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Research Article

Triangular type-2 fuzzy goal programming approach for bimatrix games

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Abstract: This paper addresses a bimatrix game in which the satisfactory degrees of the players are vague. Type-2 fuzzy goal programming technique is used to describe the game. Then, the notion of equilibrium points is introduced and an optimization problem is given to calculate them. Moreover, the special case that the type-2 fuzzy goals are triangular is investigated. Finally, an applicable example is presented to illustrate the results.

Keywords: bimatrix games, type-2 fuzzy goals, equilibrium points, triangular type-2 fuzzy numbers.

AMS Subject classification: 90C70, 91A80, 90C29

1. Introduction

In a noncooperative game containing two players: Player I and Player II, each player have a set of pure strategies. Suppose that Player I has the strategy set $X = \{1, 2, \dots, m\}$ and Player II has the strategy set $Y = \{1, 2, \dots, n\}$. The game includes two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ containing the payoffs of Players I and II, respectively. Namely, if Players I and II play respectively their ith and jth strategies, then they obtain respectively a_{ij} and b_{ij} units of profit. In spite of the fact

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that the pure strategy sets are discrete, Players can select a convex combination of these strategies. Such a strategy is called a mixed strategy. Thus the mixed strategy sets associated to Players I and II are respectively as follows:

$$\bar{X} = \{\mathbf{x} = (x_1, \ldots, x_m)^T : x_i \geq 0, i = 1, 2, \ldots, m \wedge \sum_{i=1}^{m} x_i = 1\},$$

$$\bar{Y} = \{\mathbf{y} = (y_1, \ldots, y_n)^T : y_j \geq 0, j = 1, 2, \ldots, n \wedge \sum_{j=1}^{n} y_j = 1\},$$

in which x_i and y_j denote the probability of performing the ith and jth strategies by Players I and II, respectively. When Players use the mixed strategies $\mathbf{x}$ and $\mathbf{y}$, the expected payoffs of Players I and II are respectively $\mathbf{x}^T A\mathbf{y}$ and $\mathbf{x}^T B\mathbf{y}$. The aim of solving such a game is to find a strategy profile $(\mathbf{x}^*, \mathbf{y}^*)$ so that if Players deviate from it, their payoffs don't increase, i.e,

$$\mathbf{x}^T A\mathbf{y}^* \leq (\mathbf{x}^*)^T A\mathbf{y}^* \qquad \forall \mathbf{x} \in \bar{X}, \tag{1.1a}$$

$$(\mathbf{x}^*)^T B\mathbf{y} \leq (\mathbf{x}^*)^T B\mathbf{y}^* \qquad \forall \mathbf{y} \in \bar{Y}. \tag{1.1b}$$

Such a solution is called a (Nash-)equilibrium point. By definition, deviations from an equilibrium point appear non-beneficial to any player. Such games are referred to as bimatrix games and are denoted by (A, B) in summary (see books [24, 35] for more details).

In real-world situations, it is possible that different players have different satisfactory degrees from a single payoff value. To model their satisfactory degree, one can use fuzzy goal programming approach [21, 33, 37]. Suppose that $\mu_1 : \mathbb{R} \to [0, 1]$ is a nondecreasing function in which $\mu_1(\mathbf{x}^T A\mathbf{y})$ represents the satisfactory degree of Player I from the payoff $\mathbf{x}^T A\mathbf{y}$. Similarly, let $\mu_2 : \mathbb{R} \to [0, 1]$ be a nondecreasing function in which $\mu_2(\mathbf{x}^T B\mathbf{y})$ is the satisfactory degree of Player II. According to this viewpoint, an equilibrium point is a strategy profile in which any player has no incentive to deviate from his current strategy because his satisfactory does not increase.

In various situations, it is possible that Players cannot state exactly their satisfactory degrees of the obtained payoffs. For example, this case may occur if a group of players makes decisions, instead of one player. This paper addresses such a situation in which Players' satisfactory degrees are vague and ambiguous. The notion of type-2 fuzzy goal programming approach is used to formulate the problem in this situation. The concept of equilibrium points is introduced and a method is proposed to calculate them. Moreover, a special case is considered in which the type-2 fuzzy numbers are triangular. The results are illustrated by presenting an example.

The remainder of the paper is organized as follows: Section 2 reviews the literature of the subject that address the issue explicitly. Section 3 provides the used terminology of type-1 and type-2 fuzzy sets and explains the notion of type-2 fuzzy goal programming approach. In Section 4, the bimatrix games with vague satisfactory degrees are studied. In Section 5, the case with triangular fuzzy satisfactory degrees is considered. Section 6 presents a practical example. Finally, some concluding remarks are presented in Section 7.

2. Literature review

In conventional game theory due to the ambiguous understanding of various situations by players, providing precise information is a difficult task. In these circumstances, fuzzy set theory is a very useful tool to incorporate vagueness and ambiguity of information. Generally, there are two approaches for embedding fuzziness in non-cooperative games. In the first case, players have fuzzy goals and in the second case, payoffs are represented as fuzzy numbers. In the field of games with fuzzy payoffs, a wide range of studies have been made (for example see [2, 6–9, 13, 18, 22, 23, 42–44, 47]). Jana and Roy [19] based on the similarity measure of dual hesitant fuzzy sets (DHFSs), investigated the solution procedure of fuzzy matrix games with DHFSs pay-offs. Seikh et al. [40] used the Lexicographic method to solve matrix games with hesitant fuzzy payoffs. Xue et al. [48] solved matrix games with hesitant fuzzy pay-offs applying Ambika method and applied it to resolve counter-terrorism issue. Seikh et al. [39] defined a new defuzzification function of dense fuzzy lock set and applied it to resolve the matrix games under dense fuzzy environment. Karmakar et al. [20] studied fuzzy matrix games with type-2 intuitionistic fuzzy payoffs based on normalized minkowski distance measure and applied it to resolve biogas-plant implementation problem to prevent the air pollution.

Nishizaki and Sakawa [33, 37] studied two-person games incorporating type-1 fuzzy goals in single and multi-objective environments. They defined an equilibrium solution with respect to the degree of attainment of the fuzzy goals and showed that an equilibrium can be obtained by solving several mathematical programming problems. Kumar [21] also considered a multi-objective two person zero-sum matrix game with type-1 fuzzy goals and presented some models to obtain the max-min solutions. Vidyottama et al. [45] considered a bi-matrix game with type-1 fuzzy goals. They showed that the problem is equivalent to a (crisp) non-linear programming problem in which the objective as well as all constraint functions are linear except two constraint functions, which are quadratic. Bector et al. [4] considered a two-person zero-sum matrix game with type-1 fuzzy goals and proved that it is equivalent to a primal-dual pair of fuzzy linear programming problems. Bashir et al. [3] studied a multi-criteria two-person zero-sum game with intuitionistic fuzzy (IF) goals. They showed that solving such games is equivalent to solving two crisp multi-objective linear programming problems. In [1, 28, 31], matrix games with IF goals were also considered and some mathematical programming problems were proposed to obtain solutions for such games.

