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Research Article

Exploring the precise edge irregularity strength of generalized arithmetic and geometric staircase graphs

Yeni Susanti*, **Muhammad Nurul Huda**†, **Ramadhani Latief Firmansyah**‡

Department of Mathematics, Universitas Gadjah Mada, Indonesia
*yeni_math@ugm.ac.id
†muhammadnurulhuda2000@mail.ugm.ac.id
‡ramadhani.latief1203@mail.ugm.ac.id

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Abstract: In the context of a finite undirected graph ζ, an edge irregular labelling is defined as a labelling of its vertices with some labels in such a way that each edge has a unique weight, which is determined by the sum of the labels of its endpoints. The main objective of this study is to determine the smallest positive integer n for which it is possible to assign a total edge irregular labelling to ζ with n as the biggest label. This investigation focuses particularly on cases where ζ represents the generalized arithmetic and generalized geometric staircase graphs. Within this paper, the definition of generalized geometric staircase graph is proposed. Moreover, we not only establish the edge irregularity strength of these two kind of graphs but also present a method for creating the corresponding edge irregular labelling.

Keywords: irregular labelling, staircase graph, total edge irregularity strength

AMS Subject classification: 05C78, 05C99

1. Introduction

In contemporary times, numerous scholars have expressed keen enthusiasm for graph labelling, leading to a continuous stream of new discoveries in this field each year. This achievement can be attributed not only to the intricate technical aspects of graph labelling but also to its wide array of practical applications, including electron microscopy, X-ray analysis, combinatorial optimization, sonar technology, physics research, signal processing, network design, and system configuration (see [12], [23]). Consider a connected, simple, and undirected graph ζ with V_ζ as its vertex set and E_ζ

* *Corresponding Author*

as its edge set. In this context, labelling refers to the assignment of numbers (typically positive integers) to elements of the graph, either its vertices or edges. When labels are assigned to vertices, it is called vertex labelling, and when labels are assigned to edges, it is known as edge labelling. If labels are assigned to both vertices and edges, it is termed total labelling. For an edge p-labelling $\gamma : E_\zeta \to \{1, 2, \ldots, p\}$ on ζ, the weight associated with a vertex x is determined by adding up the labels of all edges incident to x. In their work presented in reference [6] Chartrand et al. introduced the concept of edge p-labelling ensures that each vertex has a unique weight which were called irregular assignment. The minimum value of p allowing a graph ζ to possess an irregular assignment with labels not surpassing p is referred to as the irregularity strength of ζ, and denoted $s(\zeta)$. Determining the irregularity strength of a graph can be a challenging task, even for relatively simple graphs [4].

In the publication by Baca and Siddiqui, referenced as [5], they introduced the notion of total labelling within the framework of a graph ζ, with additional condition such that each vertex in the graph has a unique particular weight, where the weight of vertex x is the sum of its label and the labels of all edges incident to x. The least possible value for the biggest label that can be assigned under this labelling is considered as the total vertex irregularity strength of the graph, denoted as $tvs(\zeta)$. Additionally, a distinct type of total labelling, referred to as edge irregular total labelling, was introduced. In this labelling, each edge's weight is unique, calculated as the sum of its label and the labels of its end vertices. The smallest value that can serve as the maximum label is termed the total edge irregularity strength of the graph ζ, denoted as $tes(\zeta)$. These two parameters have garnered significant attention from researchers, as evidenced by references [2, 5, 8–11, 13–17, 24]. For a more in-depth survey of this study, one can refer to [7].

Consider a finite connected undirected graph $\zeta = (V_\zeta, E_\zeta)$ with a non-empty finite vertex set V_ζ and edge set E_ζ, a vertex $p-$labelling $\gamma : V_\zeta \to \{1, 2, \ldots, p\}$ is termed an edge irregular labelling if, for any distinct edges xy and $x'y'$ in E_ζ, the sums $\gamma(x) + \gamma(y)$ and $\gamma(x') + \gamma(y')$ are different. The weight of an edge $xy \in E_\zeta$ under labelling γ is denoted as $wt_\gamma(xy)$, calculated as $\gamma(x) + \gamma(y)$. The edge irregularity strength of the graph ζ, denoted as $eis(\zeta)$, is the smallest p for which an edge irregular vertex p-labelling on ζ is possible. Indeed, for more in-depth investigations into this particular invariant, you may find valuable insights in studies, for instance, referenced as [3, 18–22].

The following lemma, as established in [1], provides a lower bound for $eis(\zeta)$.

