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Research Article

On e-super (a, d)-edge antimagic total labeling of total graphs of paths and cycles

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Abstract: A (p, q)-graph G is (a, d)-*edge antimagic total* if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p+q\}$ such that for each edge $uv \in E(G)$, the edge weight $\Lambda(uv) = f(u) + f(uv) + f(v)$ forms an arithmetic progression with first term $a > 0$ and common difference $d \geq 0$. An (a, d)-edge antimagic total labeling in which the vertex labels are $1, 2, \ldots, p$ and edge labels are $p+1, p+2, \ldots, p+q$ is called a *super* (a, d)-*edge antimagic total labeling.* Another variant of (a, d)-edge antimagic total labeling called as e-super (a, d)-edge antimagic total labeling in which the edge labels are $1, 2, \ldots, q$ and vertex labels are $q+1, q+2, \ldots, q+p$. In this paper, we investigate the existence of e-super (a, d)-edge antimagic total labeling for total graphs of paths, copies of cycles and disjoint union of cycles.

Keywords: graph Labeling, magic Labeling, antimagic Labeling.

AMS Subject classification: 05C78, 05C76.

1. Introduction

All graphs G considered in this paper are finite, undirected, connected without any loops or multiple edges. Let $V(G)$ and $E(G)$ be the set of vertices and edges of a graph G respectively. The *order* and *size* of a graph G is denoted as $p = |V(G)|$ and $q = |E(G)|$ respectively. For general graph theoretic notions we refer to Harary [8].
A *labeling* of a graph G is a one-to-one mapping that carries the set of graph elements onto a set of numbers (usually positive or non-negative integers), called *labels.* There

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are several types of labeling and a detailed survey of many of them can be found in the dynamic survey of graph labeling by Gallian [7].
Kotzig and Rosa [10] introduced the concept of *magic labeling*. They defined an *edge-magic total labeling* of a (p, q)-graph G as a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p+q\}$ such that for all edges uv, the edge weight $f(u) + f(uv) + f(v)$ is constant.
As a natural extension of the notion of edge-magic total labeling, Hartsfield and Ringel [9] introduced the concept of an *antimagic labeling* and they defined an *antimagic labeling* of a (p, q)-graph G as a bijection from $E(G)$ to the set $\{1, 2, \ldots, q\}$ such that the sums of label of the edges incident with each vertex $v \in V(G)$ are distinct.
In 1993, Bodendiek and Walther [6] introduced the concept of an (a, d)-*antimagic labelings* and they defined a (p, q)-graph G as (a, d)-antimagic if there exist a bijection f from $E(G)$ to $\{1, 2, \ldots, q\}$ such that for each vertex $v \in V(G)$, the vertex weight $\Lambda(v) = \sum_{u \in N(v)} f(uv)$ forms an arithmetic progression with first term $a > 0$ and common difference $d \geq 0$. In [11] Lin, Miller, Simanjuntak and Slamim called this labeling as (a, d)-*vertex antimagic edge labeling*.
In 2000, Baca et al. [4] introduced the notion of (a, d)-*vertex antimagic total labeling* of a graph G as a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p+q\}$ such that for each vertex $v \in V(G)$, the vertex weight $\Lambda(v) = f(v) + \sum_{u \in N(v)} f(uv)$ forms an arithmetic progression with first term $a > 0$ and common difference $d \geq 0$. In the case where the vertices are labeled with the smallest possible integers $1, 2, \ldots, p$, the (a, d)-vertex antimagic total labeling is called a *super* (a, d)-*vertex antimagic total labeling*.
In [4] Baca et al. have proved that every super magic graph has an $(a, 1)$-vertex antimagic total labeling. They also proved that every (a, d)-antimagic graph has an $(a + q + 1, d + 1)$-vertex antimagic total labeling and an $(a + p + q, d - 1)$-vertex antimagic total labeling for $d > 1$. In the same paper they have presented labeling schemes for paths P_n, cycles C_n. They also investigated (a, d)-vertex antimagic total labeling for prisms, antiprisms and generalised Petersen graphs.
As a variation of (a, d)-vertex antimagic edge labeling, Simanjuntak et al. [12] introduced (a, d)-*edge antimagic vertex labeling* and they defined an (a, d)-*edge antimagic vertex* ((a, d)-*EAV*) *labeling* of a (p, q)-graph G as a bijection f from $V(G)$ to $\{1, 2, \ldots, p\}$ such that for each edge $uv \in E(G)$, the edge weight $\Lambda(uv) = f(u) + f(v)$ forms an arithmetic progression with first term $a > 0$ and common difference $d \geq 0$. They have also defined an (a, d)-*edge antimagic total labeling* and a *super* (a, d)-*edge antimagic total labeling* of a graph G as follows: An (a, d)-*edge antimagic total labeling* of a graph G is defined as a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p+q\}$ such that for each edge $uv \in E(G)$, the edge weight $\Lambda(uv) = f(u) + f(uv) + f(v)$ forms an arithmetic progression with first term $a > 0$ and common difference $d \geq 0$. An (a, d)-edge antimagic total labeling in which the vertex labels are $1, 2, \ldots, p$ and the edge labels are $p+1, p+2, \ldots, p+q$ is called a *super* (a, d)-*edge antimagic total* ((a, d)-*SEAT*) *labeling*.
A collection of graphs have been studied in the past that admit (a, d)-SEAT labeling. Bača et al. [1–3] have discussed the existence of (a, d)-SEAT labeling for paths, cycles, friendship graphs, fan graphs, wheel graphs, complete graphs, generalized Petersen

