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Research Article

Strong k-transitive oriented graphs with large minimum degree

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Abstract: A digraph $D = (V, E)$ is k-*transitive* if for any directed uv-path of length k, we have $(u, v) \in E$. In this paper, we study the structure of strong k-transitive oriented graphs having large minimum in- or out-degree. We show that such oriented graphs are *extended cycles*. As a consequence, we prove that Seymour's Second Neighborhood Conjecture (SSNC) holds for k-transitive oriented graphs for $k \leq 11$. Also we confirm Bermond–Thomassen Conjecture for k-transitive oriented graphs for $k \leq 11$. A characterization of k-transitive oriented graphs having a hamiltonian cycle for $k \leq 6$ is obtained immediately.

Keywords: k-transitive digraph, extended cycle, minimum degree.

AMS Subject classification: 05C20, 05C38, 05C75

1. Introduction

A digraph D is an ordered pair of two disjoint sets (V, E), where V is non-empty and $E \subset V \times V$. The set V is called the vertex set of D and is denoted by $V(D)$, while E is called the arc set of D and is denoted by $E(D)$. All the digraphs in this paper are finite and without loops (i.e. V is finite and for all $v \in V$, we have $(v, v) \notin E$). An arc (u, v) of D is symmetric if (v, u) is also an arc of D. An oriented graph D is an asymmetric digraph (with no symmetric arcs). We may write $u \to v$ and we say that u dominates v, meaning that $(u, v) \in E(D)$. We may write $u \nrightarrow v$ if u does not dominate v. The out-neighborhood of a vertex v, denoted $N_D^+(v)$, is defined as $N_D^+(v) = \{u \in V(D) \colon v \to u\}$. The second out-neighborhood of v, denoted $N_D^{++}(v)$, is defined as $N_D^{++}(v) = \{w \in V(D) \setminus N_D^+(v) \colon \exists x \in N_D^+(v), x \to w\}$. The out-degree of v is $d_D^+(v) = |N_D^+(v)|$ and its second out-degree is $d_D^{++}(v) = |N_D^{++}(v)|$. Let δ_D^+ (or δ^+) denote the minimum out-degree in D. Analogously, we define in-neighborhood, second in-neighborhood, in-degree, second in-degree and minimum in-degree. We omit the subscript when it is clear from the context. A *tournament* is an oriented graph where between any two vertices there is an arc. A *regular* n-tournament is a tournament on n vertices where n is an odd integer and every vertex has in- and out-degree $\frac{n-1}{2}$. A vertex with out-degree 0 is called a *sink*. We denote by $x_0x_1\dots x_k$ a directed x_0x_k-path of length k and we may write $x_0 \to x_1 \to \cdots \to x_k$. A directed k-cycle (C_k) is denoted by $x_0\dots x_{k-1}x_0$, and we may write $x_0 \to \cdots \to x_{k-1} \to x_0$. Throughout this paper, a path (respectively cycle) means a directed path (respectively cycle). For a path or a cycle $W = x_0x_1\dots x_k$ (the subscripts are taken modulo k if W is a cycle) we denote by x_iWx_j the subpath of W from x_i to x_j; that is $x_iWx_j = x_ix_{i+1}\dots x_j$. The length of a path (or a cycle) W is denoted by $\ell(W)$. An *acyclic* digraph is a digraph with no cycle. An *extended k-cycle*, denoted by $C[X_0,\dots,X_{k-1}]$, is obtained from a k-cycle $C = x_0\dots x_{k-1}x_0$ by replacing x_i by an independent vertex set X_i for all $i \in \{0,\dots,k-1\}$ such that every vertex in X_i dominates every vertex in X_{i+1} (subscripts taken modulo k). Figure 1 provides an example of an extended 3-cycle.

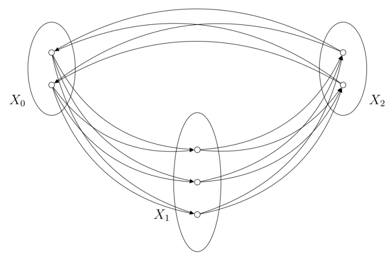

Figure 1. **An extended 3-cycle**

A digraph is *strongly connected* (or *strong*) if for every pair of vertices u and v, there exists a uv-directed path. A *strong component* of D is a maximal strong subdigraph of D. The *condensation* of D is the digraph D^* with $V(D^*)$ equals to the set of all strong components of D, and $(S,T) \in E(D^*)$ if and only if there is $(s,t) \in E(D)$ such that $s \in S$ and $t \in T$. Clearly, D^* is an acyclic digraph, and thus, it has a vertex of out-degree zero and a vertex of in-degree zero. A *terminal component* of D is a strong component T of D such that $d^+_{D^*}(T) = 0$. An *initial component* of D is a strong component I of D such that $d^-_{D^*}(I) = 0$.