Lemma 1. [1] *For any graph $\zeta = (V_\zeta, E_\zeta)$, such that the maximum vertex degree of ζ is Δ_ζ, it follows that*

$$eis(\zeta) \geq \max\left\{\left\lceil \frac{|E_\zeta| + 1}{2} \right\rceil, \Delta_\zeta\right\}.$$

This paper presents irregularity strength analyses for both the generalized arithmetic staircase graph and the generalized geometric staircase graph. Let $\alpha, \beta, n \geq 1$ be any positive integers. The generalized arithmetic staircase graph $Arsc(\alpha, \beta, n)$ consisting of n levels and with α initial grids and difference β is a graph with vertex set

$$V_{Arsc(\alpha,\beta,n)} = \{u_{i,j}|0 \leq i \leq \alpha + \beta j, 0 \leq j \leq n-1\} \cup \{u_{i,n}|0 \leq i \leq \alpha + \beta n - b\}$$

and edge set

$$E_{Arsc(\alpha,\beta,n)} = \{u_{i,j}u_{i,j+1}|0 \leq i \leq \alpha + \beta j, 0 \leq j \leq n-1\} \cup$$

$$\{u_{i,j}u_{i+1,j}|0 \leq i \leq \alpha + \beta j - 1, 0 \leq j \leq n-1\} \cup \{u_{i,n}u_{i+1,n}|0 \leq i \leq \alpha + \beta n - \beta - 1\}$$

(see [16]). Clearly, $|E_{Arsc(\alpha,\beta,n)}| = \beta n^2 + (2\alpha+1)n + \alpha - \beta$. As an example, we provide a specific generalized arithmetic staircase graph, which is illustrated in Figure 1.

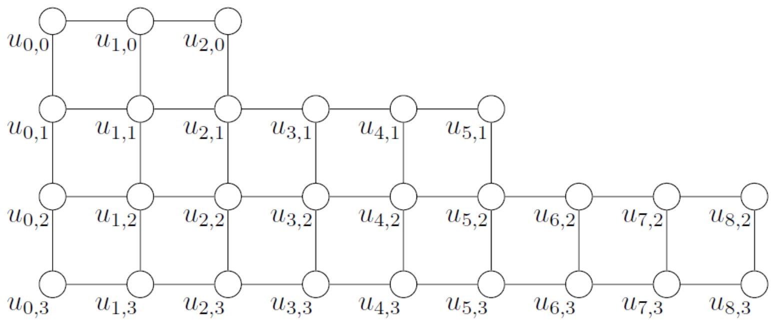

Figure 1: Generalized Arithmetic Staircase Graph $Arsc(2, 3, 3)$

Inspired by the concept of generalized arithmetic staircase graph, we propose the definition of the generalized geometric staircase graph as outlined below. Let $u, r, l \geq 1$ be arbitrary positive integers. By generalized geometric staircase graph $Geosc(u, r, l)$ of l levels with u initial grids and ratio r, we mean a graph with vertex set

$$V_{Geosc(u,r,l)} = \{a_{i,j}|0 \leq i \leq ur^j, 0 \leq j \leq l-1\} \cup \{a_{i,l}|0 \leq i \leq ur^{l-1}\}$$

and edge set

$$E_{Geosc(u,r,l)} = \{a_{i,j}a_{i,j+1}|0 \leq i \leq u, 0 \leq j \leq l-1\} \cup$$

$$\{a_{i,j}a_{i,j+1}|(u+1) \leq i \leq ur^j, 1 \leq j \leq l-1\} \cup \{a_{i,j}a_{i+1,j}|0 \leq i \leq ur^j - 1, 0 \leq j \leq l-1\}$$

$$\cup \{a_{i,n}a_{i+1,n}|0 \leq i \leq ur^{l-1} - 1\}.$$

Clearly, $|E_{Geosc(u,r,l)}| = 2u\left(\frac{r^l-1}{r-1}\right) + ur^{l-1} + l$. As an illustration, we give a particular generalized geometric staircase graph as depicted in Figure 2