Many authors are interested in games with fuzzy goals as well as fuzzy payoffs [29, 32, 34, 41, 46]. In [46], a two-person zero-sum matrix game with type-1 fuzzy goals and fuzzy payoffs was considered and its solution was conceptualized using

a suitable defuzzification function. Seikh et al. [41] proposed an aspiration level approach to solve matrix games with IF goals and IF payoffs. Nan et al. [29] also presented a method for solving bi-matrix games with IF goals and IF payoffs. In their proposed methodology, a new ranking method was proposed and the concept of IF inequalities was interpreted. Finally, an IF non-linear programming model was constructed to obtain the solution for such games. Nayak and pal [32] presented an application of IF programming to a two person bi-matrix game for obtaining mixed strategies using linear membership and non-membership functions.
We recall that any type-2 fuzzy number is a generalization of a type-1 fuzzy number in the sense that uncertainty is not only limited to the linguistic variables but also is present in the definition of the membership functions. To the best of our knowledge, the use of type-2 fuzzy numbers is not yet widespread. Figueroa-Garcia et al. [12] proposed a model for a two-player game problem involving interval type-2 fuzzy payoffs. Chakeri et al. [10] employed fuzzy logic to determine the priority of payoffs using linguistic preference relations. Then, they assigned a type-2 fuzzy set (T2FS) to each cell to determine how much a cell has possibility to have a specific degree of being equilibrium. Seikh et al. [38] developed a matrix game problem in the type-2 fuzzy environment. They used a new defuzzification model of type-2 fuzzy variables based on type reduction and applied it to consider a single-use plastic ban problem in India.
Based on our review of the literature, we observe that the matrix game theory has been experienced in various type-1 fuzzy set (T1FS) developments. But, due to different types of complex situations such as huge data sets, different assumptions presented by decision makers (here the game players), T1FS cannot describe the situation properly. On the other hand, triangular or trapezoidal fuzzy sets and their extensions are two-dimensional. While in real problems, some uncertain conditions may occur that cannot be described by such kind of two-dimensional fuzzy sets. Therefore, T1FSs become insufficient to describe some complex situation. Therefore, in that case, we consider type-2 fuzzy variables which have three dimensions (element, primary membership degree, and secondary membership degree) to discuss a matrix game.

For example, suppose two different companies are going to offer the same product in a competitive market. Both of the companies employ a team of experts to study the market and create a general plan about the future profits. Each of these expert teams determined criteria such as up-to-date technology, features, modern and cost-effective features, ... to achieve more income. Also, for each criterion, a degree of satisfaction has been considered, which must reach a aspiration level. To find the maximum profit, this scenario can be discussed through a fuzzy matrix game. Since, the experiences or experiments of experts are presented in situations where there is a lack of data or there is semantic ambiguity, therefore a T1FS is not enough to explain such a scenario, but T2FS is ideal for depicting such kind of problems.
In this paper, in order to incorporate ambiguity of judgements, we use type-2 fuzzy numbers to model Players' satisfactory degrees. Specially, we consider a bimatrix

game with type-2 fuzzy goals.

3. Preliminaries

In this section, we first provide some definitions and terminologies of type-1 and type-2 fuzzy numbers used throughout the paper. Then, we describe the type-1 and type-2 fuzzy goal programming approaches.
Let X denote a universal set. A fuzzy subset $\tilde{a}$ of X is defined by its membership function $\mu_{\tilde{a}} : X \to [0,1]$ which assigns to each element $x \in X$ a real number $\mu_{\tilde{a}}(x)$ in the interval $[0,1]$, which represents the grade of membership of x in $\tilde{a}$. A fuzzy set $\tilde{a}$ is normal if $\mu_{\tilde{a}}(x) = 1$ for some $x \in X$. A fuzzy set $\tilde{a}$ is said to be convex whenever

$$\mu_{\tilde{a}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{a}}(x_1), \mu_{\tilde{a}}(x_2)\} \qquad \forall x_1, x_2 \in X, \forall \lambda \in [0,1].$$

A fuzzy set defined on the real line $\mathbb{R}$ is called a fuzzy number if it is convex, normal, and upper semi-continuous. We denote by $F(\mathbb{R})$ the set of all fuzzy numbers and denote by $F(\mathbb{I})$ the set of fuzzy numbers defined on $\mathbb{I} = [0,1]$. A triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3)$ is a special fuzzy number whose membership function is given by

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & x \in [a_1, a_2], \\ \frac{a_3-x}{a_3-a_2} & x \in [a_2, a_3], \\ 0 & \text{otherwise}, \end{cases} \qquad \forall x \in X.$$

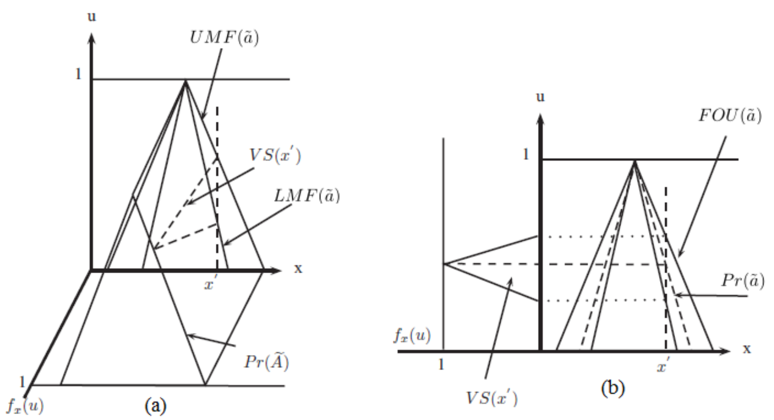

Figure 1. (a) 3D representation of a T2FS; (b) 2D representation of figure (a) with triangular vertical slices.

Definition 1. Let X be the universe of discourse. Then a type-2 fuzzy set $\tilde{a}$ can be defined in terms of its primary membership function u and the secondary membership function

$\mu_{\tilde{a}}(x,u)$ and is denoted as

$$\tilde{a} = \{((x,u), \mu_{\tilde{a}}(x,u)) : x \in X, u \in J_x \subseteq [0,1]\},$$

Note that, J_x is the the domain of the secondary membership function called the secondary domain.

A T2FS can be graphically shown in three dimensional space(3D) [17, 26]. For special value of x say x' the Vertical Slice (VS) in two dimensional (2D) plane of the u and $\mu_{\tilde{a}}(x,u)$ is defined by equation

$$\tilde{\mu}_{\tilde{a}}(x') = \mu_{\tilde{a}}(x',u) = \{(u, f_{x'}(u)) : u \in J_{x'}\},$$

where $f_{x'}(u) \in [0,1]$ is called the secondary grade. The VS is a type-1 fuzzy set (T1FS) in $[0,1]$. For this reason, we denote it by $\tilde{\mu}$. So, $\mu(.,.)$ is a real-valued bivariate function while $\tilde{\mu}(.)$ is a fuzzy-valued single-variate function. So, in a type-2 fuzzy set, each element has a primary membership function, instead of a single membership value. This primary membership function is itself fuzzy, and the secondary grade is the membership degree of this fuzziness.
It can be easily seen from Definition (1) that T2FS is defined on a three-dimensional space. To better characterize its features, Mendel and John [26] defined the information it mapped to the two-dimensional plane as the footprint of uncertainty (FOU). For any $x \in X$, the FOU of $\tilde{a}$ is the union of all the primary membership functions and can be expressed as

$$FOU(\tilde{a}) = \bigcup_{x \in X} J_x,$$

The upper and lower bounds of the FOU are called the upper membership function (UMF) and lower membership function (LMF) of $\tilde{a}$, respectively. (See Figure (1)
The principal membership function (Pr) defined as the union of all the primary memberships having secondary grades equal to 1, i.e.