A digraph D is called *transitive* if for any directed path $x_0x_1x_2$ of length 2 in D, we have $(x_0,x_2) \in E(D)$. In 2012, Galena-Sánchez and Hernández-Cruz [19] introduced the class of k-transitive digraphs as a generalization of transitive digraphs. We say that D is a *k-transitive digraph* if for every $u,v \in V(D)$, the existence of a directed uv-path of length k implies $(u,v) \in E(D)$. Since their introduction, k-transitive digraphs have received a fair amount of attention (see [12]). Strong k-transitive digraphs have been characterized for $k \in \{3,4\}$ by Hernández-Cruz [17, 18]. For $k > 4$, there are no known structural characterizations for strong k-transitive digraphs. However, there is some information about strong k-transitive digraphs for arbitrary k. For instance, Hernández-Cruz and Montellano-Ballesteros [20] characterized strong

k-transitive digraphs (general digraphs) having a cycle of length at least k.

Theorem 1. *[20] Let k be an integer, $k \geq 2$. Let D be a strong k-transitive digraph. Suppose that D contains a directed cycle of length n such that the greatest common divisor of n and $k-1$ is equal to d and $n \geq k+1$. Then the following hold:*

1. *If $d = 1$, then D is a complete digraph (that is for all x, $y \in V(D)$, we have $x \to y$ and $y \to x$).*
2. *If $d \geq 2$, then D is either a complete digraph, a complete bipartite digraph, or an extended d-cycle.*

Theorem 2. *[20] Let k be an integer, $k \geq 2$. Let D be a strong k-transitive digraph of order at least $k+1$. If D contains a directed cycle of length k, then D is a complete digraph.*

For oriented graphs, we can easily reformulate Theorems 1 and 2 as the following result.

Theorem 3. *[20] Let k be an integer such that $k \geq 3$. Let D be a strong k-transitive oriented graph of order at least $k+1$. If D contains a directed cycle of length greater than $k-1$, then D is an extended cycle.*

It is customary to consider k-transitive oriented graphs. In terms of forbidden (not necessarily induced) subdigraphs, k-transitive oriented graphs are oriented graphs with forbidden P^*_{k+1} and C_{k+1}, where P^*_{k+1} is a uv-path on $k+1$ vertices such that u and v are not adjacent. Showing that a conjecture holds on k-transitive oriented graphs, we get some information about a counterexample to this conjecture; which is that every counterexample must contain P^*_{k+1} or C_{k+1}.
In view of Theorem 3, it is convenient to find a sufficient condition for a strong k-transitive oriented graph to have a cycle of length greater than $k-1$. A relation between δ^+_D (or δ^-_D) and the length of a longest cycle in an oriented graph D is given by the following classical result.

Lemma 1. *[21] Every oriented graph D contains a directed cycle of length a least $\delta^+ + 2$.*

The bound given in Lemma 1 is best possible. Indeed, Jackson [21] constructed a family of oriented graphs that contain no directed cycles of length greater than $\delta^+ + 2$. An example of such graphs is given in Figure 2.
By Lemma 1, every strong k-transitive oriented graph with δ^+ (or δ^-) at least $k-2$ has a cycle of length at least k and hence, by Theorem 3, it has the structure of an extended cycle. The aim of this work is to improve this bound.

In this paper, we study strong k-transitive oriented graphs with $\max\{\delta^-, \delta^+\}$ at least $k-4$. We show that in most cases their structures are extended cycles. Our first result is given in the following theorem.

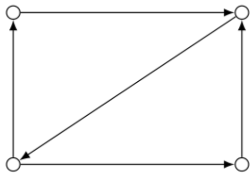

Figure 2. **An oriented graph with $\delta^+ = 1$ and longest cycle of length 3**

Theorem 4. *Let D be a strong k-transitive oriented graph.*

1. *If $k = 5$ and $\max\{\delta^-, \delta^+\} \geq 2$, then D is a regular 5-tournament or an extended cycle.*
2. *If $k = 6$ and $\max\{\delta^-, \delta^+\} \geq 3$, then D is an extended cycle.*
3. *If $k = 7$ and $\max\{\delta^-, \delta^+\} \geq 3$, then D is a regular 7-tournament or an extended cycle.*
4. *If $k \geq 8$ and $\max\{\delta^-, \delta^+\} \geq k - 4$, then D is an extended cycle.*

Note that in case of 2-transitive oriented graphs, such an oriented graph has no cycles. For $k \in \{3, 4\}$, the description is easy. In [8], we have the following result.

