* if they are adjacent. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. We denote the *degree* of v in G by $\deg_G(v) = |N_G(v)|$. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \cup_{v \in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. Moreover, the subgraph induced by S is denoted by $G[S]$. The graph $G - S$ is obtained from G by deleting all vertices in S (and all edges of G incident with vertices of S). If $S = \{v\}$, we denote $G - S$ simply by $G - v$ rather than $G - \{v\}$.
For a set $S \subseteq V(G)$ and a vertex $v \in S$, the *S-private neighborhood* $\mathrm{pn}[v, S]$ of v is the set of vertices that are in the closed neighborhood of v but not in the closed neighborhood of the set $S \setminus \{v\}$, that is, $\mathrm{pn}[v, S] = \{w \in V(G) : N_G[w] \cap S = \{v\}\}$. If $\mathrm{pn}[v, S] \neq \emptyset$, then a vertex in $\mathrm{pn}[v, S]$ is called an *S-private neighbor of v*. A set $B \subseteq V(G)$ is a *packing* of G if $N_G[u] \cap N_G[v] = \emptyset$ for any pair of distinct $u, v \in B$.
We denote a *path*, a *cycle*, and a *complete graph* on n vertices by P_n, C_n, and K_n, respectively. A complete graph K_3 we call a *triangle*. A *fan* of order $n \geq 5$, denoted F_n, is the graph obtained from a path P_{n-1} by adding a new vertex v and joining it to all vertices of the path. We say that the fan F_n is *centered at v*.
For an integer $k \geq 1$, we use the standard notation $[k] = \{1, \ldots, k\}$ and $[k]_0 = [k] \cup \{0\} = \{0, 1, \ldots, k\}$.

2. Main result

In this section, we characterize all mops G with $\gamma_{\mathrm{pr2}}(G) = 2\gamma(G)$. We shall prove the following result, where the class $\mathcal{G}(\mathcal{F}, \mathcal{H})$ is defined in Section 4.

Theorem 2. *If G is a mop, then $\gamma_{\mathrm{pr2}}(G) \leq 2\gamma(G)$, with equality if and only if $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$.*

We proceed as follows. In Section 3, we present some preliminary results. In Section 4 we define the class $\mathcal{G}(\mathcal{F}, \mathcal{H})$ and prove that every graph G in the class $\mathcal{G}(\mathcal{F}, \mathcal{H})$ is a mop satisfying $\gamma_{\mathrm{pr2}}(G) = 2\gamma(G)$. Finally, in Section 4 we prove that if G is a mop satisfying $\gamma_{\mathrm{pr2}}(G) = 2\gamma(G)$, then $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$. The result of Theorem 2 follows immediately from Observation 1, and the results presented in Section 4 and Section 5.

3. Preliminary Results

In this section, we present some preliminary results on mops.

Observation 3. If v is a vertex in a mop G, then the induced subgraph $G[N_G[v]]$ is a fan centered at v, and so $G[N_G(v)]$ is a path.

We shall need the following property of mops observed by O'Rourke [18] (see Lemma 7.2, p. 169) and others.

Observation 4. ([18]) Every mop has a unique Hamiltonian cycle.

Following the notation of [17], for simplicity we refer to an edge that belongs to the Hamiltonian cycle of a mop as a *Hamiltonian edge*, and to every other edge of the mop as a *diagonal*. We shall also need the following property of mops due to Allgeier [1].

Lemma 1. ([1]) *If H is a 2-connected induced subgraph of a mop G, then H is a mop.*

4. The Class $\mathcal{G}(\mathcal{F}, \mathcal{H})$

Let $\ell \ge 2$ be an integer. Let F_i be a fan of order at least 5 and let x_i be the center of F_i for $i \in [\ell]$. Let $\mathcal{F}' = \{F_1, F_2, \ldots, F_\ell\}$. We say that the fans in $\mathcal{F}'$ are *linked* in a graph G if, for each $i \in [\ell]$, there exist exactly two consecutive vertices y_i and z_i on the path $F_i - x_i$ such that $\{y_i, z_i : i \in [\ell]\}$ induces a mop in G, which we call a *linked mop* associated with $\mathcal{F}'$. We say also that F_i is *linked* in $\mathcal{F}'$ via the edge $y_i z_i$. Moreover, we call $y_i z_i$ the *linked edge* of F_i in $\mathcal{F}'$. For example, Figure 1 illustrates a

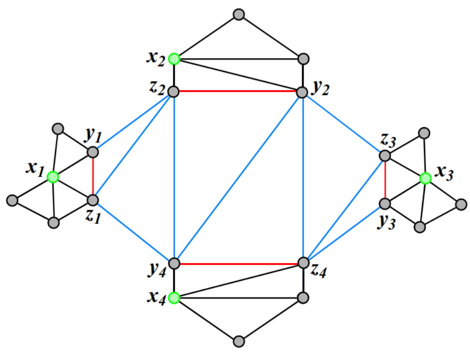

Figure 1. **Linked fans**

family $\mathcal{F}'$ with four linked fans F_1, F_2, F_3, F_4, where the linked edges of G are colored red and the center of each fan is colored green. Further the edges, different from the linked (red) edges, in the mop of G induced by the set $\{y_i, z_i : i \in [4]\}$ are colored blue in Figure 1. Thus the red and blue edges in Figure 1 form the linked mop of G associated with $\mathcal{F}'$.

Let $\mathcal{F}_1, \ldots, \mathcal{F}_d$ be $d \geq 1$ families of linked fans in a mop G, and let F be a common fan that belongs to each family $\mathcal{F}_j$ and is linked to the fans in $\mathcal{F}_j$ by the linked edge $a_j b_j \in E(F)$ for all $j \in [d]$ (see Figure 2).