$$Pr(\tilde{a}) = \{(x,u) : f_x(u) = 1, x \in X\}.$$

In a parallel viewpoint, a type-2 fuzzy set $\tilde{a}$ is characterised by a fuzzy set whose membership grades are type-1 fuzzy sets on $[0,1]$ [50]. In a special case, we suppose that all membership grades are type-1 fuzzy numbers. So any type-2 fuzzy set is determined by its membership function as $\tilde{\mu} : X \to F(\mathbb{I})$ which $\tilde{\mu}(x)$ is a type-1 fuzzy number representing fuzzy membership grade of x. For simplifying the notations, we denote a type-2 fuzzy set by its fuzzy membership function. As a special case, if membership grades are type-1 triangular fuzzy number, then the set is referred to as a triangular type-2 fuzzy set [25]. A more specific type of triangular type-2 fuzzy sets is depicted in Figure 2. It has the fuzzy membership function

$$\tilde{\mu}_{\tilde{a}}(x) = \begin{cases} (0,0,0) & x < \alpha_L, \\ (0,0,\frac{x-\alpha_L}{c-\alpha_L}) & \alpha_L \le x < \alpha, \\ (0,\frac{x-\alpha}{c-\alpha},\frac{x-\alpha_L}{c-\alpha_L}) & \alpha \le x < \alpha_U, \\ (\frac{x-\alpha_U}{c-\alpha_U},\frac{x-\alpha}{c-\alpha},\frac{x-\alpha_L}{c-\alpha_L}) & \alpha_U \le x < c, \\ (\frac{x-\beta_L}{c-\beta_L},\frac{x-\beta}{c-\beta},\frac{x-\beta_U}{c-\beta_U}) & c \le x < \beta_L, \\ (0,\frac{x-\beta}{c-\beta},\frac{x-\beta_U}{c-\beta_U}) & \beta_L \le x < \beta, \\ (0,0,\frac{x-\beta_U}{c-\beta_U}) & \beta \le x < \beta_U, \\ (0,0,0) & \beta_U \le c, \end{cases} \tag{3.1}$$

and is denoted by $(\alpha_L, \alpha, \alpha_U, c, \beta_L, \beta, \beta_U)$. In the case that $\beta_L = \beta = \beta_U = +\infty$, we denote the set by $(\alpha_L, \alpha, \alpha_U, c)$.

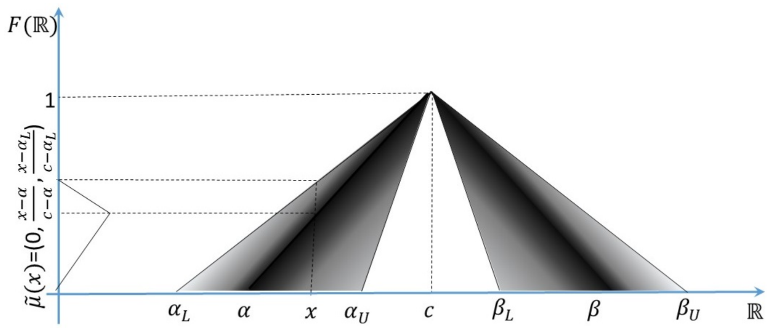

Figure 2. **An instance of triangular type-2 fuzzy set.**

3.1. Type-1 and type-2 fuzzy goal programming approach

Goal programming (GP) is one of the favorite techniques for solving multi-objective optimization problems which was first introduced by Charnes and Cooper [11]. In GP, the decision maker (DM) determines an aspiration level for each objective. Using an optimization problem, a solution is looked for to minimize the deviations between objective values and their aspiration levels, i.e., the aggregated deviation of the objectives from the aspiration levels.

A fault of GP is that it does not consider directly the satisfactory value of the decision maker in fullfilling the goals. For this reason, fuzzy goal programming approach arose [16, 30], in which a function $\mu : \mathbb{R} \to [0,1]$ is defined for any goal whose values reflect the satisfactory degrees of the decision maker in satisfying the goal. The technique looks for a solution having the maximum satisfactory degree of the decision maker. For more illustration, consider a problem containing three types of goals: one of them is in the equality form and the others are in the inequality forms $\le$ and $\ge$ as follows:

$$f_1(x) = z^1, \quad (3.2a)$$

$$f_2(x) \leq z^2, \quad (3.2b)$$

$$f_3(x) \geq z^3, \quad (3.2c)$$

in which z^i, $i = 1, 2, 3$, is the aspiration level associated with the ith goal. If the

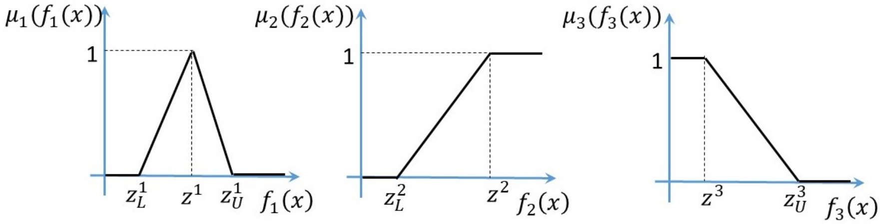

Figure 3. **The satisfactory functions in (type-1) fuzzy goal programming approach.**

inequality system has at least one solution x^*, then it is a solution which satisfies completely the decision maker's expectations. If not, then we have to present a solution which satisfies his expectations as much as possible. To this end, we ask the decision maker to introduce a function μ_i for the i-th goal which reflects his satisfactory degree. It is natural that the satisfactory functions associated with the second and third goals are respectively non-decreasing and non-increasing while that of the first goal has a global maximum in z^1 and it is nondecreasing on $(-\infty, z^1]$ and nonincreasing on $[z^1, +\infty)$. Figure 3 shows the instances of some linear functions μ_i proposed by DM. By the linearity assumption, it is easy to prove that the fuzzy goal programming approach is converted into a standard goal programming in which the weight associated with the deviation from the ith goal is proportional to the slop of line μ_i [27].
In spite of the fact that fuzzy goal programming approach can well models the satisfactory degree of DM, it does not take into account vagueness in the satisfactory degree of DM. A reasonable remedy is applying type-2 fuzzy sets to model the problem. This leads to a natural extension of fuzzy goal programming approach, called type-2 fuzzy goal programming approach. To the best of our knowledge, there is only a paper by Patino-callejas et al. [36] which presented an interval type-2 fuzzy goal programming model for problems where resources are defined by the opinion of multiple experts. In this paper, we present type-2 fuzzy goal programming approach in the general case.
Suppose that the DM's satisfactory degree is not determined clearly but it can be represented by a type-1 fuzzy number. For more clarity, when we ask the DM to determine his satisfactory degree from aspiration level z^1, he says "1". However, if we ask him to determine his satisfactory degree from $z^1 \pm \Delta$, he says "*almost* 0.9", i.e., he proposes a fuzzy number for his satisfactory degree. So the satisfactory degree functions μ_i are fuzzy-valued. Hereafter, we denote these functions by $\tilde{\mu}_i : \mathbb{R} \longrightarrow F(\mathbb{I})$. Figure 4 shows the fuzzy-valued functions $\tilde{\mu}_i$ for the goals (3.2). Since the functions $\tilde{\mu}_i$ can be regarded as membership functions of type-2 fuzzy numbers [50], the approach

is referred to as type-2 fuzzy goal programming.