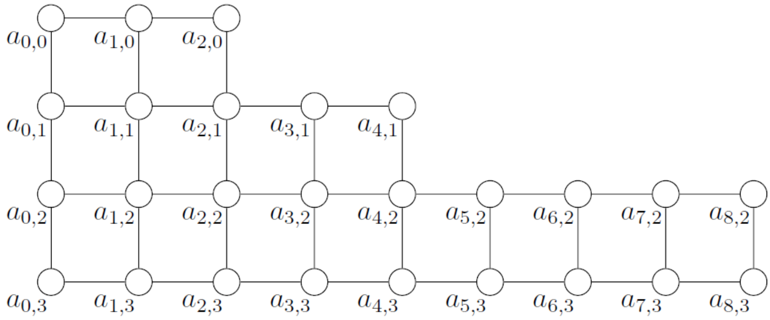

Figure 2: Generalized Geometric Staircase Graph $Geosc(2, 2, 3)$

2. Results

In this section, we find the precise value of the total edge irregularity strength for these graphs by creating the corresponding vertex irregular p-labelling, where the value of p satisfies the lower bound for these graphs. We also provide examples of these graphs, including labelled ones, for better understanding.
It is readily apparent that $\Delta(Arsc(\alpha, \beta, n)) = \Delta(Geosc(u, r, l)) = 4$ and this value remains smaller than or equal to $|E(Arsc(\alpha, \beta, n))|$ and $|E(Geosc(u, r, l))|$. Equality is achieved only when $\alpha = n = u = l = 1$. Therefore, Lemma 1 implies Lemma 2 and Lemma 5.

Lemma 2. *For any positive integers $\alpha, \beta, n \geq 1$, we have*

$$eis(Arsc(\alpha, \beta, n)) \geq \left\lceil \frac{\beta n^2 + (2\alpha + 1)n + \alpha - \beta + 1}{2} \right\rceil.$$

In order to ascertain the precise value of $eis(Arsc(\alpha, \beta, n))$, it is imperative to demonstrate its equivalence to the lower bound. This task can be accomplished through the establishment of an edge-irregular vertex p-labelling scheme for the graph, with p equating to the lower bound.
The subsequent lemma serves as a tool to prove Lemma 4.

Lemma 3. *Let x and y be odd integers. Then $\lceil \frac{x}{2} \rceil + \lceil \frac{y}{2} \rceil = \lceil \frac{x+y}{2} + 1 \rceil$.*

Proof. Let $x = 2m + 1$ and $y = 2n + 1$ for some integers m and n. Then

$$\left\lceil \frac{x}{2} \right\rceil + \left\lceil \frac{y}{2} \right\rceil = \left\lceil \frac{2m+1}{2} \right\rceil + \left\lceil \frac{2n+1}{2} \right\rceil = m + n + 2 = \left\lceil \frac{2(m+n+2)}{2} \right\rceil = \left\lceil \frac{x+y}{2} + 1 \right\rceil.$$

□

The following lemma will be a valuable instrument for ascertaining the lower bound of $eis(Arsc(\alpha, \beta, n))$.

Lemma 4. *For any three positive integers $\alpha, \beta, n \geq 1$, let*

$$\kappa_{i,j} = \begin{cases} 1 + j\alpha + \frac{(j-1)j\beta}{2} + \left\lceil \frac{j-1}{2} \right\rceil + \left\lfloor \frac{i}{2} \right\rfloor, & j \text{ even} \\ 1 + j\alpha + \frac{(j-1)j\beta}{2} + \left\lceil \frac{j-1}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil, & j \text{ odd.} \end{cases}$$

It follows that

$$\kappa_{\alpha+(n-1)\beta,n} = \left\lceil \frac{\beta n^2 + (2\alpha+1)n + \alpha - \beta + 1}{2} \right\rceil.$$

Proof. Let $i = \alpha + (n-1)\beta$ and $j = n$. We consider the following three cases:

a. If n is an odd integer, then

$$\begin{aligned} \kappa_{\alpha+(n-1)\beta,n} &= 1 + n\alpha + \frac{(n-1)n\beta}{2} + \frac{n-1}{2} + \left\lceil \frac{\alpha+(n-1)\beta}{2} \right\rceil \\ &= \left\lceil 1 + n\alpha + \frac{(n-1)n\beta}{2} + \frac{n-1}{2} + \frac{\alpha+(n-1)\beta}{2} \right\rceil \\ &= \left\lceil \frac{\beta n^2 + (2\alpha+1)n + \alpha - \beta + 1}{2} \right\rceil. \end{aligned}$$