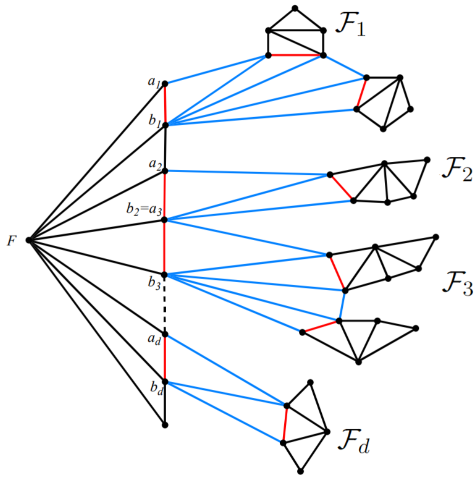

Figure 2. **A common fan F in $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots, \mathcal{F}_d$**

Further, let $\mathcal{F}_r \cap \mathcal{F}_s = \{F\}$ for all r and s where $1 \leq r < s \leq d$. Hence, all the fans from all the families $\mathcal{F}_1, \ldots, \mathcal{F}_d$ are vertex disjoint, except for the fan F which belongs to each family $\mathcal{F}_j$ for all $j \in [d]$. We define a *linked set* J_F of the fan F as follows:

(a) If $a_j b_j$ is the edge of F that is linked to the family $\mathcal{F}_j$ and $|\mathcal{F}_j| \geq 3$, then we add both a_j and b_j to J_F for all $j \in [d]$.

(b) If $a_j b_j$ is the edge of F that is linked to the family $\mathcal{F}_j$ and $|\mathcal{F}_j| \leq 2$, then we add only one of a_i or b_i to J_F.

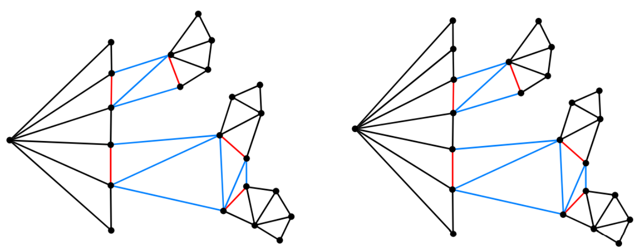

Figure 3. **A fan without property $\mathcal{J}$ (left) and a fan with property $\mathcal{J}$ (right)**

We say that the fan F has property $\mathcal{J}$ if none of the linked sets J_F of F is a dominating set of F (see Figure 3).

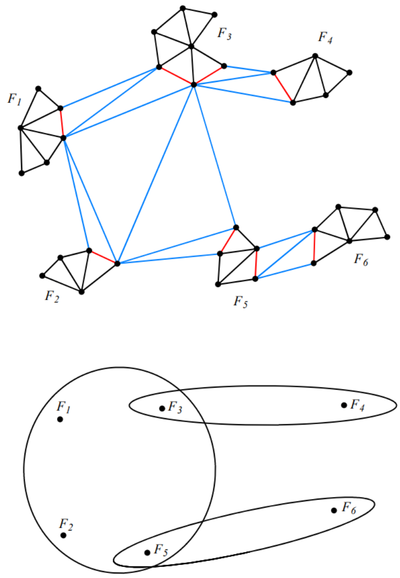

Figure 4. **The hypergraph $\mathcal{H}$ corresponding to the fans $F_1, \dots, F_6$**

Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a family of vertex disjoint fans satisfying $|V(F_i)| \geq 5$ for all $i \in [k]$. Let $\mathcal{H}$ be a hypergraph having vertex set $V(\mathcal{H}) = \mathcal{F}$ and edge set $E(\mathcal{H}) = \{e_1, e_2, \dots, e_m\}$ satisfying the following properties:

(a) $\bigcup_{i=1}^{m} e_i = \mathcal{F}$;

(b) $|e_i \cap e_j| \leq 1$ for $1 \leq i < j \leq m$;

(c) The hypergraph $\mathcal{H}$ contains no hypercycle, that is, $\mathcal{H}$ is a linear hypertree where the vertices of $\mathcal{H}$ correspond to the fans F_i of $\mathcal{F}$ for $i \in [k]$ (see Figure 4).

We call $\mathcal{H}$ the linear hypertree associated with the family $\mathcal{F} = \{F_1, \dots, F_k\}$ of fans. Let G be the graph obtained from the vertex disjoint union of the fans $F_1, \dots, F_k$ as follows. For each edge $e_i = \{F_{i_1}, \dots, F_{i_j}\}$, we add edges joining the fans in the family $\mathcal{F}_i = \{F_{i_1}, \dots, F_{i_j}\}$ in such a way that the fans in $\mathcal{F}_i$ are linked and resulting linked mop consists of these added edges and the linked edges from each fan in the family $\mathcal{F}_i$, for all $i \in [m]$. Further, we construct the graph G in such a way that every fan has property $\mathcal{J}$. Moreover, for each $F_i \in \mathcal{F}$, if F_i is linked in the families $\mathcal{F}_j$ and $\mathcal{F}_{j'}$ via the linked edges $y_j z_j$ and $y_{j'} z_{j'}$, respectively, then $y_j z_j \neq y_{j'} z_{j'}$. Let $\mathcal{G}(\mathcal{F}, \mathcal{H})$ be the class of all such graphs G constructed in this manner from the family of fans $\mathcal{F} = \{F_1, \dots, F_k\}$ and the linear hypertree $\mathcal{H}$ associated with the family $\mathcal{F}$.
We will prove that every graph G in the class $\mathcal{G}(\mathcal{F}, \mathcal{H})$ is a mop satisfying $\gamma_{\mathrm{pr2}}(G) = \gamma_{\mathrm{pr}}(G) = 2\gamma(G)$.

Theorem 5. *If $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$, then G is a mop with $\gamma_{\mathrm{pr2}}(G) = 2\gamma(G)$.*

Proof. Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a family of fans satisfying $|V(F_i)| \geq 5$ for all $i \in [k]$, and let $\mathcal{H}$ be the linear hypertree associated with the family $\mathcal{F}$. Let x_i be the center of the fan F_i for $i \in [k]$. Let $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$. We proceed further by proving two claims.

Claim 1. *The graph G satisfies $\gamma_{\rm pr2}(G) = 2\gamma(G)$.*

Proof. The center of the fans dominate the graph G, that is, $\{x_1, x_2, \ldots, x_k\}$ is a dominating set of G. Thus, $\gamma(G) \leq k$, implying by Observation 1 that $\gamma_{\rm pr2}(G) \leq 2\gamma(G) \leq 2k$. Hence it suffices for us to show that $\gamma_{\rm pr2}(G) \geq 2k$. Among all $\gamma_{\rm pr2}$-sets of G, let S be chosen so that $|S \cap \{x_1, x_2, \ldots, x_k\}|$ is maximum.