Proposition 1. *[8] If D is a 3-transitive oriented graph, then $\delta^+ \leq 1$. (Since the converse of D is 3-transitive, we get also $\delta^- \leq 1$).*

If D is a strong 4-transitive oriented graph with $\max\{\delta^-, \delta^+\} \geq 2$, then D has a cycle of length at least 4 and $|V(D)| \geq 5$. Hence, by Theorem 3 D is an extended cycle. One can think about the least integer $f(k)$ such that if $\max\{\delta^-, \delta^+\} \geq f(k)$, then every strong k-transitive oriented graph is an extended cycle. We conjecture the following.

Conjecture 1.1. Let D be a strong k-transitive oriented graph having at least $k+1$ vertices. If $\max\{\delta^-, \delta^+\} \geq \frac{k-1}{2}$, then D is an extended cycle.

In Section 2, we prove Theorem 4. In Section 3, we use the characterizations given in Theorem 4 to immediately prove Seymour's second neighborhood conjecture and Bermond–Thomassen conjecture for some cases of k-transitive oriented graphs as well as to obtain a characterization of k-transitive oriented graphs having a hamiltonian cycle for $k \leq 6$.

2. Main result

Lemma 2. *Let D be a strong k-transitive digraph having two disjoint cycles C_m and C_n of lengths at most $k - 1$. If $m + n \geq k + 1$, then we have the following properties.*

1. *Each vertex in C_m dominates some vertex in C_n and vice versa.*

2. *Each vertex in C_m is dominated by some vertex in C_n and vice versa.*

3. *There exists a cycle in D of length greater than* $\max\{m, n\}$.

Proof. 1. Set $C_m = x_0 \cdots x_{m-1}x_0$ and $C_n = y_0 \cdots y_{n-1}y_0$. Since D is strong, there is a path P from x_i to y_j for some i and j such that $V(P) \cap V(C_m) = \{x_i\}$ and $V(P) \cap V(C_n) = \{y_j\}$. We may assume that $\ell(P) < k$, otherwise we can find, by k-transitivity, a path of length at most $k-1$ from x_i to y_j. As $m + n \geq k + 1$, there exist an integer $s \in \{0, \ldots, m-1\}$ and an integer $t \in \{0, \ldots, n-1\}$ such that $x_s C_m x_i P y_j C_n y_t$ is a path of length k. Hence $x_s \to y_t$. Clearly, there exists an integer $r \in \{0, \ldots, n-1\}$ such that $y_t C_n y_r$ is a path of length $k - m$ since $m + n \geq k + 1$. Now, $x_{s+1} C_m x_s y_t C_n y_r$ is a path of length k implying that $x_{s+1} \to y_r$. Therefore, by induction, each vertex in C_m dominates some vertex in C_n. Similarly, we show that each vertex in C_n dominates some vertex in C_m. 2. Let $x \in V(C_m)$. Let D' be the converse of D. Note that D' is also k-transitive. By 1., there exists $y \in V(C_n)$ such that $(x, y) \in E(D')$. Hence $(y, x) \in E(D)$. Similarly, we show that each vertex in C_n is dominated by some vertex in C_m.
3. We can assume that $m \geq n$. Without loss of generality, by 1. and 2., assume that $x_0 \to y_0$ and $y_i \to x_1$ for some $i \in \{0, \ldots, n\}$. So $x_0 y_0 C_n y_i x_1 C_m x_0$ is a cycle of length at least $m + 1$. □

Lemma 3. *Let D be a strong k-transitive oriented graph with $k \geq 5$. If $\delta^+ = k - 3$, then D has a cycle of length greater than $k - 1$.*

Proof. Suppose to the contrary that the length of a longest cycle in D is at most $k-1$. Since $\delta^+ = k - 3$, there exists a cycle of length at least $k - 1$. Hence a longest cycle in D has length $k - 1$. Let $C = x_0 \cdots x_{k-2} x_0$ be a longest cycle in D. Set $H = D - C$. The oriented graph H is acyclic since otherwise, by Lemma 2, there exists a cycle of length greater than $k - 1$ in D, which is a contradiction. Let u be a sink in H. Since D is strong k-transitive and $\ell(C) = k - 1$, there exists some $x_i \in V(C)$ such that $x_i \to u$. We may assume that $x_0 \to u$. Note that $N^+(u) \subseteq V(C)$. We have $u \nrightarrow x_1$ since otherwise $x_0 u x_1 C x_0$ is a cycle of length k, which is a contradiction. Hence $N^+(u) = \{x_2, \ldots, x_{k-2}\}$ since $\delta^+ = k - 3$ and $N^+(u) \subseteq V(C)$. Let $i \in \{1, \ldots, k-3\}$. We have $x_i \nrightarrow y$ for all $y \in V(H)$ because otherwise $u x_{i+1} C x_i y$ is a path of length k implying that $u \to y$, which contradicts the fact that u is a sink in H. Hence $N^+(x_i) \subseteq V(C)$. It follows that $N^+(x_i) = V(C) \setminus \{x_{i-1}, x_i\}$ for all $i \in \{1, \ldots, k-3\}$. For $k \geq 6$, we get $N^+(x_1) = V(C) \setminus \{x_0, x_1\}$ and $N^+(x_2) = V(C) \setminus \{x_1, x_2\}$ as well as $N^+(x_3) = V(C) \setminus \{x_2, x_3\}$. Thus $x_1 \to x_3$ and $x_3 \to x_1$, which is a contradiction. It is easy to check that the case for $k = 5$ also leads to a contradiction. In fact, for $k = 5$, we must have $N^+(x_1) = \{x_2, x_3\}$ and $N^+(x_2) = \{x_0, x_3\}$. Hence x_3 must have some out-neighbor $w \in V(H)$ since $d^+(x_3) \geq 2$. Now $u x_2 x_0 x_1 x_3 w$ is a path of length 5 implying that $u \to w$, a contradiction.