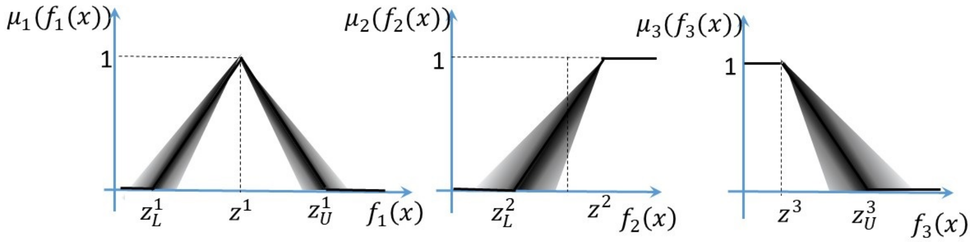

Figure 4. **The satisfactory functions in type-2 fuzzy goal programming approach.**

4. Bimatrix Games using type-2 fuzzy goal programming approach

Consider a non-cooperative game containing two players. Suppose that A and B are the payoff matrices of Players I and II, respectively. We recall that their payoffs are respectively $\mathbf{x}^T A\mathbf{y}$ and $\mathbf{x}^T B\mathbf{y}$ for strategy profile $(\mathbf{x},\mathbf{y}) \in \bar{X} \times \bar{Y}$. In various situations, it is possible that any player has vague satisfactory degree of his payoff. This may occur specially if a team make decisions instead of a person. We denote by $\tilde{\mu}_1$ and $\tilde{\mu}_2$ the fuzzy-valued satisfactory functions corresponding to Players I and II, respectively. These functions can be regarded as type-2 fuzzy numbers. Indeed, we here use type-2 goal programming approach to model ambiguous satisfactory degrees of Players. So $\tilde{\mu}_1(\mathbf{x}^T A\mathbf{y})$ is a type-1 fuzzy number to state the satisfactory degree of Player I for the payoff $\mathbf{x}^T A\mathbf{y}$. Similarly, $\tilde{\mu}_2(\mathbf{x}^T B\mathbf{y})$ denotes the satisfactory degree of Player II. Such the game is referred to as the game with vague satisfactory degrees (GVSD). In this section, our aim is to extend the notion of equilibrium points to such the games and present an approach to find them. Throughout this paper, we denote an instance of GVSD by $(A, B, \tilde{\mu}_1, \tilde{\mu}_2)$.

In order to avoid reducing fuzzy satisfactory degrees to real ones, a partial order (a reflexive, antisymmetric and transitive binary relation) has to be considered for comparing fuzzy numbers [14, 15]. Let $\preceq$ be a partial order on the set of fuzzy numbers $F(\mathbb{R})$. We define

$$\tilde{x} \succeq \tilde{y} \text{ if } \tilde{y} \preceq \tilde{x}, \tag{4.1a}$$

$$\tilde{x} \prec \tilde{y} \text{ if } \tilde{x} \preceq \tilde{y} \wedge \tilde{x} \neq \tilde{y}, \tag{4.1b}$$

$$\tilde{x} \succ \tilde{y} \text{ if } \tilde{y} \preceq \tilde{x} \wedge \tilde{x} \neq \tilde{y}, \tag{4.1c}$$

for every $\tilde{x}, \tilde{y} \in F(\mathbb{R})$.

Using the partial order $\preceq$, we compare the fuzzy satisfactory degrees of Players. For example, if $\tilde{\mu}_1(\mathbf{x}^T A\mathbf{y}) \preceq \tilde{\mu}_1(\bar{\mathbf{x}}^T A\bar{\mathbf{y}})$, then Player I prefers the profit obtained from the strategy profile $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ to that of $(\mathbf{x}, \mathbf{y})$. If neither $\tilde{\mu}_1(\bar{\mathbf{x}}^T A\bar{\mathbf{y}}) \preceq \tilde{\mu}_1(\mathbf{x}^T A\mathbf{y})$ nor $\tilde{\mu}_1(\mathbf{x}^T A\mathbf{y}) \preceq \tilde{\mu}_1(\bar{\mathbf{x}}^T A\bar{\mathbf{y}})$, the strategy profiles $(\mathbf{x}, \mathbf{y})$ and $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ cannot be compared (because $\preceq$ is not a total order); namely, Player I cannot prefer one to the other. In this case, we say that such strategy profiles are incomparable.

Remark 1. In this paper, we use the same partial order for both the players to simplify notations. However, the results remain valid even if any player has an individual partial order.

The satisfactory functions $\tilde{\mu}_1$ and $\tilde{\mu}_2$ have to be nondecreasing with respect to the partial order $\preceq$ due to their inherent nature. Namely, for each strategy profiles $(\mathbf{x}, \mathbf{y})$ and $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$,

$$\text{if } \mathbf{x}^T A\mathbf{y} < \bar{\mathbf{x}}^T A\bar{\mathbf{y}}, \text{ then } \tilde{\mu}_1(\mathbf{x}^T A\mathbf{y}) \preceq \tilde{\mu}_1(\bar{\mathbf{x}}^T A\bar{\mathbf{y}}) \text{ or they are incomparable,}$$

$$\text{if } \mathbf{x}^T B\mathbf{y} < \bar{\mathbf{x}}^T B\bar{\mathbf{y}}, \text{ then } \tilde{\mu}_2(\mathbf{x}^T B\mathbf{y}) \preceq \tilde{\mu}_2(\bar{\mathbf{x}}^T B\bar{\mathbf{y}}) \text{ or they are incomparable.}$$

This property states that the players are not less satisfied whenever their payoffs are increasing.
Let us now introduce the notion of equilibrium points for such the games.

Definition 2. Given an instance $(A, B, \tilde{\mu}_1, \tilde{\mu}_2)$ of GVSD, a strategy profile $(\mathbf{x}^*, \mathbf{y}^*) \in \bar{X} \times \bar{Y}$ is an equilibrium point if

$$\nexists \mathbf{x} \in \bar{X}, \ \ \tilde{\mu}_1((\mathbf{x}^*)^T A\mathbf{y}^*) \prec \tilde{\mu}_1(\mathbf{x}^T A\mathbf{y}^*); \tag{4.2}$$

$$\nexists \mathbf{y} \in \bar{Y}, \ \ \tilde{\mu}_2((\mathbf{x}^*)^T B\mathbf{y}^*) \prec \tilde{\mu}_2((\mathbf{x}^*)^T B\mathbf{y}). \tag{4.3}$$

By definition, an equilibrium point is a situation which neither of Players becomes more satisfied if he deviates from his current strategy. So Players have not incentive to change their strategies.
The nondecreasing property yields a relationship between equilibrium points of a GVSD and those of the corresponding ordinary game.

Lemma 1. *Given a GVSD $(A, B, \tilde{\mu}_1, \tilde{\mu}_2)$, if a strategy profile $(\mathbf{x}^*, \mathbf{y}^*)$ satisfies (1.1) then, it is an equilibrium point for the GVSD.*

Proof. The proof is straightforward. □

For a GVSD, Lemma 1 implies that an equilibrium point of the corresponding ordinary game is also an equilibrium point of GVSD but the reverse is not valid in general (see Example 1).

By Definition 2, it is easy to see that a strategy profile $(\mathbf{x}^*, \mathbf{y}^*) \in \bar{X} \times \bar{Y}$ is an equilibrium point if and only if $\mathbf{x}^*$ and $\mathbf{y}^*$ are respectively efficient solutions for two following optimization problems:

$$\max_{\mathbf{x}\in\bar{X}} \tilde{\mu}_1(\mathbf{x}^T A\mathbf{y}^*), \tag{4.4}$$

$$\max_{\mathbf{y}\in\bar{Y}} \tilde{\mu}_2((\mathbf{x}^*)^T B\mathbf{y}). \tag{4.5}$$

Note that we have used the word "efficient" instead of "optimal" because an optimization problem defined on a partially ordered set, namely $F(\mathbb{R})$, cannot admit an optimal solution in general.
In order to find the efficient solutions of the problems (4.4) and (4.5), a customary method is using scalarizing functions that are consistent with the order $\preceq$ in the following sense.