graphs and trees. Sugeng et al. [13, 15, 16] have studied various properties of (a,d)-SEAT labeling and proved several results on ladders, prisms and caterpillars. For a detailed survey about super edge antimagic graphs one can refer to [5].
Another variant of (a,d)-edge antimagic total labeling called as e-super (a,d)-edge antimagic total labeling was introduced by Sugeng et al. [14]. Similar to (a,d)-edge antimagic total labeling, they defined an *e-super* (a,d)-*edge antimagic total labeling* of a graph G as a bijection f from $V(G) \cup E(G)$ to $\{1,2,\dots,q+p\}$ such that for each edge $uv \in E(G)$, the edge weight $\Lambda(uv) = f(u) + f(uv) + f(v)$ forms an arithmetic progression $a, a+d, \dots, a+(q-1)d$ with an additional property that the edge labels are $1,2,\dots,q$ and the vertex labels are $q+1, q+2, \dots, q+p$.
Sugeng et al. [14] have proved that the generalized Petersen graph $P(m,n)$ has an e-super (a,d)-edge antimagic total labeling for odd $n \geq 3$,$m \in \left\{1, 2, \frac{n-1}{2}\right\}$ and $d \in \{0,1,2\}$. They also proved that every caterpillar has an e-super $(a,0)$-edge antimagic total labeling and an e-super $(a,2)$-edge antimagic total labeling for any number of vertices $p \geq 3$ and has an e-super $(a,1)$-edge antimagic total labeling for even number of vertices $p \geq 4$. Further the relationship between (a,d)-EAV labeling and e-super (a,d)-edge antimagic total labeling are also obtained in [14].
The *total graph* of a graph G denoted by $T(G)$ is defined as a graph in which the set of vertices is both the set of vertices and edges of G and any two vertices in $T(G)$ are adjacent if and only if their corresponding elements are either adjacent or incident in G.
In this paper, we investigate the existence of e-super (a,d)-edge antimagic total labeling for total graphs of paths, copies of cycles and disjoint union of cycles.

2. Properties of e-super (a,d)-edge antimagic total labeling

The following theorem gives an upper bound for d of an e-super (a,d)-edge antimagic total labeling.

Theorem 2.1 . *If a graph G has an e-super (a,d)-edge antimagic total labeling, then* $d \leq \frac{2p+q-5}{q-1}$.

Proof. Let us assume that the graph G has an e-super (a,d)-edge antimagic total labeling. Then by definition, there exist a bijection $f : V(G) \cup E(G) \to \{1,2,\dots,q+p\}$ such that

(i) $f(E(G)) = \{1,2,\dots,q\}$

(ii) $f(V(G)) = \{q+1, q+2, \dots, q+p\}$ and

(iii) for any edge $uv \in E(G)$, the set of edge weight

$$\Lambda(uv) = \{a, a+d, a+2d, \dots, a+(q-1)d\}.$$

Clearly the minimum possible edge weight is $(q+1)+1+(q+2) = 2q+4$. Thus, we have

$$a \geq 2q+4. \tag{2.1}$$

Also, the maximum possible edge weight is $(q+p-1)+q+(q+p) = 3q+2p-1$ Thus, we have

$$a+(q-1)d \leq 3q+2p-1 \Rightarrow a \leq 3q+2p-1-(q-1)d. \tag{2.2}$$

From (2.1) and (2.2) we get, $2q+4 \leq 3q+2p-1-(q-1)d$ implying that $(q-1)d \leq 3q+2p-1-2q-4$. Hence, d $\leq \frac{2p+q-5}{(q-1)}$. □

The following theorem provides a relationship between e-super $(a,0)$-edge antimagic total labeling and e-super $(b,2)$-edge antimagic total labeling of a graph G.