This proves that a longest cycle in D must have length greater than $k-1$. □

Lemma 4. *Let D be a strong k-transitive oriented graph with $k \geq 7$. If $\delta^+ = k-4$, then D has a cycle of length greater than $k-1$.*

Proof. Suppose to the contrary that the length of a longest cycle in D is at most $k-1$. Since $\delta^+ = k-4$, there exists a cycle of length at least $k-2$. Let C be a longest cycle in D. Hence the length of C is $k-2$ or $k-1$. Set $H = D - C$. By Lemma 2, we must have that H is an acyclic digraph since otherwise there will be a cycle of length greater than $\ell(C)$, which is a contradiction.

Case 1. $\ell(C) = k-2$.
Set $C = x_0 \cdots x_{k-3} x_0$.

Claim 1. There is a sink x in H and there is $x_i \in V(C)$ such that $x_i \to x$.
Proof of Claim 1. Let y be a sink in H. Since D is strong, there exists a path P from x_i to y for some $i \in \{0, \ldots, k-3\}$. We may assume that $P \cap C = \{x_0\}$. If $\ell(P) \geq 3$, then there is a path of length k from some $x_i \in V(C)$ to y. Hence $x_i \to y$. If $\ell(P) = 1$, then there is nothing to prove. Assume now that $\ell(P) = 2$. Set $P = x_0 w y$. We have $y \nrightarrow x_i$ for each $i \in \{1, 2\}$ since otherwise $x_0 w y x_i C x_0$ is a cycle of length greater than $\ell(C)$, which is a contradiction. It follows that $N^+(y) = V(C) \setminus \{x_1, x_2\}$ since $\delta^+ = k-4$ and y is a sink in H. Assume that $N^+(x_2) \subseteq V(C)$ and $N^+(x_3) \subseteq V(C)$. Hence $N^+(x_2) = V(C) \setminus \{x_1, x_2\}$ and $N^+(x_3) = V(C) \setminus \{x_2, x_3\}$ since $\delta^+ = k-4$. Now $y x_3 x_1 x_2 x_4 C x_0 w y$ is a cycle of length greater than $\ell(C)$, which is a contradiction. Thus there exists, outside C, some out-neighbor of x_2 or x_3. It is easy to show that if $x_2 \to x$ for some $x \in V(H)$, then x is a sink in H. In fact, suppose that there is $x' \in V(H)$ such that $x \to x'$. We have $x' \neq y$ since otherwise $x_0 x_1 x_2 x y x_3 C x_0$ is a cycle of length greater than $\ell(C)$, which is a contradiction. Now $y x_3 C x_2 x x'$ is a path of length k implying that $y \to x'$, which is a contradiction. Thus x must be a sink in H. Similarly, we show that if $x_3 \to x$ for some $x \in V(H)$, then x must be a sink in H. ♦

In view of Claim 1, we may assume that $x_0 \to x$ where x is a sink in H. So we have $N^+(x) = V(C) \setminus \{x_0, x_1\}$. We will show that $N^+(x_2) \subseteq V(C)$. On the contrary, suppose that there exists $x' \in V(H)$ such that $x_2 \to x'$. Clearly, $x' \nrightarrow x$ since otherwise $x' x x_3 C x_2 x'$ is a cycle of length greater than $\ell(C)$, which is a contradiction. Also, $x' \nrightarrow y$ for all $y \in V(H)$ because otherwise $x x_3 C x_2 x' y$ is a path of length k implying that $x \to y$, which is a contradiction. It follows that x' is a sink in H, and hence $N^+(x') = V(C) \setminus \{x_2, x_3\}$. We must have $x_1 \nrightarrow x_3$ since otherwise $x x_2 x' x_1 x_3 C x_0 x$ is a cycle of length greater than $\ell(C)$, which is a contradiction. As $d^+(x_1) \geq k-4$, there exists $y \in V(H)$ such that $x_1 \to y$. Clearly, we have $y \neq x$ since otherwise $x_0 x_1 x x_2 C x_0$ is a cycle of length greater than $\ell(C)$, which is a contradiction. Also, $y \nrightarrow y'$ for all $y' \in V(H)$ because otherwise $x x_2 C x_1 y y'$ is a path of length k implying that $x \to y'$, which is a contradiction. It follows that y is a sink in H, and hence $N^+(y) = V(C) \setminus \{x_1, x_2\}$. Now $x x_2 x' x_1 y x_3 C x_0 x$ is a cycle of length greater than $\ell(C)$, which is a contradiction. We conclude that such an x' does not exist. Therefore