Definition 3. A function $u : F(\mathbb{R}) \to \mathbb{R}$ is consistent with the fuzzy partial order $\preceq$ when the following condition hold.

- If $\tilde{a} \preceq \tilde{b}$ then $u(\tilde{a}) \leq u(\tilde{b})$ for any $\tilde{a}, \tilde{b} \in F(\mathbb{R})$.

Note that the scalarizing functions define a total order on fuzzy numbers. This helps us to convert the problems (4.4) and (4.5) into ordinary optimization problems. Suppose that $u_1 : F(\mathbb{R}) \to \mathbb{R}$ and $u_2 : F(\mathbb{R}) \to \mathbb{R}$ are the scalarizing functions consistent with $\preceq$ corresponding to Players I and II, respectively. Then, Players can obtain the optimal solutions of the following optimization problems instead of solving the problems (4.4) and (4.5).

$$\max_{\mathbf{x}\in\bar{X}} u_1(\tilde{\mu}_1(\mathbf{x}^T A\mathbf{y}^*)), \tag{4.6}$$

$$\max_{\mathbf{y}\in\bar{Y}} u_2(\tilde{\mu}_2((\mathbf{x}^*)^T B\mathbf{y})). \tag{4.7}$$

The problem (4.6) determines a best-response strategy of Player I whenever Player II chooses $\mathbf{y}^*$. Also, a similar argument is valid on the problem (4.7). We now propose equivalent conditions to equilibrium points, which lead us to obtain an equilibrium point only by solving an optimization problem. Using two scalarizing functions, the conditions of Definition 2 can be written as

$$u_1(\tilde{\mu}_1(\mathbf{x}^T A\mathbf{y}^*)) \leq u_1(\tilde{\mu}_1((\mathbf{x}^*)^T A\mathbf{y}^*)) \qquad \forall \mathbf{x} \in \bar{X} \tag{4.8}$$

$$u_2(\tilde{\mu}_2((\mathbf{x}^*)^T B\mathbf{y})) \leq u_2(\tilde{\mu}_2((\mathbf{x}^*)^T B\mathbf{y}^*)) \qquad \forall \mathbf{y} \in \bar{Y}. \tag{4.9}$$

By setting $\mathbf{x} = [0\ldots0\underbrace{1}_{ith}0\ldots0]^T$ and $\mathbf{y} = [0\ldots0\underbrace{1}_{jth}0\ldots0]^T$ in the above conditions, we have

$$u_1(\tilde{\mu}_1((A\mathbf{y}^*)_i) \leq u_1(\tilde{\mu}_1((\mathbf{x}^*)^T A\mathbf{y}^*)) \qquad i = 1, 2, \ldots, m \tag{4.10}$$

$$u_2(\tilde{\mu}_2(((\mathbf{x}^*)^T B)_j) \leq u_2(\tilde{\mu}_2((\mathbf{x}^*)^T B\mathbf{y}^*)) \qquad j = 1, 2, \ldots, n. \tag{4.11}$$

Then, the conditions (4.10) and (4.11) are necessary for any equilibrium point. This conditions are also sufficient if we impose an additional assumption.

Assumption 1. *$u_1 o \tilde{\mu}_1$ is convex, that is,*

$$u_1(\tilde{\mu}_1(\sum_{i=1}^{n} \lambda_i v_i)) \leq \sum_{i=1}^{n} \lambda_i u_1(\tilde{\mu}_1(v_i))$$

for $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ with $v_i \in [\min_{i,j} a_{ij}, \max_{i,j} a_{ij}]$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_i \geq 0$, $i = 1, 2, \ldots, n$, and $\sum_i \lambda_i = 1$.

Assumption 1 is stated only for vectors whose elements belong to $[\min_{i,j} a_{ij}$ $, \max_{i,j} a_{ij}]$ because any payoff of Player I belongs to this interval. Similarly, we suppose that the convexity assumption also holds for Player II.

Lemma 2. *Under convexity assumption, A strategy profile $(\mathbf{x}^*, \mathbf{y}^*)$ is an equilibrium point if and only if it satisfies the conditions (4.10) and (4.11).*

Proof. Based on the preceding argument, we only prove the sufficiency. Assume that $(\mathbf{x}^*, \mathbf{y}^*)$ satisfies the conditions (4.10). Let $\mathbf{x} \in \bar{X}$ be arbitrary. By multiplying the ith condition (4.10) by x_i and summation over i, we have

$$\sum_i x_i u_1(\tilde{\mu}_1((A\mathbf{y}^*)_i)) \leq \sum_i x_i u_1(\tilde{\mu}_1((\mathbf{x}^*)^T A\mathbf{y}^*)) = u_1(\tilde{\mu}_1((\mathbf{x}^*)^T A\mathbf{y}^*)). \tag{4.12}$$

On the other hand, based on Assumption 1, we have

$$u_1(\tilde{\mu}_1(\mathbf{x}^T A\mathbf{y}^*)) \leq \sum_i x_i u_1(\tilde{\mu}_1((A\mathbf{y}^*)_i))$$

which together with (4.12) imply that

$$u_1(\tilde{\mu}_1(\mathbf{x}^T A\mathbf{y}^*)) \leq u_1(\tilde{\mu}_1((\mathbf{x}^*)^T A\mathbf{y}^*)).$$

A similar proof shows that the conditions (4.9) and (4.11) are equivalent. This completes the proof. □

Now we are ready to state the main result of this section.

Theorem 1. *A strategy profile $(\mathbf{x}^*, \mathbf{y}^*)$ is an equilibrium point if and only if $(\mathbf{x}^*, \mathbf{y}^*)$ is an optimal solution to the following optimization problem:*

$$\min z = \alpha + \beta - u_1(\tilde{\mu}_1(\mathbf{x}^T A\mathbf{y})) - u_2(\tilde{\mu}_2(\mathbf{x}^T B\mathbf{y})) \tag{4.13a}$$
$$u_1(\tilde{\mu}_1((A\mathbf{y})_i)) \leq \alpha \qquad i = 1, 2, \ldots, m, \tag{4.13b}$$
$$u_2(\tilde{\mu}_2((\mathbf{x}^T B)_j)) \leq \beta \qquad j = 1, 2, \ldots, n, \tag{4.13c}$$
$$\mathbf{x} \in \bar{X}, \mathbf{y} \in \bar{Y}, \tag{4.13d}$$
$$\alpha, \beta \in \mathbb{R}. \tag{4.13e}$$

Proof. Suppose that $(\mathbf{x}^*, \mathbf{y}^*)$ is an equilibrium point and moreover, set $\alpha^* = u_1(\tilde{\mu}_1((\mathbf{x}^*)^T A\mathbf{y}^*))$ and $\beta^* = u_2(\tilde{\mu}_2((\mathbf{x}^*)^T B\mathbf{y}^*))$. Based on Lemma 2, it is easy to see that $(\mathbf{x}^*, \mathbf{y}^*, \alpha^*, \beta^*)$ is a feasible solution to the problem with the objective value $z^* = 0$. To prove the optimality of $(\mathbf{x}^*, \mathbf{y}^*, \alpha^*, \beta^*)$, assume that $(\mathbf{x}, \mathbf{y}, \alpha, \beta)$ is an arbitrary feasible solution. Thus it satisfies the constraints (4.13b) and (4.13c). By multiplying the ith constraint (4.13b) by x_i and then summation, we have