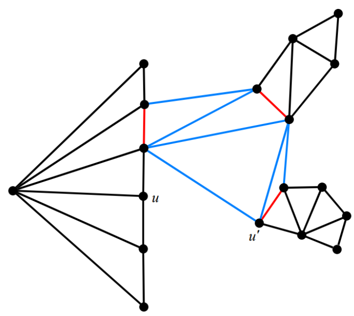

Figure 5. **The vertices u and u'**

We show that $\{x_1, x_2, \ldots, x_k\} \subseteq S$. Suppose, to the contrary, that $x_i \notin S$ for some $i \in [k]$. If $S \cap V(F_i) \subseteq J_{F_i}$, then, by property $\mathcal{J}$, the set S does not dominate F_i, contradicting the fact that S is a semi-PD-set of G. Hence there exists a vertex $u \in S \cap V(F_i)$ such that $u \notin J_{F_i}$ (see Figure 5).
Let u' be the vertex that is S-paired with u. If $u' \in V(F_i)$, then $S' = (S \setminus \{u\}) \cup \{x_i\}$ is a semi-PD-set of G with u' S'-paired with x_i, and with the S'-pairings of all other pairs of vertices in S' unchanged from their S-pairings. We note that $|S'| = |S|$, and so S' is a $\gamma_{\rm pr2}$-set of G. However, $|S' \cap \{x_1, x_2, \ldots, x_k\}| > |S \cap \{x_1, x_2, \ldots, x_k\}|$, contradicting our choice of the set S. Hence, $u' \notin V(F_i)$. Since $u \notin J_{F_i}$ and u' and u are S'-paired, it follows that $d_G(u', x_i) = 2$. As before, $S' = (S \setminus \{u\}) \cup \{x_i\}$ is a semi-PD-set of G with u' S'-paired with x_i, and with the S'-pairings of all other pairs of vertices in S' unchanged from their S-pairings. Thus, S' is a $\gamma_{\rm pr2}$-sets of G and $|S' \cap \{x_1, x_2, \ldots, x_k\}| > |S \cap \{x_1, x_2, \ldots, x_k\}|$, a contradiction.
Hence, $\{x_1, x_2, \ldots, x_k\} \subseteq S$. Let x'_i be the S-partner of x_i for $i \in [k]$, and so $d_G(x_i, x'_i) \leq 2$. The central vertices of the fans are pairwise at distance at least 3 apart, that is, $d_G(x_i, x_j) \geq 3$ for all i and j where $1 \leq i < j \leq k$. This implies that $x'_i \neq x'_j$ for $1 \leq i < j \leq k$. Therefore, $\{x'_1, x'_2, \ldots, x'_k\} \subseteq S$ and $\{x_1, x_2, \ldots, x_k\} \cap \{x'_1, x'_2, \ldots, x'_k\} = \emptyset$, implying that $|S| \geq 2k$. □

Claim 2. *The graph G is a mop.*

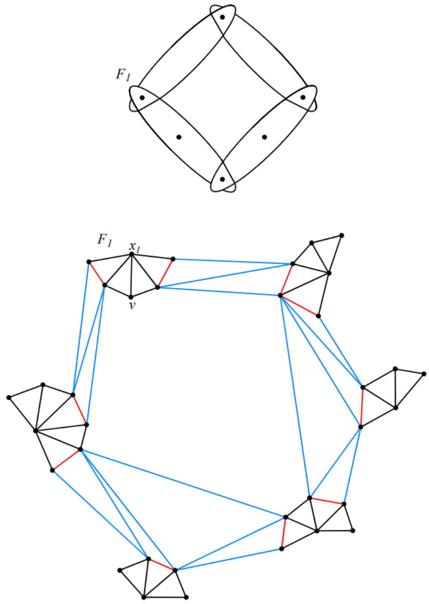

Figure 6. **The corresponding hypergraph which is a hypercycle**

Proof. Suppose, to the contrary, that G is not a mop. By construction, G is a planar graph and all interior faces of G are triangles. Thus, G has a vertex v which does not lie on the outer face. Renaming vertices if necessary, we may assume that $v \in V(F_1) \setminus \{x_1\}$. Since v does not lie on the outer face, it follows that v is enclosed by maximal outerplanar subgraphs corresponding to the edges $e_1, e_2, \dots, e_d$ of $\mathcal{H}$ where $F_1 \in e_1 \cap e_d$ and $e_i \cap e_{i+1} \neq \emptyset$ for $i \in [d-1]$. If $d \geq 3$, then $\mathcal{H}$ has a hypercycle (see Figure 6). Thus, $d = 2$. In this case, v still lies on the outer face if $e_1 \cap e_2 = \{F_1\}$. Thus, $|e_1 \cap e_2| \geq 2$, contradicting the fact that $\mathcal{H}$ is linear. □

The proof of Theorem 5 now follows from Claims 1 and 2. □

5. Mops G satisfying $\gamma_{\rm pr2}(G) = \gamma(G)$

In this section, we will show that if G is a mop satisfying $\gamma_{\rm pr2}(G) = 2\gamma(G)$, then $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$.

Theorem 6. *If G is a mop satisfying $\gamma_{\rm pr2}(G) = 2\gamma(G)$, then $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$.*

Proof. Let G be a mop satisfying $\gamma_{\rm pr2}(G) = 2\gamma(G) = 2k$. If $k = 1$, then the graph G contains a dominating vertex x (that is adjacent to every other vertex in G, and $G = G[N_G[x]]$ is a fan centered at x. In this case, trivially $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$. Hence we may assume that $k \geq 2$. Let $S = \{x_1, x_2, \dots, x_k\}$ be a γ-set of G. Since $\gamma_{\rm pr2}(G) = 2k$, it follows that $d_G(x_i, x_j) \geq 3$ for all $1 \leq i < j \leq k$. Thus, $N_G[x_i] \cap N_G[x_j] = \emptyset$. Clearly, $G[N_G[x_i]]$ is a fan centered at x_i. Let $F_i = G[N_G[x_i]]$ for $i \in k$ and let $\mathcal{F} = \{F_i \colon i \in [k]\}$. Further, let $V(F_i) = \{x_i, x_i^1, x_i^2, \ldots, x_i^{\ell_i}\}$ where the vertices are labelled clockwise (see Figure 7). Moreover, we let $P(F_i) = x_i^1 x_i^2 \ldots x_i^{\ell_i}$ be the path

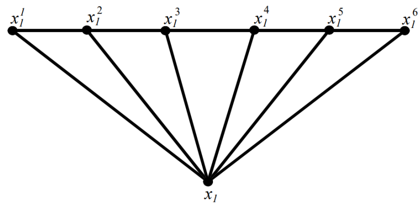

Figure 7. **The clockwise labelling**

$F_i - x_i$. Since S is a dominating set and $N_G[x_i] \cap N_G[x_j] = \emptyset$, it follows that S is a packing of G. Moreover since G is connected, there are edges of G that join vertices in one fan F_i to a different fan F_j for some i and j where $1 \leq i < j \leq k$. We proceed further by proving a series of claims establishing properties of the graph G.