Theorem 2.2 . *If a graph G has an e-super $(a_1,0)$-edge antimagic total labeling then it has an e-super $(a_2,2)$-edge antimagic total labeling where $a_2 = a_1+1-q$.*

Proof. Let us assume that the graph G has an e-super $(a_1,0)$-edge antimagic total labeling. Then by definition, there exist a bijection $f : V(G) \cup E(G) \rightarrow \{1,2,\ldots,q+p\}$ such that

(i) $f(E(G)) = \{1,2,\ldots,q\}$

(ii) $f(V(G)) = \{q+1,q+2,\ldots,q+p\}$ and

(iii) for every edge $uv \in E(G)$, $f(u)+f(uv)+f(v) = a_1$.

Let us define an induced function $g : V(G) \cup E(G) \rightarrow \{1,2,\ldots,q+p\}$ as follows:

(i) for every vertex $v \in V(G)$, $g(v) = f(v)$

(ii) for every edge $uv \in E(G)$, $g(uv) = q+1-f(uv)$.

Then, we have

(i) $g(E(G)) = \{1,2,\ldots,q\}$

(ii) $g(V(G)) = \{q+1,q+2,\ldots,q+p\}$

and for any edge $uv \in E(G)$,

$$\begin{aligned} g(u)+g(uv)+g(v) &= f(u)+q+1-f(uv)+f(v) \\ &= q+1+f(u)+f(uv)+f(v)-2f(uv) \\ &= q+1+a_1-2f(uv) \\ &= (a_1+1-q)+2(q-f(uv)). \end{aligned}$$

Since $f(E(G)) = \{1, 2, \ldots, q\}$, for any edge $uv \in E(G)$, we have the set of edge weights as

$$g(u) + g(uv) + g(v) = \left\{ \begin{array}{c} (a_1 + 1 - q) + 2(q-1), (a_1 + 1 - q) + 2(q-2), \\ \ldots, (a_1 + 1 - q) + 2(q-q) \end{array} \right\}$$
$$= \{a_2, a_2 + 2(1), \ldots, a_2 + 2(q-1)\}, \text{ where } a_2 = a_1 + 1 - q.$$

Thus, g is an e-super $(a_2, 2)$-edge antimagic total labeling of G.
Hence, if G has an e-super $(a_1, 0)$-edge antimagic total labeling then it has an e-super $(a_2, 2)$-edge antimagic total labeling where $a_2 = a_1 + 1 - q$. □

3. Total graph of paths P_n

In this section we establish the e-super (a, d)-edge antimagic total labeling for the total graph of paths P_n.
Let $\{v_1, v_2, \ldots, v_n\}$ and $\{e_i = v_i v_{i+1} : 1 \leq i \leq n-1\}$ be the set of vertices and edges respectively of a path P_n. Then we have,
$V(T[P_n]) = \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n-1\}$ and $E(T[P_n]) = E_1 \cup E_2 \cup E_3 \cup E_4$ where

$$E_1 = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$$

$$E_2 = \{v_i e_i : 1 \leq i \leq n-1\}$$

$$E_3 = \{v_i e_{i-1} : 2 \leq i \leq n\}$$

$$E_4 = \{e_i e_{i+1} : 1 \leq i \leq n-2\}.$$

It is clear that, for the graph $T[P_n]$, $p = 2n - 1$ and $q = 4n - 5$.
By Theorem 2.1 , the following lemma is immediate.

Lemma 3.1. *If the graph $T[P_n]$, $n \geq 3$, has an e-super (a, d)-edge antimagic total labeling, then $d \leq 2$.* □

Lemma 3.2. *For every path P_n, $n \geq 3$, the graph $G = T[P_n]$ has an e-super $(a, 0)$-edge antimagic total labeling.*

Proof. Let us define a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, q+p\}$ as follows:

(i) $f(v_i v_{i+1}) = 4n - 6 - 4(i-1)$; for $1 \leq i \leq n-1$

(ii) $f(v_i e_i) = 4n - 5 - 4(i-1)$; for $1 \leq i \leq n-1$

(iii) $f(v_i e_{i-1}) = 4n - 7 - 4(i-2)$; for $2 \leq i \leq n$

(iv) $f(e_i e_{i+1}) = 4n - 8 - 4(i-1)$; for $1 \leq i \leq n-2$

(v) $f(v_i) = 4n - 6 + 2i$; for $1 \leq i \leq n$

(vi) $f(e_i) = 4n - 5 + 2i$; for $1 \leq i \leq n - 1$.