$$\sum_i x_i u_1(\tilde{\mu}_1((A\mathbf{y})_i)) \leq \alpha,$$

and consequently, based on Assumption 1,

$$u_1(\tilde{\mu}_1(\mathbf{x}^T A\mathbf{y})) \leq \alpha. \tag{4.14}$$

Similarly, one can prove that

$$u_2(\tilde{\mu}_2(\mathbf{x}^T B\mathbf{y})) \leq \beta. \tag{4.15}$$

The inequalities (4.14) and (4.15) guarantee that the objective value of any feasible solution $(\mathbf{x}, \mathbf{y}, \alpha, \beta)$ is nonnegative. Hence, $(\mathbf{x}^*, \mathbf{y}^*, \alpha^*, \beta^*)$ is an optimal solution.
Now suppose that $(\mathbf{x}^*, \mathbf{y}^*, \alpha^*, \beta^*)$ is an optimal solution to the problem (4.13). According to the proof of the necessity, the optimal value z^* is equal to zero. A similar proof shows that the solution $(\mathbf{x}^*, \mathbf{y}^*, \alpha, \beta)$ satisfies the inequalities (4.14) and (4.15). This together with the equality $z^* = 0$ imply that $\alpha = u_1(\tilde{\mu}_1((\mathbf{x}^*)^T A\mathbf{y}^*))$ and $\beta = u_2(\tilde{\mu}_2((\mathbf{x}^*)^T B\mathbf{y}^*))$. By substituting these relations in the constraints (4.13b) and (4.13c), the conditions (4.10) and (4.11) are acquired. These conditions establish that $(\mathbf{x}^*, \mathbf{y}^*)$ is an equilibrium point (see Lemma 2). □

5. Type-2 triangular fuzzy satisfactory degrees

In this section we consider a special case of GVSD in which $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are type-2 triangular fuzzy numbers and apply the partial order $\preceq$ defined as

$$\tilde{x} \preceq \tilde{y} \text{ iff } x_k \leq y_k \quad k = 1, 2, 3 \tag{5.1}$$

for each (type-1) triangular fuzzy numbers $\tilde{x} = (x_1, x_2, x_3)$ and $\tilde{y} = (y_1, y_2, y_3)$. However, the results remain valid for various partial orders. We recall that the notations $\succeq, \succ$, and $\prec$ are defined with respect to $\preceq$ by (4.1).
Suppose that $\tilde{\mu}_1 = (c^1, \alpha_L^1, \alpha^1, \alpha_U^1)$ and $\tilde{\mu}_2 = (c^2, \alpha_L^2, \alpha^2, \alpha_U^2)$ are type-2 triangular fuzzy numbers which determine the satisfactory degree of Players I and II, respectively. For instance, the satisfactory degree $\tilde{\mu}_1$ of Player I is depicted in Figure 5.

The membership functions $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are defined as follows:

$$\tilde{\mu}_1(\mathbf{x}^T A\mathbf{y}) = \begin{cases} (0,0,0) & \mathbf{x}^T A\mathbf{y} < \alpha_L^1, \\ (0,0,\frac{\mathbf{x}^T A\mathbf{y}-\alpha_L^1}{c^1-\alpha_L^1}) & \alpha_L^1 \le \mathbf{x}^T A\mathbf{y} < \alpha^1, \\ (0,\frac{\mathbf{x}^T A\mathbf{y}-\alpha^1}{c^1-\alpha^1},\frac{\mathbf{x}^T A\mathbf{y}-\alpha_L^1}{c^1-\alpha_L^1}) & \alpha^1 \le \mathbf{x}^T A\mathbf{y} < \alpha_U^1, \\ (\frac{\mathbf{x}^T A\mathbf{y}-\alpha_U^1}{c^1-\alpha_U^1},\frac{\mathbf{x}^T A\mathbf{y}-\alpha^1}{c^1-\alpha^1},\frac{\mathbf{x}^T A\mathbf{y}-\alpha_L^1}{c^1-\alpha_L^1}) & \alpha_U^1 \le \mathbf{x}^T A\mathbf{y} < c^1, \\ (1,1,1) & c^1 \le \mathbf{x}^T A\mathbf{y}, \end{cases} \tag{5.2}$$

and

$$\tilde{\mu}_2(\mathbf{x}^T B\mathbf{y}) = \begin{cases} (0,0,0) & \mathbf{x}^T B\mathbf{y} < \alpha_L^2. \\ (0,0,\frac{\mathbf{x}^T B\mathbf{y}-\alpha_L^2}{c^2-\alpha_L^2}) & \alpha_L^2 \le \mathbf{x}^T B\mathbf{y} < \alpha^2, \\ (0,\frac{\mathbf{x}^T B\mathbf{y}-\alpha^2}{c^2-\alpha^2},\frac{\mathbf{x}^T B\mathbf{y}-\alpha_L^2}{c^2-\alpha_L^2}) & \alpha^2 \le \mathbf{x}^T B\mathbf{y} < \alpha_U^2, \\ (\frac{\mathbf{x}^T B\mathbf{y}-\alpha_U^2}{c^2-\alpha_U^2},\frac{\mathbf{x}^T B\mathbf{y}-\alpha^2}{c^2-\alpha^2},\frac{\mathbf{x}^T B\mathbf{y}-\alpha_L^2}{c^2-\alpha_L^2}) & \alpha_U^2 \le \mathbf{x}^T B\mathbf{y} < c^2, \\ (1,1,1) & c^2 \le \mathbf{x}^T B\mathbf{y}, \end{cases} \tag{5.3}$$

Notice that this special case of membership functions occur in practice whenever the

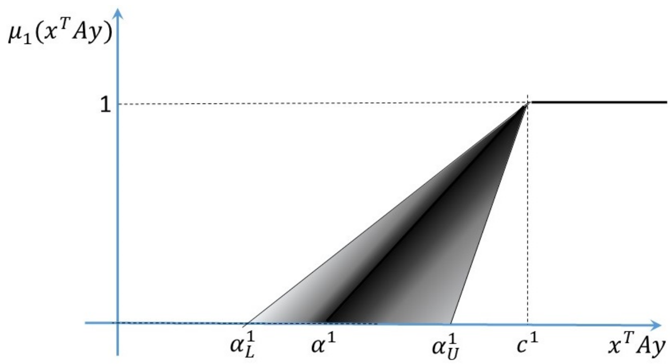

Figure 5. **The type-2 triangular satisfactory function $\tilde{\mu}_1$.**

proportionality (linearity) property exists in partial decision making.
Recall that the problems (4.4) and (4.5) can be regarded as multiobjective optimization problems. Then, we can apply Benson's approach to check the efficiency of solutions [5]. This gives us a method to check whether or not a strategy profile is equilibrium.

Proposition 1. *A strategy profile* $(\mathbf{x}^*, \mathbf{y}^*)$ *is an equilibrium point for a GVSD, if and* *only if the optimal value of the following optimization problem is equal to zero.*

$$\max z = s_1 + s_2 + s_3 + s'_1 + s'_2 + s'_3 \tag{5.4}$$
$$(\tilde{\mu}_1(\mathbf{x}^T A\mathbf{y}^*))_i - s_i = (\tilde{\mu}_1((\mathbf{x}^*)^T A\mathbf{y}^*))_i \quad i = 1, 2, 3 \tag{5.5}$$
$$(\tilde{\mu}_2((\mathbf{x}^*)^T B\mathbf{y}))_i - s'_i = (\tilde{\mu}_2((\mathbf{x}^*)^T B\mathbf{y}^*))_i \quad i = 1, 2, 3 \tag{5.6}$$
$$\mathbf{1}^T\mathbf{x} = 1, \tag{5.7}$$
$$\mathbf{1}^T\mathbf{y} = 1, \tag{5.8}$$
$$\mathbf{x}, \mathbf{y} \geq 0, \tag{5.9}$$
$$s_i, s'_i \geq 0 \quad i = 1, 2, 3. \tag{5.10}$$

Proof. The proof is straightforward based on the notion of equilibrium points and the used partial order. □

Let us now state the relationship between equilibrium points of a GVSD and those of the corresponding ordinary bimatrix game.