Claim 3. $\deg_G(x_i) = \deg_{F_i}(x_i)$ *for all* $i \in [k]$.

Proof. Suppose that $\deg_G(x_i) > \deg_{F_i}(x_i)$ for all $i \in [k]$. Thus, the vertex x_i is adjacent to a vertex of F_j for some j with $i \neq j$ and $j \in [k]$, implying that $d_G(x_i, x_j) \leq 2$, contradicting our earlier observation that the set S is a packing in G. □

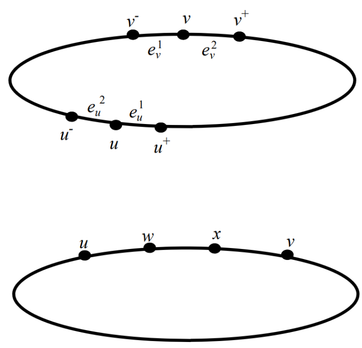

Figure 8. **The edges $e_v^1 = v^- v$ and $e_v^2 = vv^+$ (top) and the path $u\overline{C}v = uwxv$ (bottom)**

By Claim 3, the vertex x_i is not adjacent to any vertex of F_j for all $i, j \in [k]$ where $i \neq j$. Before we prove the following claims, we introduce some additional terminology and notation. Let C be the unique Hamiltonian cycle of G. We write $\overrightarrow{C}$ to indicate the clockwise orientation on C. Moreover, the *successor* of a vertex v on $\overrightarrow{C}$ we denote by v^+ and the *predecessor* we denote by v^-. Observe that every vertex v is incident to exactly two Hamiltonian edges v^-v and vv^+. For notational convenience, we denote $e_v^1 = v^-v$ and $e_v^2 = vv^+$. For two vertices u and v on C, we use $u\overrightarrow{C}v$ to indicate the (u, v)-path that follows the orientation on $\overrightarrow{C}$ and all edges of the path are Hamiltonian edges (see Figure 8). A cycle in a mop is *alternating* if its edges alternate between Hamiltonian edges and diagonal.

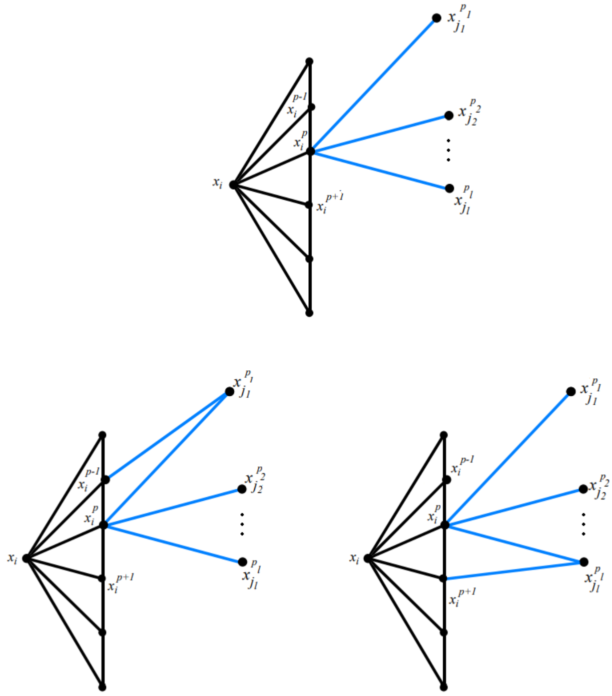

Figure 9. **The graph when $x_i^{p-1}x_{j_1}^{p_1} \in E(G)$ or $x_i^{p+1}x_{j_\ell}^{p_\ell} \in E(G)$**

Claim 4. *Let x_i^p be a vertex of F_i which is adjacent to vertices $x_{j_1}^{p_1}, x_{j_2}^{p_2}, \ldots, x_{j_\ell}^{p_\ell} \in V(G) \setminus V(F_i)$ where these vertices occur in the clockwise orientation on $\overrightarrow{C}$. (It is possible that $j_s = j_{s+1}$ for some s, implying that in this case the vertices $x_{j_s}^{p_s}$ and $x_{j_{s+1}}^{p_s+1}$ are in the same fan). Then, $x_i^{p-1}x_{j_1}^{p_1} \in E(G)$ or $x_i^{p+1}x_{j_\ell}^{p_\ell} \in E(G)$ (see Figure 9).*

Proof. Suppose $x_i^{p-1}x_{j_1}^{p_1} \notin E(G)$ and $x_i^{p+1}x_{j_\ell}^{p_\ell} \notin E(G)$. Since G is 2-connected, there exists an $(x_i^{p-1}, x_{j_1}^{p_1})$-path or an $(x_{j_\ell}^{p_\ell}, x_i^{p+1})$-path, neither of which contains the vertex x_i^p. Suppose, without loss of generality, that there exists an $(x_i^{p-1}, x_{j_1}^{p_1})$-path P that does not contain the vertex x_i^p (see Figure 10). Clearly, $G' = G[V(P) \cup \{x_i^p\}]$ is a 2-connected subgraph. By Lemma 1, the graph G' is a mop. Because $x_i^{p-1}x_{j_1}^{p_1} \notin E(G)$, it follows that x_i^p is adjacent to a vertex in $V(P) \setminus \{x_i^{p-1}, x_{j_1}^{p_1}\}$, contradicting the fact that $x_{j_1}^{p_1}$ is the first vertex that x_i^p is adjacent to on $C - V(F_i)$. Thus, $x_i^{p-1}x_{j_1}^{p_1} \in E(G)$. The case $x_i^{p+1}x_{j_\ell}^{p_\ell} \in E(G)$ can be proved similarly and this completes the proof. $\square$

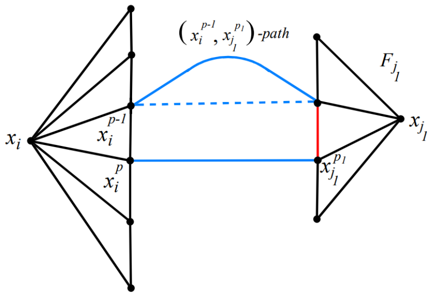

Figure 10. **An $(x_i^{p-1}, x_{j_1}^{p_1})$-path P that does not contain the vertex x_i^p**

By Claim 4, renaming vertices if necessary, we may assume that $x_i^{p-1}x_{j_1}^{p_1} \in E(G)$. With this assumption, we have the following claim.