b. If n is an even integer and α, β have the same parity, then

$$\begin{aligned} \kappa_{\alpha+(n-1)\beta,n} &= 1 + n\alpha + \frac{(n-1)n\beta}{2} + \left\lceil \frac{n-1}{2} \right\rceil + \left\lfloor \frac{\alpha+(n-1)\beta}{2} \right\rfloor \\ &= 1 + n\alpha + \frac{(n-1)n\beta}{2} + \left\lceil \frac{n-1}{2} \right\rceil + \frac{\alpha+(n-1)\beta}{2} \\ &= \left\lceil 1 + n\alpha + \frac{(n-1)n\beta}{2} + \frac{n-1}{2} + \frac{\alpha+(n-1)\beta}{2} \right\rceil \\ &= \left\lceil \frac{\beta n^2 + (2\alpha+1)n + \alpha - \beta + 1}{2} \right\rceil. \end{aligned}$$

c. If n is an even integer and α, β have distinct parity, then

$$\begin{aligned} \kappa_{\alpha+(n-1)\beta,n} &= 1 + n\alpha + \frac{(n-1)n\beta}{2} + \left\lceil \frac{n-1}{2} \right\rceil + \left\lfloor \frac{\alpha+(n-1)\beta}{2} \right\rfloor \\ &= n\alpha + \frac{(n-1)n\beta}{2} + \left\lceil \frac{n-1}{2} \right\rceil + \left\lceil \frac{\alpha+(n-1)\beta}{2} \right\rceil. \end{aligned}$$

Lemma 3 implies $\left\lceil \frac{n-1}{2} \right\rceil + \left\lceil \frac{\alpha+(n-1)\beta}{2} \right\rceil = \left\lceil \frac{(n-1)+\alpha+(n-1)\beta+2}{2} \right\rceil$. Therefore,

$$\begin{aligned} \kappa_{\alpha+(n-1)\beta,n} &= n\alpha + \frac{(n-1)n\beta}{2} + \left\lceil \frac{(n-1)+\alpha+(n-1)\beta+2}{2} \right\rceil \\ &= \left\lceil n\alpha + \frac{(n-1)n\beta}{2} + \frac{n+1+\alpha+(n-1)\beta}{2} \right\rceil \\ &= \left\lceil \frac{\beta n^2 + (2\alpha+1)n + \alpha - \beta + 1}{2} \right\rceil. \end{aligned}$$

□

The following result shows that the lower bound of $eis(Arsc(\alpha,\beta,n))$ given in Lemma 2 is sharp.

Theorem 1. *Given three positive integers $\alpha,\beta,n \geq 1$. Let $Arsc(\alpha,\beta,n)$ be the generalized arithmetic staircase graph of n levels with α initial grids and difference β. The edge irregularity strength of $Arsc(\alpha,\beta,n)$ is*

$$eis(Arsc(\alpha,\beta,n)) = \left\lceil \frac{\beta n^2 + (2\alpha+1)n + \alpha - \beta + 1}{2} \right\rceil.$$

Proof. To prove that the upper bound meets the lower bound given in Lemma 2, it is constructed a vertex p-labelling $f : V(Arsc(\alpha,\beta,n)) \to \{1,2,\ldots,p\}$ where p is equal to the lower bound given on Lemma 1, defined by

$$f(u_{i,j}) = \kappa_{i,j} = \begin{cases} 1 + j\alpha + \frac{(j-1)j\beta}{2} + \left\lceil \frac{j-1}{2} \right\rceil + \left\lfloor \frac{i}{2} \right\rfloor, & j \text{ even} \\ 1 + j\alpha + \frac{(j-1)j\beta}{2} + \left\lceil \frac{j-1}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil, & j \text{ odd,} \end{cases}$$

as given in Lemma 4. Clearly, f is a non decreasing function and again by Lemma 4, the biggest label is $p = \left\lceil \frac{\beta n^2+(2\alpha+1)n+\alpha-\beta+1}{2} \right\rceil$. This confirms that f is indeed a vertex p-labelling with $p = \left\lceil \frac{\beta n^2+(2\alpha+1)n+\alpha-\beta+1}{2} \right\rceil$. The weight of each edge is indeed as follows:

(i) For j that is even, we have
$wt_f(u_{i,j}u_{i,j+1}) = \kappa_{i,j} + \kappa_{i,j+1} = 1 + j\alpha + \frac{(j-1)j\beta}{2} + \left\lceil \frac{j-1}{2} \right\rceil + \left\lfloor \frac{i}{2} \right\rfloor + 1 + (j+1)\alpha + \frac{j(j+1)\beta}{2} + \left\lceil \frac{j}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil = 2 + (2j+1)\alpha + j^2\beta + j + i.$