$N^+(x_2) = V(C) \setminus \{x_1, x_2\}$. Suppose that x_1 has some out-neighbor y outside of C. As before, y must be a sink in H. Hence $N^+(y) = V(C) \setminus \{x_1, x_2\}$. If $x_{k-3} \to w$ for some $w \in V(H)$, then $xx_2x_0x_1yx_3Cx_{k-3}w$ is a path of length k. Hence $x \to w$, which is a contradiction. It follows that $N^+(x_{k-3}) = V(C) \setminus \{x_{k-4}, x_{k-3}\}$. Now $xx_3Cx_{k-3}x_1x_2x_0x$ is a cycle of length greater than $\ell(C)$, which is a contradiction. Suppose now that $N^+(x_1) \subseteq V(C)$. So $N^+(x_1) = V(C) \setminus \{x_0, x_1\}$. As $x_1 \to x_{k-3}$ and $\delta^+ = k-4$, there exists $w \in V(H)$ such that $x_{k-3} \to w$. If $w \to x$, then $xx_2x_0x_1x_3Cx_{k-3}wx$ is a cycle of length k, a contradiction. If $w \to w'$ for some w' in H, then $xx_2x_0x_1x_3Cx_{k-3}ww'$ is a path of length k, and hence $x \to w'$, a contradiction. It follows that w must be a sink in H. Thus $N^+(w) = V(C) \setminus \{x_0, x_{k-3}\}$. Now $xx_3Cx_{k-3}wx_1x_2x_0x$ is a cycle of length greater than $\ell(C)$, which is a contradiction.

Case 2. $\ell(C) = k-1$.

Set $C = x_0 \cdots x_{k-2}x_0$. Let x be a sink in H. It is clear that there exists $x_i \in V(C)$ such that $x_i \to x$ since $\ell(C) = k-1$ and D is a strong k-transitive oriented graph. Assume, without loss of generality, that $x_0 \to x$. Note that $x \nrightarrow x_1$ since otherwise there will be a cycle of length greater than $\ell(C)$.

We claim that if $x \to x_i$ for some $i \in \{2, \ldots, k-2\}$, then $N^+(x_{i-1}) \subseteq V(C)$. Indeed, let $x \to x_i$ for some $i \in \{2, \ldots, k-2\}$. We have xx_iCx_{i-1} is a path of length $k-1$. Hence $x_{i-1} \nrightarrow x$ since otherwise there will be a cycle of length k, which is a contradiction. Also, $x_{i-1} \nrightarrow y$ for all $y \in V(H)$ since otherwise there will be a path of length k from x to y implying that $x \to y$, which is a contradiction. Thus $N^+(x_{i-1}) \subseteq V(C)$ as claimed.

Subcase 2.1. $x \nrightarrow x_2$.