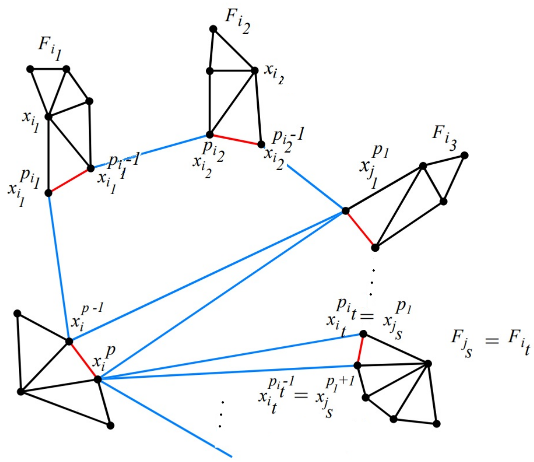

Figure 11. **The mop induced by $\{x_i^p, x_i^{p-1}, x_{i_1}^{p_{i_1}}, x_{i_1}^{p_{i_1}-1}, x_{i_2}^{p_{i_2}}, x_{i_2}^{p_{i_2}-1}, \ldots, x_{i_t}^{p_{i_t}}, x_{i_t}^{p_{i_t}-1}\}$**

Claim 5. *If $x_i^{p-1}x_{j_1}^{p_1} \in E(G)$, then there exist fans $F_{i_1}, F_{i_2}, \ldots, F_{i_t}$ such that $F_{i_t} = F_{j_s}$ for some $s \in [\ell]$ such that, for each fan F_{i_q}, there are exactly two vertices $x_{i_q}^{p_{i_q}}$ and $x_{i_q}^{p_{i_q}-1}$ of F_{i_q} such that $\{x_i^p, x_i^{p-1}, x_{i_1}^{p_{i_1}}, x_{i_1}^{p_{i_1}-1}, x_{i_2}^{p_{i_2}}, x_{i_2}^{p_{i_2}-1}, \ldots, x_{i_t}^{p_{i_t}}, x_{i_t}^{p_{i_t}-1}\}$ induced a mop (see Figure 11).*

Proof. We will find an alternating cycle by the following method. We start the cycle from the path $x_i^p x_i^{p-1}$. From the vertex x_i^{p-1}, we follow the Hamiltonian edge $e^2_{x_i^{p-1}}$.

Since S is a packing in G, the edge $e^2_{x_i^{p-1}}$ is incident to a vertex of some fan in $\mathcal{F}$, F_{i_1} say. Because $d_G(x_i, x_j) \geq 3$ for all $1 \leq i < j \leq k$, the edge $e^2_{x_i^{p-1}}$ is incident to a vertex $x_{i_1}^{p_{i_1}}$ which is not the center of F_{i_1} (see Figure 12). We first construct the

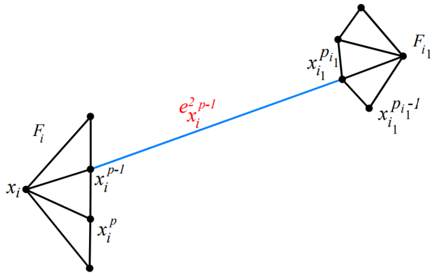

Figure 12. **The edge** $e^2_{x_i^{p-1}}$

alternating path

$$x_i^p x_i^{p-1} x_{i_1}^{p_{i_1}} x_{i_1}^{p_{i_1}-1}.$$

From $x_{i_1}^{p_{i_1}-1}$, again, we follow the Hamiltonian edge $e^2_{x_{i_1}^{p_{i_1}-1}}$. If $x_{i_1}^{p_{i_1}-1}$ is not adjacent to any vertex in $V(G) \setminus V(F_{i_1})$, then, by the orientation,

$$e^2_{x_{i_1}^{p_{i_1}-1}} = x_{i_1}^{p_{i_1}-1} x_{i_1}^{p_{i_1}}$$

but $x_{i_1}^{p_{i_1}}$ has occurred once on the edge $e^2_{x_i^{p-1}}$. This contradicts the fact that C is a Hamiltonian cycle. Thus, $e^2_{x_{i_1}^{p_{i_1}-1}}$ is incident to a vertex $x_{i_2}^{p_{i_2}}$ which is not the center of a fan $F_{i_2} \in \mathcal{F} \setminus \{F_{i_1}\}$. We now add $x_{i_2}^{p_{i_2}}$ and $x_{i_2}^{p_{i_2}-1}$ to the path (see Figure 13), yielding the alternating path

$$x_i^p x_i^{p-1} x_{i_1}^{p_{i_1}} x_{i_1}^{p_{i_1}-1} x_{i_2}^{p_{i_2}} x_{i_2}^{p_{i_2}-1}.$$

Similarly, the vertex $x_{i_2}^{p_{i_2}-1}$ is adjacent to a vertex $x_{i_3}^{p_{i_3}} \in V(F_{i_3})$ via the Hamiltonian edge $e^2_{x_{i_2}^{p_{i_2}-1}}$ where $x_{i_3}^{p_{i_3}}$ is not the center of F_{i_3} and we add $x_{i_3}^{p_{i_3}}$ and $x_{i_3}^{p_{i_3}-1}$ to the path. We keep applying this method until we meet the fan F_i again. Because C is the (unique) Hamiltonian cycle in the mop G, by this choice, the path will return to F_i. If the path contains a Hamiltonian edge that joins the vertex $x_{i_t}^{p_{i_t}-1}$ to the vertex in F_i which is not x_i^p, then x_i^p does not lie on the outer face, contradicting the fact that G is an outerplanar graph. Thus, $x_{i_t}^{p_{i_t}-1}$ is adjacent to x_i^p via $e^2_{x_{i_t}^{p_{i_t}-1}}$. Thus,

$$C' : x_i^p x_i^{p-1} x_{i_1}^{p_{i_1}} x_{i_1}^{p_{i_1}-1} x_{i_2}^{p_{i_2}} x_{i_2}^{p_{i_2}-1} \ldots x_{i_t}^{p_{i_t}} x_{i_t}^{p_{i_t}-1} x_i^p$$