(ii) For j that is odd, we have
$wt_f(u_{i,j}u_{i,j+1}) = 1 + j\alpha + \frac{(j-1)j\beta}{2} + \left\lceil \frac{j-1}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil + 1 + (j+1)\alpha + \frac{j(j+1)\beta}{2} + \left\lceil \frac{j}{2} \right\rceil + \left\lfloor \frac{i}{2} \right\rfloor = 2 + (2j+1)\alpha + j^2\beta + j + i.$

(iii) For j that is even, we have
$wt_f(u_{i,j}u_{i+1,j}) = \kappa_{i,j} + \kappa_{i+1,j} = 1 + j\alpha + \frac{(j-1)j\beta}{2} + \left\lceil \frac{j-1}{2} \right\rceil + \left\lfloor \frac{i}{2} \right\rfloor + 1 + j\alpha + \frac{(j-1)j\beta}{2} + \left\lceil \frac{j-1}{2} \right\rceil + \left\lfloor \frac{i+1}{2} \right\rfloor = 2 + 2j\alpha + (j-1)j\beta + j + i.$

(iv) For j that is odd, we have
$wt_f(u_{i,j}u_{i+1,j}) = \kappa_{i,j} + \kappa_{i+1,j} = 1 + j\alpha + \frac{(j-1)j\beta}{2} + \left\lceil \frac{j-1}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil + 1 + j\alpha + \frac{(j-1)j\beta}{2} + \left\lceil \frac{j-1}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil = 2 + 2j\alpha + (j-1)j\beta + j + i.$

It is straightforward to verify that all edge weights with respect to function f are distinct. □

Figure 3 gives an illustration on the establishment of the edge irregular labelling showing the edge irregularity strength of a particular generalized arithmetic staircase graph. The numbers given in blue color are the weight of the edges.

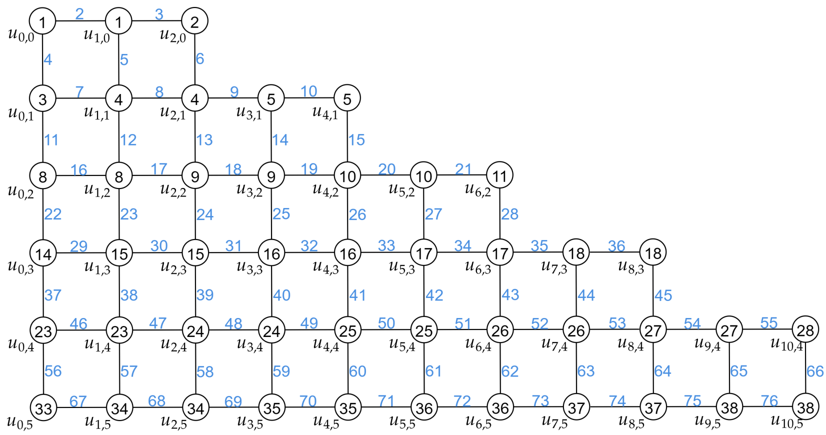

Figure 3: Edge irregular 38-labelling on $Arsc(2, 2, 5)$ with biggest label meets the $eis(Arsc(2, 2, 5))$

Next, we turn our consideration to the second graph in question, which is the generalized geometric staircase graph. The subsequent lemma, which establishes the minimum edge irregularity strength $eis(Geosc(u, r, l))$, can be directly inferred from the definition of the generalized staircase graph and Lemma 1.

Lemma 5. *For any positive integers $u, r, l \geq 1$, it follows that*

$$eis(Geosc(u, r, l)) \geq \left\lceil \frac{2u\left(\frac{r^l-1}{r-1}\right) + ur^{l-1} + l + 1}{2} \right\rceil .$$

The following result demonstrates that the lower bound for $eis(Geosc(u, r, l))$ corresponds to the actual value of $eis(Geosc(u, r, l))$.

Theorem 2. *Let $u, r, l \geq 1$ be natural numbers. Then the edge irregularity strength of $Geosc(u, r, l)$ is*

$$eis(Geosc(u, r, l)) = \left\lceil \frac{2u\left(\frac{r^l-1}{r-1}\right) + ur^{l-1} + l + 1}{2} \right\rceil .$$