In this case, we have $N^+(x) = V(C) \setminus \{x_0, x_1, x_2\}$. We will prove that $N^+(x_{k-2}) \subseteq V(C)$. On the contrary, suppose that there exists $y \in V(H)$ such that $x_{k-2} \to y$. By the above claim, for all $i \in \{2, \ldots k-3\}$, we have $N^+(x_i) \subseteq V(C)$. If $x_2 \to x_4$, then $N^+(x_4) = V(C) \setminus \{x_2, x_3, x_4\}$ since $d^+(x_4) \geq k-4$ and $N^+(x_4) \subseteq V(C)$. Hence $x_4 \to x_0$. So $x_1x_2x_3x_4x_0xx_5Cx_{k-2}y$ is a path of length k implying that $x_1 \to y$. If $x_2 \nrightarrow x_4$, then we must have $N^+(x_2) = V(C) \setminus \{x_1, x_2, x_4\}$, and hence $x_2 \to x_0$. Now $x_1x_2x_0xx_3Cx_{k-2}y$ is a path of length k implying that $x_1 \to y$. We have $y \nrightarrow x$ since otherwise yxx_3Cx_1y is a cycle of length k, which is a contradiction. Also $y \nrightarrow y'$ for all $y' \in V(H)$ because otherwise xx_3Cx_1yy' is a path of length k implying that $x \to y'$, which is a contradiction. It follows that y is a sink in H. Clearly, we have $y \nrightarrow x_0$ and $y \nrightarrow x_2$ since otherwise there is a cycle $yx_0Cx_{k-2}y$ or $x_0x_1yx_2Cx_0$ of length k, which is a contradiction. Thus $N^+(y) \subseteq V(C) \setminus \{x_{k-2}, x_0, x_1, x_2\}$, and therefore $d^+(y) \leq k-5 < \delta^+$, which is a contradiction. We conclude that $N^+(x_{k-2}) \subseteq V(C)$. Now we will show that $x_{k-2} \nrightarrow x_1$. Suppose to the contrary that $x_{k-2} \to x_1$. If $x_2 \to x_4$, then $N^+(x_4) = V(C) \setminus \{x_2, x_3, x_4\}$, and hence $x_4 \to x_0$. So $x_{k-2}x_1x_2x_3x_4x_0xx_5Cx_{k-2}$ is a cycle of length k, which is a contradiction. If $x_2 \nrightarrow x_4$, then we must have $N^+(x_2) = V(C) \setminus \{x_1, x_2, x_4\}$, and hence $x_2 \to x_0$. Now $x_{k-2}x_1x_2x_0xx_3Cx_{k-2}$ is a cycle of length k, which is a contradiction. Thus $x_{k-2} \nrightarrow x_1$, and therefore $N^+(x_{k-2}) = V(C) \setminus \{x_{k-3}, x_{k-2}, x_1\}$. Hence $x_{k-2} \to x_3$, and thereby $N^+(x_3) = V(C) \setminus \{x_{k-2}, x_2, x_3\}$. So $x_3 \to x_1$. As $x_{k-2} \to x_2$, this forces $x_2 \to x_0$. Now

$x_3x_1x_2x_0xx_4Cx_{k-2}x_3$ is a cycle of length k, which is a contradiction.

Subcase 2.2. $x \to x_2$.

Let us show that $x_i \nrightarrow x_1$ for all $i \in \{3, \ldots, k-2\}$. Note that $N^+(x_1) \subseteq V(C)$ since $x \to x_2$. If $x_i \to x_1$ for some $i \in \{3, \ldots, k-3\}$, then $N^+(x_1) = V(C) \setminus \{x_0, x_1, x_i\}$. Hence $x_1 \to x_{i+1}$. Now $x_0xx_2Cx_ix_1x_{i+1}Cx_0$ is a cycle of length k, which is a contradiction. Thus $x_i \nrightarrow x_1$ for all $i \in \{3, \ldots, k-3\}$. Since $\delta^+ = k-4 \geq 7$, there exists $s \in \{4, \ldots, k-2\}$ such that $x \to x_s$. Hence $N^+(x_{s-1}) \subseteq V(C)$. Note that $x_{s-1} \nrightarrow x_1$ as $s-1 \in \{3, \ldots, k-3\}$. This forces $x_{s-1} \to x_0$. If $x_{k-2} \to x_1$, then $x_{k-2}x_1Cx_{s-1}x_0xx_sCx_{k-2}$ is a cycle of length k, which is a contradiction. Thus $x_{k-2} \nrightarrow x_1$. Therefore $x_i \nrightarrow x_1$ for all $i \in \{3, \ldots, k-2\}$. It is easy to show that $N^+(x_{k-2}) \subset V(C)$. In fact, if $x_{k-2} \to y$ for some $y \in V(H)$, then $x_1Cx_{s-1}x_0xx_sCx_{k-2}y$ is a path of length k. This gives $x_1 \to y$, which contradicts $N^+(x_1) \subseteq V(C)$. As $x_{k-2} \nrightarrow x_1$, we have $N^+(x_{k-2}) = V(C) \setminus \{x_{k-3}, x_{k-2}, x_1\}$. And as $x_3 \nrightarrow x_1$, we have $N^+(x_3) = V(C) \setminus \{x_1, x_2, x_3\}$. Hence $x_{k-2} \to x_3$ and $x_3 \to x_{k-2}$, which is a contradiction.

This proves that a longest cycle in D must have length greater than $k-1$. □

Note that we can replace δ^+ by δ^- in the statements of Lemmas 3 and 4 since the converse of D is also a k-transitive digraph.

Now, we are ready to prove Theorem 4.

Proof of Theorem 4. For $\max\{\delta^-, \delta^+\} \geq k-2$, it is clear that D has a cycle of length at least k. Hence, what remains is to check the cases $\max\{\delta^-, \delta^+\} = k-3$ and $\max\{\delta^-, \delta^+\} = k-4$. It is well-known that $|V(D)| \geq 2\max\{\delta^-, \delta^+\} + 1$.

1. For $k = 5$. If $|V(D)| = 5$ and $\max\{\delta^-, \delta^+\} \geq 2$, then we must have $\max\{\delta^-, \delta^+\} = 2$. Thus D is a regular 5-tournament. If $|V(D)| \geq 6$ and $\max\{\delta^-, \delta^+\} \geq 2$, then D has a cycle of length at least 5 by Lemma 3. Hence, by Theorem 3, D is an extended cycle.