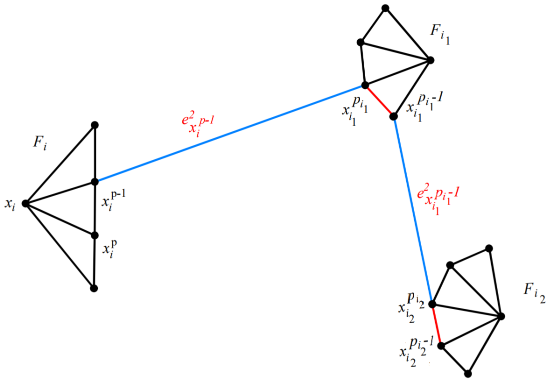

Figure 13. **The alternating path** $x_i^p x_i^{p-1} x_{i_1}^{p_{i_1}} x_{i_1}^{p_{i_1}-1} x_{i_2}^{p_{i_2}} x_{i_2}^{p_{i_2}-1}$

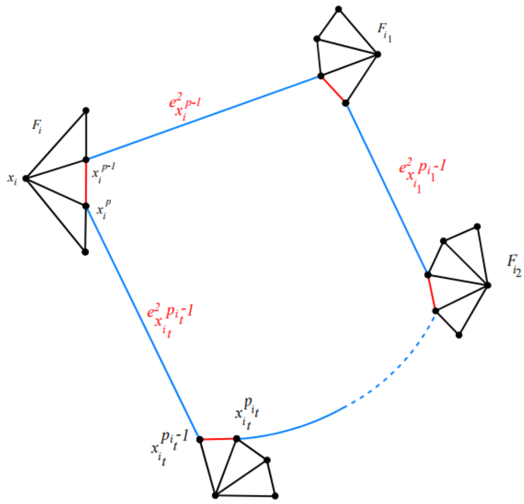

Figure 14. **The alternating cycle** $C' \colon x_i^p x_i^{p-1} x_{i_1}^{p_{i_1}} x_{i_1}^{p_{i_1}-1} x_{i_2}^{p_{i_2}} x_{i_2}^{p_{i_2}-1} \dots x_{i_t}^{p_{i_t}} x_{i_t}^{p_{i_t}-1} x_i^p$

is an alternating cycle as required (see Figure 14). Since $G[V(C')]$ induced a 2-connected subgraph of G, by Lemma 1 the graph $G[V(C')]$ is a mop. This completes the proof of Claim 5. □

We now use Claim 5 to construct a hypergraph. For $F_{i_1}, F_{i_2}, \dots, F_{i_\ell}$, if there are two vertices $x_{i_j}^{p_j}, x_{i_j}^{p_j-1} \in V(F_{i_j})$ for $j \in [\ell]$ such that $G[\{x_{i_j}^{p_j}, x_{i_j}^{p_j-1} \colon j \in [\ell]\}]$ is a mop, then we say that $F_{i_1}, F_{i_2}, \dots, F_{i_\ell}$ are *linked*. In the following, we construct the corresponding hypergraph H_G of a mop G satisfying $2\gamma(G) = \gamma_{\rm pr2}(G)$. Let $V(H_G) = \{F_i \colon i \in [k]\}$ and $E(H_G) = \{e = \{F_{i_1}, F_{i_2}, \dots, F_{i_r}\} \colon F_{i_1}, F_{i_2}, \dots, F_{i_r}$ are linked in $G\}$. We call H_G the *linked hypergraph* of G. Let B be a mop and let C_B

denote the (unique) Hamiltonian cycle in B, and let $a, b \in V(B)$. For notational convenience, we let aBb denote the path $a\overrightarrow{C_B}b$.

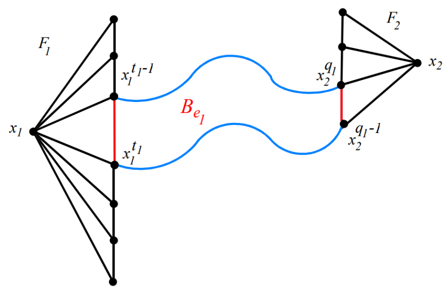

Figure 15. **The mop B_{e_1}**

Claim 6. *If G is a mop with H_G as the linked hypergraph, then H_G is a linear hypertree.*

Proof. We first show that H_G is linear. Suppose to the contrary that there exists two hyperedges $e_1, e_2 \in E(H_G)$ such that $|e_1 \cap e_2| \geq 2$. Renaming vertices if necessary, we may assume that $\{F_1, F_2\} \subseteq e_1 \cap e_2$. Thus, F_1 and F_2 are linked twice by two different maximal outerplanar subgraphs. So, there exist $x_1^{t_1}, x_1^{t_1-1} \in V(F_1)$ and $x_2^{q_1}, x_2^{q_1-1} \in V(F_2)$ such that, for some $W_1 \subseteq V(G) \setminus (V(F_1) \cup V(F_2))$, the graph $G[\{x_1^{t_1}, x_1^{t_1-1}, x_2^{q_1}, x_2^{q_1-1}\} \cup W_1]$ is a mop corresponding to the hyperedge e_1 and we call this mop B_{e_1} (see Figure 15).
Similarly, there exist $x_1^{t_2}, x_1^{t_2-1} \in V(F_1)$ and $x_2^{q_2}, x_2^{q_2-1} \in V(F_2)$ such that the induce subgraph $G[\{x_1^{t_2}, x_1^{t_2-1}, x_2^{q_2}, x_2^{q_2-1}\} \cup W_2]$ for some $W_2 \subseteq V(G) \setminus (V(F_1) \cup V(F_2))$ is a mop corresponding to the edge e_2 and we call this mop B_{e_2}. Without loss of generality, let $t_1 < t_2$.
First, we may assume that $q_1 < q_2$. Recall that aGb is a path from the vertex a to the vertex b passing Hamiltonian edges of G in clockwise direction. Clearly, x_1 is enclosed by the cycle

$$x_1^{t_2-1}B_{e_2}x_2^{q_2}F_2x_2F_2x_2^{q_1-1}B_{e_1}x_1^{t_1}F_1x_1^{t_2-1} \text{ (see Figure 16)}$$

or x_2 is enclosed by the cycle

$$x_2^{q_2-1}B_{e_2}x_1^{t_2}F_1x_1F_1x_1^{t_1-1}B_{e_1}x_1^{q_1}F_2x_2^{q_2-1} \text{ (see Figure 17).}$$