Lemma 3. *For $i = 1, 2$ and $s, t \in \mathbb{R}$, we have*

1. $\tilde{\mu}_i(s) \preceq \tilde{\mu}_i(t)$ *if and only if* $s \leq t$;
2. *If* $\tilde{\mu}_i(s) \prec \tilde{\mu}_i(t)$, *then* $s < t$;
3. *If* $\alpha^i_L < s < t < c^i$, *then* $\tilde{\mu}_i(s) \prec \tilde{\mu}_i(t)$.

Proof. The membership functions $\tilde{\mu}_i(t)$ can be written as

$$\tilde{\mu}_i(t) = (\min\{1, \max\{0, \frac{t - \alpha^i_U}{c^i - \alpha^i_U}\}\},$$
$$\min\{1, \max\{0, \frac{t - \alpha^i}{c^i - \alpha^i}\}\}, \min\{1, \max\{0, \frac{t - \alpha^i_L}{c^i - \alpha^i_L}\}\}).$$

Notice that three components of $\tilde{\mu}_i(t)$ are nondecreasing functions of t. So the part 1 is immediate.
To prove 2, let $\tilde{\mu}_i(s) \prec \tilde{\mu}_i(t)$. It is easy to see that the inequality $(\tilde{\mu}_i(s))_3 < (\tilde{\mu}_i(t))_3$ holds necessarily. So one of the following cases occurs

Case I: $(\tilde{\mu}_i(s))_3 = 0$ and $(\tilde{\mu}_i(t))_3 = 1$.

Case II: $(\tilde{\mu}_i(s))_3 = 0$ and $(\tilde{\mu}_i(t))_3 \in (0, 1)$.

Case III: $(\tilde{\mu}_i(s))_3 \in (0, 1)$ and $(\tilde{\mu}_i(t))_3 \in (0, 1)$.

Case IV: $(\tilde{\mu}_i(s))_3 \in (0, 1)$ and $(\tilde{\mu}_i(t))_3 = 1$.

We consider only cases I and III. The others can be proved similarly. In Case I, $s < \alpha_L^i$, and $t > c^i$. Since $c^i \geq \alpha_L^i$, it follows that $s < t$. In Case III, we have

$$\frac{s - \alpha_L^i}{c^i - \alpha_L^i} = (\tilde{\mu}_i(s))_3 < (\tilde{\mu}_i(t))_3 = \frac{t - \alpha_L^i}{c^i - \alpha_L^i}$$

which implies that $s < t$.
To prove the part 3, note that the inequalities $\alpha_L^i < s < c^i$ and $\alpha_L^i < t < c^i$ guarantee that

$$(\tilde{\mu}_i(s))_3 = \frac{s - \alpha_L^i}{c^i - \alpha_L^i} \text{ and } (\tilde{\mu}_i(t))_3 = \frac{t - \alpha_L^i}{c^i - \alpha_L^i}$$

and since $s < t$, we have

$$(\tilde{\mu}_i(s))_3 < (\tilde{\mu}_i(t))_3. \tag{5.11}$$

On the other hand, based on the part 1, $\tilde{\mu}_i(s) \preceq \tilde{\mu}_i(t)$. This together with (5.11) complete the proof. □

An obvious and important result of Lemma 3 is that there is not any two strategy profiles which are incomparable from Players' viewpoints.
Based on Lemma 1, the set of equilibrium points of a GVSD contains any equilibrium point of the corresponding ordinary game. The following example shows that the reverse result is not true in general.

Example 1. Consider a bimatrix game with

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix}.$$

Suppose that the satisfactory degrees are determined by the type-2 triangular fuzzy numbers $\tilde{\mu}_1 = \tilde{\mu}_2 = (1, 2, 3, 4)$. Consider two strategy profiles $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = ([1, 0]^T, [0, 1]^T)$ and $(\bar{\mathbf{x}}, \hat{\mathbf{y}}) = ([0, 1]^T, [0, 1]^T)$. The satisfactory degrees of the strategy profiles are

$$\tilde{\mu}_1(\hat{\mathbf{x}}^T A\hat{\mathbf{y}}) = \tilde{\mu}_1(\bar{\mathbf{x}}^T A\hat{\mathbf{y}}) = (1, 1, 1),$$

$$\tilde{\mu}_2(\hat{\mathbf{x}}^T B\hat{\mathbf{y}}) = \tilde{\mu}_2(\bar{\mathbf{x}}^T B\hat{\mathbf{y}}) = (1, 1, 1).$$

So both the strategy profiles satisfy completely Players' satisfactions. Consequently, they are equilibrium points of GVSD. On the other hand, we have

$$\hat{\mathbf{x}}^T A\hat{\mathbf{y}} = 4 < 5 = \bar{\mathbf{x}}^T A\hat{\mathbf{y}},$$
$$\hat{\mathbf{x}}^T B\hat{\mathbf{y}} = 4 = \bar{\mathbf{x}}^T B\hat{\mathbf{y}},$$

which imply that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is not an ordinary equilibrium point.

Theorem 2. *Let $(A, B, \tilde{\mu}_1, \tilde{\mu}_2)$ be a GVSD where $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are defined by (5.2) and (5.3), respectively, and (A, B) is the corresponding ordinary bimatrix game. If $(\mathbf{x}^*, \mathbf{y}^*)$ is an equilibrium point of the GVSD $(A, B, \tilde{\mu}_1, \tilde{\mu}_2)$ and*

$$(0,0,0) \prec \tilde{\mu}_1((\mathbf{x}^*)^T A\mathbf{y}^*) \prec (1,1,1) \wedge (0,0,0) \prec \tilde{\mu}_2((\mathbf{x}^*)^T B\mathbf{y}^*) \prec (1,1,1),$$

then it is also an equilibrium point of the ordinary bimatrix game (A, B) (defined by (1.1)).

Proof. Suppose that $(\mathbf{x}^*, \mathbf{y}^*)$ is not an equilibrium point of the bimatrix game (A, B). Without loss of generality, we assume that there is a vector $\bar{\mathbf{x}}$ so that

$$(\mathbf{x}^*)^T A\mathbf{y}^* < \bar{\mathbf{x}}^T A\mathbf{y}^*. \tag{5.12}$$

If $\bar{\mathbf{x}}^T A\mathbf{y}^* > c^1$, then

$$\tilde{\mu}_1((\mathbf{x}^*)^T A\mathbf{y}^*) \prec (1,1,1) = \tilde{\mu}_1(\bar{\mathbf{x}}^T A\mathbf{y}^*)$$

which contradicts the fact that $(\mathbf{x}^*, \mathbf{y}^*)$ is an equilibrium point of the GVSD $(A, B, \tilde{\mu}_1, \tilde{\mu}_2)$.
If $\bar{\mathbf{x}}^T A\mathbf{y}^* \leq c^1$, then we have

$$\frac{(\mathbf{x}^*)^T A\mathbf{y}^* - \alpha_U^1}{c^1 - \alpha_U^1} < \frac{\bar{\mathbf{x}}^T A\mathbf{y}^* - \alpha_U^1}{c^1 - \alpha_U^1},$$
$$\frac{(\mathbf{x}^*)^T A\mathbf{y}^* - \alpha^1}{c^1 - \alpha^1} < \frac{\bar{\mathbf{x}}^T A\mathbf{y}^* - \alpha^1}{c^1 - \alpha^1},$$
$$\frac{(\mathbf{x}^*)^T A\mathbf{y}^* - \alpha_L^1}{c^1 - \alpha_L^1} < \frac{\bar{\mathbf{x}}^T A\mathbf{y}^* - \alpha_L^1}{c^1 - \alpha_L^1}.$$