One can easily observe that the edge labels form the set
$$\{1, 2, \ldots, 4n - 5\} = \{1, 2, \ldots, q\}$$

and the vertex labels form the set
$$\{(4n - 5) + 1, (4n - 5) + 2, \ldots, (4n - 5) + (2n - 1)\} = \{q + 1, q + 2, \ldots, q + p\}.$$

To complete the proof, we have to prove that for any edge $uv \in E(G)$, $\Lambda(uv)$ is a constant.
For $1 \leq i \leq n - 1$,
$$\begin{aligned}\Lambda(v_i v_{i+1}) &= f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) \\ &= (4n - 6 + 2i) + (4n - 6 - 4(i - 1)) + (4n - 6 + 2(i + 1)) \\ &= 12n - 12 = 12(n - 1).\end{aligned}$$
For $1 \leq i \leq n - 1$,
$$\begin{aligned}\Lambda(v_i e_i) &= f(v_i) + f(v_i e_i) + f(e_i) \\ &= (4n - 6 + 2i) + (4n - 5 - 4(i - 1)) + (4n - 5 + 2i) \\ &= 12n - 12 = 12(n - 1).\end{aligned}$$
For $2 \leq i \leq n$,
$$\begin{aligned}\Lambda(v_i e_{i-1}) &= f(v_i) + f(v_i e_{i-1}) + f(e_{i-1}) \\ &= (4n - 6 + 2i) + (4n - 7 - 4(i - 2)) + (4n - 5 + 2(i - 1)) \\ &= 12n - 12 = 12(n - 1).\end{aligned}$$
For $1 \leq i \leq n - 2$,
$$\begin{aligned}\Lambda(e_i e_{i+1}) &= f(e_i) + f(e_i e_{i+1}) + f(e_{i+1}) \\ &= (4n - 5 + 2i) + (4n - 8 - 4(i - 1)) + (4n - 5 + 2(i + 1)) \\ &= 12n - 12 = 12(n - 1).\end{aligned}$$
Thus, for any edge $uv \in E(G)$, we have $\Lambda(uv) = 12(n - 1)$.
Hence, f is an e-super $(a, 0)$-edge antimagic total labeling of $T[P_n]$ where $a = 12(n - 1)$. □

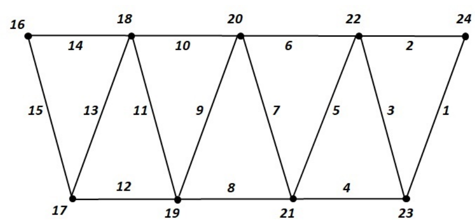

Figure 1. **e-Super $(48, 0)$-edge antimagic total labeling of $T[P_5]$**

Lemma 3.3. *For every path P_n, $n \geq 3$, the graph $G = T[P_n]$ has an e-super $(a,1)$-edge antimagic total labeling.*

Proof. Let us define a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, q+p\}$ as follows:

(i) $f(v_iv_{i+1}) = 4n - 3 - 2i$; for $1 \leq i \leq n-1$

(ii) $f(v_ie_i) = 2n - 2i$; for $1 \leq i \leq n-1$

(iii) $f(v_ie_{i-1}) = 2n + 1 - 2i$; for $2 \leq i \leq n$

(iv) $f(e_ie_{i+1}) = 4n - 4 - 2i$; for $1 \leq i \leq n-2$

(v) $f(v_i) = 4n - 6 + 2i$; for $1 \leq i \leq n$

(vi) $f(e_i) = 4n - 5 + 2i$; for $1 \leq i \leq n-1$.

One can easily observe that the edge labels form the set

$$\{1, 2, \ldots, 4n-5\} = \{1, 2, \ldots, q\}$$

and the vertex labels form the set

$$\{(4n-5)+1, (4n-5)+2, \ldots, (4n-5)+(2n-1)\} = \{q+1, q+2, \ldots, q+p\}.$$

To complete the proof, we have to prove that the edge weights $\Lambda(uv)$ form an arithmetic sequence $\{a, a+1, \ldots, a+(q-1)\}$.
For $1 \leq i \leq n-1$,

$$\begin{aligned}\Lambda(v_iv_{i+1}) &= f(v_i) + f(v_iv_{i+1}) + f(v_{i+1})\\ &= (4n-6+2i) + (4n-3-2i) + (4n-6+2(i+1))\\ &= 12n - 13 + 2i = (10n-9) + 2(n+i) - 4.\end{aligned}$$