Therefore, x_1 or x_2 does not lie on the outer face, contradicting the fact that G is an outerplanar graph. Thus, we may assume that $q_2 < q_1$. Clearly, x_1 and x_2 are enclosed by the cycle

$$x_1^{t_2-1}B_{e_2}x_2^{q_2}F_2x_2^{q_1-1}B_{e_1}x_1^{t_1}F_1x_1^{t_2-1} \text{ (see Figure 18)}$$

or $x_1^{t_1}, x_1^{t_2-1}, x_2^{q_2}, x_2^{q_1-1}$ are enclosed by the cycle

$$x_1^{t_1-1}B_{e_1}x_2^{q_1}F_2x_2F_2x_2^{q_2-1}B_{e_2}x_1^{t_2}F_1x_1F_1x_1^{t_1-1} \text{ (see Figure 19).}$$

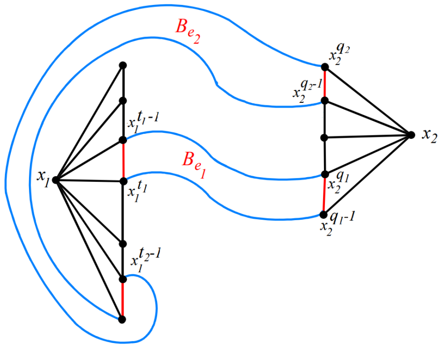

Figure 16. **The cycle that encloses x_1**

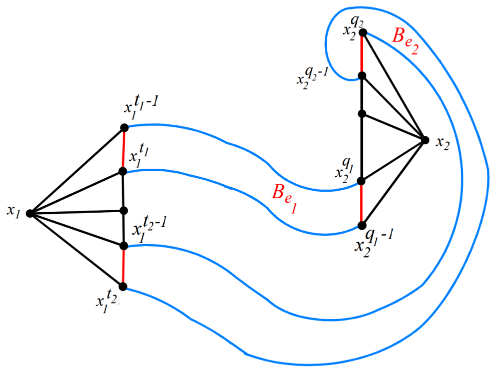

Figure 17. **The cycle that encloses x_2**

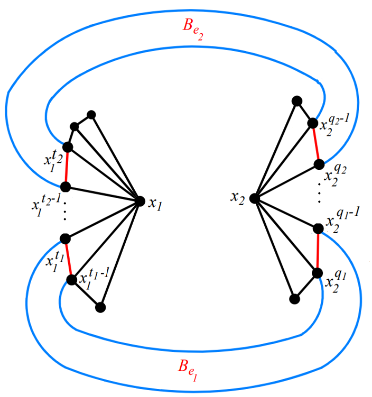

Figure 18. **The cycle that encloses x_1 and x_2**

Therefore, x_1 and x_2 or $x_1^{t_1}, x_1^{t_2-1}, x_2^{q_2}, x_2^{q_1-1}$ do not lie on the outer face, contradicting that fact that G is an outerplanar graph. Hence, H_G is linear. We show next that

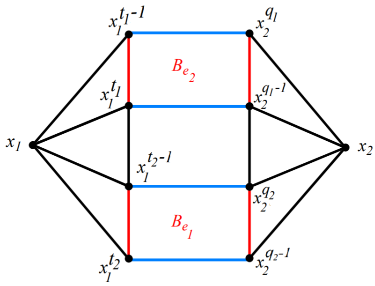

Figure 19. **The cycle that encloses $x_1^{t_1}, x_1^{t_2-1}, x_2^{q_2}, x_2^{q_1-1}$**

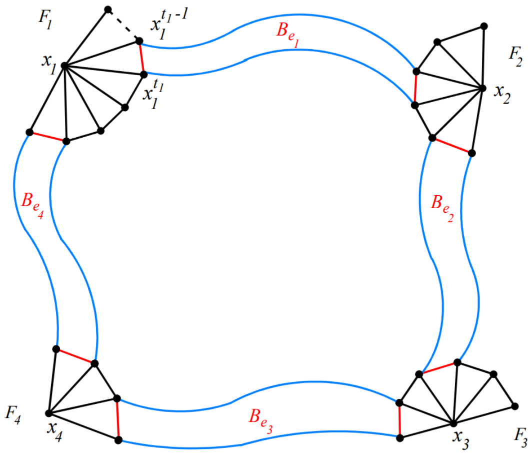

Figure 20. **The cycle that encloses $x_1^{t_1}$**

H_G is a hypertree. Suppose, to the contrary, that H_G is not a hypertree. Thus, there exists a hypercycle C_{H_G} which is a subhypergraph of H_G. We may let $E(C_{H_G}) = \{e_1, e_2, \ldots, e_\ell\}, \{F_{i+1}\} = e_i \cap e_{i+1}$ for all $i \in [\ell - 1]$ and $\{F_1\} = e_\ell \cap e_1$. By the definition of e_i, there exists the maximal outerplanar subgraph B_{e_i} which contains a fan from each of F_i and F_{i+1}, and a maximal outerplanar subgraph B_{e_ℓ} which contains a fan from each of F_ℓ and F_1. Renaming vertices if necessary, we may let $x_i^{t_i-1}, x_i^{t_i} \in V(B_{e_i}) \cap V(F_i)$ and $x_{i+1}^{p_i}, x_{i+1}^{p_i-1} \in V(B_{e_i}) \cap V(F_{i+1})$ and $x_\ell^{t_\ell-1}, x_\ell^{t_\ell} \in V(B_{e_\ell}) \cap V(F_\ell)$ and $x_1^{p_\ell}, x_1^{p_\ell-1} \in V(B_{e_\ell}) \cap V(F_1)$. Since $x_1, x_2, \ldots, x_\ell$ are in the Hamiltonian cycle C, the vertex $x_1^{t_1}$ is enclosed by the cycle

$$x_1 F_1 x_1^{t_1-1} B_{e_1} x_2^{p_1} F_2 x_2 F_2 x_2^{t_2-1} B_{e_2} x_3^{p_2} \ldots x_i^{t_i-1} B_{e_i} x_{i+1}^{p_i} F_{i+1} x_{i+1} F_{i+1} \ldots x_\ell^{t_\ell-1} B_{e_\ell} x_1^{p_\ell} F_1 x_1,$$

(see Figure 20) contradicting that fact that G is an outerplanar graph. This establishes Claim 6. □

Claim 7. *Each F_i has at least five vertices.*

Proof. Suppose, to the contrary, that $|V(F_i)| \leq 4$. Let $F_j \in e \setminus \{F_i\}$ be a fan which is linked with F_i in e. We pair x_j with x_i^2 and each vertex $x_q \in S \setminus \{x_j, x_i\}$ with its own private neighbor x'_q in F_q. Thus, $(S \setminus \{x_i\}) \cup \{x_i^2\} \cup \{x'_q : q \in [k] \setminus \{i\}\}$ is a semi-PD-set of G. Clearly, $\gamma_{\rm pr2}(G) < 2k$, a contradiction. Therefore, $|V(F_i)| \geq 5$. $\square$

The following claim follows readily from the property that G is a mop.