2. For $k = 6$. By Lemma 3, D has a cycle of length at least 6. As $\max\{\delta^-, \delta^+\} \geq 3$, we must have $|V(D)| \geq 7$. Thus, by Theorem 3, D is an extended cycle.

3. For $k = 7$. If $|V(D)| = 7$ and $\max\{\delta^-, \delta^+\} \geq 3$, then we must have $\max\{\delta^-, \delta^+\} = 3$. Thus D is a regular 7-tournament. If $|V(D)| \geq 8$ and $\max\{\delta^-, \delta^+\} \geq 3$, then D has a cycle of length at least 7 by Lemmas 3 and 4. Hence, by Theorem 3, D is an extended cycle.

4. For $k \geq 8$. By Lemmas 3 and 4, D has a cycle of length at least k. As $\max\{\delta^-, \delta^+\} \geq k-4$ and $k \geq 8$, we must have $|V(D)| \geq k+1$. Thus, by Theorem 3, D is an extended cycle. □

3. Applications to some problems

3.1. Seymour's Second Neighborhood Conjecture

We say that v is a Seymour vertex if $d^{++}(v) \geq d^+(v)$. In 1990, Paul Seymour proposed the following conjecture.

Conjecture 3.1 (SSNC). In every finite oriented graph, there exists a Seymour vertex.

The first non-trivial case of SSNC was proved in 1996 by Fisher [11] for the class of tournaments. Since then, SSNC was proven only for some very specific classes of oriented graphs (e.g. [1, 5, 6, 9, 10, 14–16, 22]).
In 2001, Kaneko and Locke proved the following result.

Theorem 5. *[22] Let D be an oriented graph. If $\delta^+ \leq 6$, then D has a Seymour vertex.*

In 2017, García-Vásquez and Hernández-Cruz [13] proved SSNC for 4-transitive oriented graphs using a characterization of strong 4-transitive digraphs. Recently, in [8], SSNC has been proved, by combinatorial methods, for k-transitive oriented graphs for $k \leq 6$. It is seen that the difficulty of SSNC for k-transitive digraphs is increasing with respect to k, but the existence of characterizations, descriptions or properties for such class of digraphs, may reduce the difficulty of the problem. For instance, using the characterization given by Hernández-Cruz and Montellano-Ballesteros [20], SSNC has been proved in [7] for k-transitive oriented graphs for $k \leq 9$. Here, we use the characterization obtained in Theorem 4 to confirm SSNC for k-transitive oriented graphs for $k \in \{10, 11\}$.
We need the following two lemmas.

Lemma 5. *[7] Let D be an oriented graph. Let T be a terminal strong component of D. If v is a Seymour vertex in the subdigraph $D[T]$ induced by T, then v is a Seymour vertex in D.*

Proof. For all $x \in T$, we have $N_T^+(x) = N_D^+(x)$ since T is a terminal strong component of D. Hence $d_T^+(v) = d_D^+(v)$ and $d_T^{++}(v) = d_D^{++}(v)$. □

Lemma 6. *[7] If n is an integer at least 3, then every extended n-cycle $C[V_0, V_1, ..., V_{n-1}]$ has at least two Seymour vertices.*

Proof. Let V_i be a smallest set of the partition $\{V_0, V_1, ..., V_{n-1}\}$, that is $|V_i| \leq |V_j|$ for all $0 \leq j \leq n-1$. Note that for all $0 \leq j \leq n-1$, we have $|V_j| \geq 1$. Let $x \in V_{i-1}$, where the subscripts are taken modulo n. We have $d^+(x) = |V_i| \leq |V_{i+1}| = d^{++}(x)$, and hence x is a Seymour vertex. If $|V_{i-1}| \geq 2$, then there are at least two Seymour vertices. If $|V_{i-1}| = 1$, then $|V_i| = 1$. Let $y \in V_{i-2}$. We have $d^+(y) = |V_{i-1}| = 1 = |V_i| = d^{++}(y)$. Therefore x and y are two Seymour vertices in $C[V_0, V_1, ..., V_{n-1}]$. □

In [7, 8], SSNC is proved for k-transitive oriented graph for $k \leq 9$. Moreover, for $k \leq 6$ and $\delta^+ > 0$, at least two Seymour vertices were found. Here, we obtain the following results.