These inequalities simply imply that $\tilde{\mu}_1((\mathbf{x}^*)^T A\mathbf{y}^*) \prec \tilde{\mu}_1(\bar{\mathbf{x}}^T A\mathbf{y}^*)$. So $(\mathbf{x}^*, \mathbf{y}^*)$ is not an equilibrium point of the GVSD. □

6. A practical example

We consider a problem of wargaming as follows. Consider a situation between blue and red forces. Blue force is attacker and red force is defender. Blue force would like to destroy two specified targets of red side by its unmanned air vehicles. It is obvious that blue and red forces have two pure strategies: (I) sending their all forces to the first target, (II) sending their all forces to the second target. We denote by $x \in [0, 1]$ the mixed strategy of blue force in which x percent of its force is sent to the first target and $1 - x$ percent to the second target. Similarly, we denote by $y \in [0, 1]$ the mixed strategy of red force. For every strategy profile (x, y), suppose that $b(x, y)$ and $r(x, y)$ are respectively payoffs of blue and red forces. The values $b(x, y)$ and $r(x, y)$

are respectively proportional to the total amount of battle survivors of Blue and Red forces. Let

$$A_b = \begin{bmatrix} 5 & 5 \\ 3 & 8 \end{bmatrix}, A_r = \begin{bmatrix} 4 & 5 \\ 7 & 6 \end{bmatrix}$$

be the payoff matrices of blue and red forces, respectively. Since blue force has a group of commanders for decision making, they have different satisfactory degrees of $b(x, y) = [x, 1-x]A_b[y, 1-y]^T$: a pessimistic viewpoint $p_1(b(x, y))$, a middle viewpoint $p_2(b(x, y))$, an optimistic viewpoint $p_3(b(x, y))$ in which $p_1(b(x, y)) \leq p_2(b(x, y)) \leq p_3(b(x, y))$. We can construct a TFN $(p_1(b(x, y)), p_2(b(x, y)), p_3(b(x, y)))$, for every $x, y \in [0, 1]$. So type-2 fuzzy number $\tilde{\mu}_b(b(x, y)) = (p_1(b(x, y)), p_2(b(x, y)), p_3(b(x, y)))$ is the satisfactory degree function of Blue force. Similarly, we denote by $\tilde{\mu}_r(r(x, y)) = (q_1(r(x, y)),\ q_2(r(x, y)), q_3(r(x, y)))$ the satisfactory degree of red commanders from the payoff $r(x, y)$. Here, suppose that $\tilde{\mu}_b = (4, 5, 7, 10)$ and $\tilde{\mu}_r = (3, 4, 6, 10)$. So

$$\begin{aligned}
&\tilde{\mu}_b(b(x, y)) = (\min\{1, \max\{0, \tfrac{b(x,y)-7}{3}\}\},\\
&\min\{1, \max\{0, \tfrac{b(x,y)-5}{5}\}\}, \min\{1, \max\{0, \tfrac{b(x,y)-4}{6}\}\}),\\
&\tilde{\mu}_r(r(x, y)) = (\min\{1, \max\{0, \tfrac{r(x,y)-6}{4}\}\},\\
&\min\{1, \max\{0, \tfrac{r(x,y)-4}{6}\}\}, \min\{1, \max\{0, \tfrac{r(x,y)-3}{7}\}\}).
\end{aligned}$$

In order to consider more effect of middle viewpoint, we use Yager's ranking function $u_1(x_1, x_2, x_3) = u_2(x_1, x_2, x_3) = \frac{x_1+4x_2+x_3}{6}$ [49]. So the problem (4.13) can be written as follows:

$$\begin{aligned}
\min\quad & z = \alpha + \beta - \tfrac{1}{6}(\min\{1, \max\{0, \tfrac{5xy-5y-3x+1}{3}\}\} +\\
& 4\min\{1, \max\{0, \tfrac{5xy-5y-3x+3}{5}\}\} + \min\{1, \max\{0, \tfrac{5xy-5y-3x+4}{6}\}\}) -\\
& \tfrac{1}{6}(\min\{1, \max\{0, \tfrac{2xy+x-y-12}{4}\}\} + 4\min\{1, \max\{0, \tfrac{2xy+x-y-10}{6}\}\} +\\
& \min\{1, \max\{0, \tfrac{2xy+x-y-9}{7}\}\}), \qquad (6.1)\\
\text{s.t.}\quad & \tfrac{1}{6}(\min\{1, \max\{0, \tfrac{-5y+1}{3}\}\} + 4\min\{1, \max\{0, \tfrac{-5y+3}{5}\}\} + \min\{1, \max\{0, \tfrac{-5y+4}{6}\}\}) \leq \alpha,\\
& \tfrac{1}{6}(\min\{1, \max\{0, \tfrac{-2}{3}\}\} + 4\min\{1, \max\{0, \tfrac{0}{5}\}\} + \min\{1, \max\{0, \tfrac{1}{6}\}\}) \leq \alpha,\\
& \tfrac{1}{6}(\min\{1, \max\{0, \tfrac{x-12}{4}\}\} + 4\min\{1, \max\{0, \tfrac{x-10}{6}\}\} + \min\{1, \max\{0, \tfrac{x-9}{7}\}\}) \leq \beta,\\
& \tfrac{1}{6}(\min\{1, \max\{0, \tfrac{3x-13}{4}\}\} + 4\min\{1, \max\{0, \tfrac{3x-11}{6}\}\} + \min\{1, \max\{0, \tfrac{3x-10}{7}\}\}) \leq \beta,\\
& x, y \in [0, 1].
\end{aligned}$$

We used Lingo version 18.0.44 to solve the problem. An equilibrium strategy profile is $(\mathbf{x}^*, \mathbf{y}^*) = (0.5, 0.6)$. The payoffs of blue and red forces are respectively $b(x^*, y^*) = 5$ and $r(x^*, y^*) = 5.5$ whose satisfactory degrees are $(0, 0, 0.16666)$ and $(0, 0.25, 0.35714)$. Hence, in spite of the fact that the payoffs of both sides are approximately the same, the expectations of blue commanders are satisfied less than those of red commanders.

7. Conclusion

In this paper, a bimatrix game was considered in which the players have vague satisfactory degrees. A type-2 fuzzy goal programming approach was applied to describe

these vagueness. The notion of equilibrium points was stated and an optimization problem was presented for calculating an equilibrium point. Furthermore, the special case that the type-2 fuzzy numbers are triangular was considered.
As future works, it will be meaningful to consider games with type-2 fuzzy goals containing fuzzy payoffs. Furthermore, the type-2 fuzzy goal programming approach used in this paper can be convenient to describe vague satisfactory of players in other (cooperative and noncooperative) games.

List of abbreviations

2D: Two Dimensional Space
3D: Three Dimensional Space
DHFS: Dual Hesitant Fuzzy Set
DM: Decision Maker
FOU: Footprint Of Uncertainty
GP: Goal Programmi ng
GVSD: game with Vague Satisfactory Degrees
IF: Intuitionistic Fuzzy
LMF: Lower Membership Function
T1FS: Type-1 Fuzzy Set
T2FS: Type-2 Fuzzy Set
UMF: Upper Membership Function
VS: Vertical Slice

Conflict of Interest: The authors declare that they have no conflict of interest.

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