Claim 8. *If F_i is linked in e_j and $e_{j'}$ by the edges $x_i^{t-1}x_i^t$ and $x_i^{p-1}x_i^p$, respectively, then $x_i^{t-1}x_i^t \neq x_i^{p-1}x_i^p$.*

Finally, we need only prove that every fan F_i has property $\mathcal{J}$ defined in Section 4.

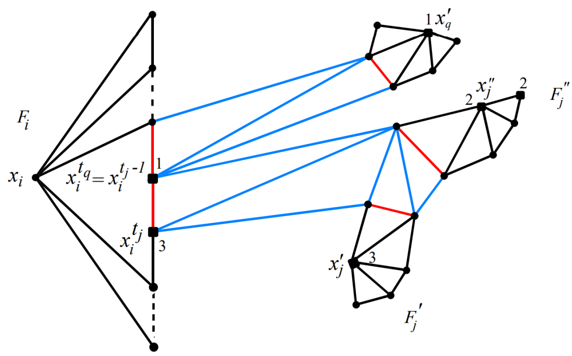

Figure 21. **Pairing when $x_i^{t_j-1} = x_i^{t_q}$ and $x_i^{t_q}$ is already paired with x'_q**

Claim 9. *Every fan F_i has property $\mathcal{J}$.*

Proof. Suppose, to the contrary, that there exists a linked set J_{F_i} such that J_{F_i} dominates F_i. Let $e_1, e_2, \ldots, e_p \in E(H_G)$ be all subfamilies of $\mathcal{F}$ that are linked to F_i. Renaming hyperedges if necessary, we assume that $|e_j| = 2$ for all $j \in [r]$ and $|e_i| \geq 3$ for all $r < j \leq p$. We, further, let $x_i^{t_j-1}x_i^{t_j}$ be the linked edge of F_i in e_j. For the case when $1 \leq j \leq r$, we let $\{F'_j\} = e_j \setminus \{F_i\}$. By the construction on J_{F_i} in Section 4(b), we let $\{v_j\} = \{x_i^{t_j-1}, x_i^{t_j}\} \cap J_{F_i}$. For the case when $r < j \leq p$, we focus on the clockwise orientation on B_{e_j}, and we let F'_j and F''_j be the fans in $e_j \setminus \{F_i\}$ that occur consecutively before and after F_i, respectively. Clearly,

$$J_{F_i} = \{v_j : j \in [r]\} \cup \{x_i^{t_j-1}, x_i^{t_j} : r < j \leq p\}.$$

We let x'_j and x''_j be the centers of F'_j and F''_j, respectively. We pair x'_j with v_j for all $j \in [r]$ and we pair $x_i^{t_j}$ and $x_i^{t_j-1}$ with x'_j and x''_j, respectively. We remark that if $x_i^{t_j-1} = x_i^{t_q}$ for some $q \in [\ell]$ and $x_i^{t_q}$ is already paired with x'_q, then we pair

x''_j with its own private neighbor in F''_j and still pair $x_i^{t_j}$ with x'_j (see Figure 21). We note from the figure that the vertices that are assigned the same number are paired. We let Z be the set of these private neighbors. Observe that, if all vertices in $\{v_j : j \in [r]\} \cup \{x_i^{t_j-1}, x_i^{t_j} : r < j \leq p\}$ are distinct, then $Z = \emptyset$. Moreover, $|J_{F_i} \cup Z| = r + 2(p - r) = 2p - r$. We let $X = \{x'_j : j \in [p]\} \cup \{x''_j : r < j \leq p\}$.

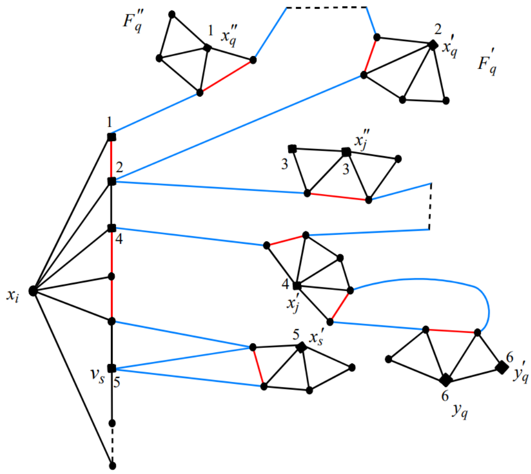

Figure 22. **Pairing y_q with own neighbor y'_q**

Now we have that $|S \setminus (X \cup \{x_i\})| = k - (r + 2(p - r) + 1) = k - 2p + r - 1$. For notational convenience, we rename the vertices in $S \setminus (X \cup \{x_i\})$ to be $y_1, y_2, \ldots, y_\ell$ where $\ell = k - 2p + r - 1$. We pair each vertex $y_q \in S \setminus (X \cup \{x_i\})$ with its own private neighbor y'_q in the same fan (see Figure 22). We let $Y = \{y'_q : q \in [\ell]\}$, and note that $(S \setminus \{x_i\}) \cup Y \cup (J_{F_i} \cup Z)$ is a semi-PD-set of G. Moreover,

$$\gamma_{\rm pr2}(G) \leq |(S \setminus \{x_i\}) \cup Y \cup (J_{F_i} \cup Z)| = (k - 1) + \ell + 2p - r = 2k - 2,$$

contradicting the fact that $\gamma_{\rm pr2}(G) = 2\gamma(G)$. Thus, F_i has the property $\mathcal{J}$ and this completes the proof of Claim 9. □

By the properties of the graph G established in Claims 3–9, we infer that $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$. This completes the proof of Theorem 6. □

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