Theorem 6. *Let D be a k-transitive oriented graph with $k \geq 7$. If $\delta^+ \geq k-4$, then D has at least two Seymour vertices.*

Proof. Let T be a terminal strong component of D. Note that $D[T]$ is also a k-transitive digraph with $\delta_T^+ \geq \delta^+ \geq k-4$. Hence, by Theorem 4, we have $D[T]$ is an extended cycle or a regular 7-tournament. If $D[T]$ is a regular 7-tournament, then $D[T]$ has at least two Seymour vertices (it is a well-known result and easy to check). If $D[T]$ is an extended cycle, then $D[T]$ has at least two Seymour vertices by Lemma 6. Therefore, by Lemma 5, D has at least two Seymour vertices. □

Corollary 1. *Let D be a k-transitive oriented graph. If $k \leq 11$, then D has a Seymour vertex.*

Proof. In [7], SSNC is proved for $k \leq 9$. Let $k \in \{10, 11\}$. If $\delta^+ \geq k-4$, then SSNC holds by Theorem 6. If $\delta^+ \leq k-5$, then $\delta^+ \leq 6$. Therefore, by Theorem 5, D has a Seymour vertex. □

3.2. Bermond–Thomassen Conjecture

In 1981, Bermond and Thomassen [4] proposed the following conjecture.

Conjecture 3.2 (BTC). [4] If a digraph D has minimum out-degree at least $2r-1$, then D contains r disjoint cycles.

For $r = 1$, BTC is trivial. In 1983, Thomassen [25] proved it for $r = 2$.

Theorem 7. *[25] Every digraph with $\delta^+ \geq 3$ contains two disjoint cycles.*

In 2009, Lichiardopol, Por and Sereni [24] proved it for $r = 3$.

Theorem 8. *[24] Every digraph with $\delta^+ \geq 5$ contains three disjoint cycles.*

For $r \geq 4$, BTC still remains open. In 2014, Bang-Jensen, Bessy and Thomassé [3] proved BTC for tournaments. In 2015, Bai, Li, and Li [2] proved the conjecture for bipartite tournaments. In 2020, R. Li et al. [23] proved BTC for local tournaments. Here, we consider BTC for k-transitive oriented graphs, and we obtain the following result.

Theorem 9. *Let D be a k-transitive oriented graph with $3 \leq k \leq 11$. If $\delta^+ \geq 2r-1$, then D contains r disjoint cycles.*

Proof. If $\delta^+ < 7$, then $r \in \{1, 2, 3\}$. Hence the proof follows from Theorems 7, 8. For $\delta^+ \geq 7$, we consider T a terminal strong component of D. Clearly, we have $\delta^+_{D[T]} \geq \delta^+ \geq 7$. Hence, by Theorem 4, we have $D[T]$ is an extended cycle. Let V_0 be a smallest set of the cyclical partition of $D[T]$. So $|V_0| = \delta^+_{D[T]}$. It is easily seen that $D[T]$ contains a collection of disjoint cycles; each visits the set V_0 once. Thus, D contains at least $\delta^+_{D[T]}$ disjoint cycles. □

3.3. Hamiltonian Cycle

Recall that a hamiltonian cycle of a digraph D is a directed cycle passing through all the vertices of D. In this case we say that the digraph D is hamiltonian. Evidently, an extended cycle $C[X_0, \ldots, X_s]$ is hamiltonian if and only if all X_i's have the same size, that is, if and only if $C[X_0, \ldots, X_s]$ is a regular digraph. Note that, for regular digraphs, the concepts of connectedness and strong connectedness coincide. Hence by Theorem 4, a k-transitive oriented graph with sufficiently large minimum in- or out-degree is hamiltonian if and only if it is a connected regular oriented graph. Therefore, to consider the hamiltonian problem for k-transitive oriented graphs, it suffices to study the cases for small minimum in- or out-degree.

It is easily seen that a 3-transitive oriented graph is hamiltonian if and only if it is connected and 1-regular, that is, if and only if it is a directed triangle since δ^+ and δ^- are at most 1.

For $k \geq 4$ with $|V(D)|$ at least $k+1$, the regularity and the hamiltonicity of a k-transitive oriented graph D force $\delta^+ \geq 2$ and $\delta^- \geq 2$. Thus by Theorem 4, for $k \in \{4, 5\}$, we have D is hamiltonian if and only if D is an extended cycle, that is, if and only if D is connected and regular.

For $k = 6$. Since a 6-transitive oriented graph D is an extended cycle when $\delta^+ \geq 3$, it only remains to verify that if D is connected and 2-regular, then D is hamiltonian. The proof of this case is straightforward. Actually, using Lemma 2 and the fact that D is 6-transitive as well as D is 2-regular, we proved that D has a cycle of length greater than 6, which implies that D is an extended cycle and therefore D is hamiltonian since it is regular. Another shorter proof of this case is obtained by using the well-known fact that a regular oriented graph has a cycle factor (a collection of vertex-disjoint cycles that covers the vertex set of the digraph).

For future research, we propose the following conjecture.

Conjecture 3.3. Let k be an integer such that $k \geq 4$ and let D be a k-transitive oriented graph with $|V(D)| \geq k+1$. There exists a hamiltonian cycle in D if and only if D is connected and regular.

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Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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