Proof. To prove the equality, we construct a function $f : V(Geosc(u,r,l)) \to \{1,2,\ldots,p\}$ with $p = \left\lceil \frac{2u\left(\frac{r^l-1}{r-1}\right)+ur^{l-1}+l+1}{2} \right\rceil$, defined by

$$\begin{aligned}
f(a_{i,0}) &= \left\lceil \frac{i+1}{2} \right\rceil, \quad 0 \le i \le u \\
f(a_{0,j}) &= \left\lfloor \frac{\omega_{j-1}+ur^{j-1}+1}{2} \right\rfloor, \quad 1 \le j \le l, \\
f(a_{i,j}) &= \left\lceil \frac{\omega_{j-1}+ur^{j-1}+i}{2} \right\rceil, \quad 1 \le i \le ur^j,\ 1 \le j \le l-1, \\
f\left(a_{i,l}\right) &= \left\lceil \frac{\omega_{l-1}+ur^{l-1}+i}{2} \right\rceil, \quad 1 \le i \le ur^{l-1},
\end{aligned}$$

where $\omega_j = 2u(\frac{r^j-1}{r-1}) + ur^j + j + 2$. Clearly, f is a non-decreasing function and the largest label is $p = \left\lceil \frac{2u\left(\frac{r^l-1}{r-1}\right)+ur^{l-1}+l+1}{2} \right\rceil$. This shows that f is a vertex p-labelling. Moreover, the weight of each edge is indeed as follows:

(i) For $0 \le i \le ur^j$ and $0 \le j \le l-1$, we have

$$wt_f(a_{i,j}a_{i,j+1}) = \left\lceil \frac{\omega_{j-1}+\omega_j+u(r^{j-1}+r^j)+2i}{2} \right\rceil = \left\lceil \frac{2u\left(\frac{r^{j+1}+r^j-2}{r-1}\right)+2(i+j)+3}{2} \right\rceil.$$

(ii) For $0 \le j \le l$, we have

$$wt_f(a_{0,j}a_{1,j}) = \omega_{j-1} + ur^{j-1} + 1 = 2u\left(\frac{r^j-1}{r-1}\right) + j + 2.$$

(iii) For $1 \le i \le ur^{l-1} - 1$ and $0 \le j \le l-1$, we have

$$wt_f(a_{i,j}a_{i+1,j}) = \left\lceil \frac{2\omega_{j-1}+2ur^{j-1}+2i+1}{2} \right\rceil = \left\lceil \frac{4u\left(\frac{r^j-1}{r-1}\right)+2(i+j)+3}{2} \right\rceil.$$

(iv) For $1 \le i \le ur^{l-1} - 1$, we have

$$wt_f(a_{i,l}a_{i+1,l}) = \left\lceil \frac{2\omega_{l-1}+2ur^{l-1}+2i+1}{2} \right\rceil = \left\lceil \frac{4u\left(\frac{r^l-1}{r-1}\right)+2(i+l)+3}{2} \right\rceil.$$

This completes the proof. □

We close this discussion by providing Figure 4 as an illustration on the construction of an edge irregular labelling, showcasing the edge irregularity strength of a specific generalized geometric staircase graph. The numbers highlighted in blue represent the edge weights.

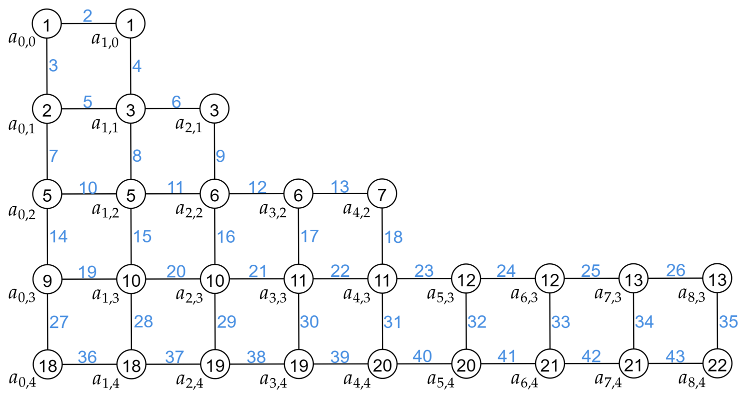

Figure 4: Edge irregular 22-labelling on $Geosc(1, 2, 4)$ with biggest label meets the $eis(Geosc(1, 2, 4))$

3. Conclusion

In this paper, we considered the concept of edge irregularity strength, which is a variant of the established irregularity strength, total edge irregularity strength, and total vertex irregularity strength. We successfully determined the exact values for the edge irregularity strength of specific graphs, including the generalized arithmetic staircase graphs and generalized geometric staircase graphs. It is worth noting that uncovering the precise edge irregularity strength for families of graphs remains a formidable challenge.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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