For $1 \leq i \leq n-1$,

$$\begin{aligned}\Lambda(v_ie_i) &= f(v_i) + f(v_ie_i) + f(e_i)\\ &= (4n-6+2i) + (2n-2i) + (4n-5+2i)\\ &= 10n - 11 + 2i = (10n-9) + 2(i-1).\end{aligned}$$

For $2 \leq i \leq n$,

$$\begin{aligned}\Lambda(v_ie_{i-1}) &= f(v_i) + f(v_ie_{i-1}) + f(e_{i-1})\\ &= (4n-6+2i) + (2n+1-2i) + (4n-5+2(i-1))\\ &= 10n - 12 + 2i = (10n-9) + 2(i-1) - 1.\end{aligned}$$

For $1 \leq i \leq n-2$,

$$\begin{aligned}\Lambda(e_ie_{i+1}) &= f(e_i) + f(e_ie_{i+1}) + f(e_{i+1})\\ &= (4n-5+2i) + (4n-4-2i) + (4n-5+2(i+1))\\ &= 12n - 12 + 2i = (10n-9) + 2(n+i) - 3.\end{aligned}$$

Thus, the edge weights are

$$(10n-9), (10n-9)+1, \ldots, (10n-9)+(4n-6).$$

Hence, f is an e-super $(a,1)$-edge antimagic total labeling of $T[P_n]$ where $a = 10n - 9$. □

By Lemmas 3.1, 3.2, 3.3 and Theorem 2.2 , we have the following theorem:

Theorem 3.3 . *The graph $T[P_n]$, $n \geq 3$, has an e-super (a,d)-edge antimagic total labeling if and only if $d \in \{0,1,2\}$.*

4. Total graph of copies of cycles C_n

This section deals with the e-super (a,d)-edge antimagic total labeling of total graph of copies of cycles C_n.
Let $\{v_j^i : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and $\{e_j^i = v_j^i v_{j+1}^i : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ (where the subscripts i and j are taken modulo m and modulo n respectively) be the set of vertices and edges of the disjoint union of m copies of cycles C_n. Then for the total graph of m copies of C_n, we have
$V(T[mC_n]) = \{v_j^i : 1 \leq i \leq m,\ 1 \leq j \leq n\} \cup \{e_j^i : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and $E(T[mC_n]) = E_1 \cup E_2 \cup E_3 \cup E_4$ where

$$E_1 = \{v_j^i v_{j+1}^i : 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

$$E_2 = \{v_j^i e_j^i : 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

$$E_3 = \{v_j^i e_{j+1}^i : 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

$$E_4 = \{e_j^i e_{j+1}^i : 1 \leq i \leq m,\ 1 \leq j \leq n\}.$$

It is clear that, for the graph $T[mC_n]$, $p = 2mn$ and $q = 4mn$.
By Theorem 2.1 , the following lemma is immediate.

Lemma 4.4. *If the graph $T[mC_n]$, $m \geq 1$, $n \geq 3$ has an e-super (a,d)-edge antimagic total labeling, then $d < 2$.*

Lemma 4.5. *For every disjoint union of m copies of cycles C_n, $m \geq 1$, $n \geq 3$, the graph $G = T[mC_n]$, has no e-super $(a,0)$-edge antimagic total labeling.*

Proof. Suppose G has an e-super $(a,0)$-edge antimagic total labeling.
Then by definition, there exist a bijection $f : V(G) \cup E(G) \to \{1,2,\ldots,q+p\}$ such that

(i) $f(E(G)) = \{1,2,\ldots,q\}$

(ii) $f(V(G)) = \{q+1, q+2, \ldots, q+p\}$ and

(iii) for all edge $uv \in E(G)$, $\Lambda(uv) = a$.

Since G is a 4-regular graph, we have the sum of all edge weights is equal to

$$4\sum_{v\in V(G)} f(v) + \sum_{e\in E(G)} f(e) = 4\sum_{j=1}^{2mn}(4mn+j) + \sum_{i=1}^{4mn} i = 48m^2n^2 + 6mn. \tag{4.1}$$

Also, since G has an e-super $(a,0)$-edge antimagic total labeling, the sum of all edge weights is equal to

$$\sum_{i=1}^{4mn} a = 4mna. \tag{4.2}$$

From (4.1) and (4.2) we get, $4mna = 48m^2n^2 + 6mn$ implying that $a = 12mn + \dfrac{3}{2}$ which is not an integer. Hence, for the graph $T[mC_n]$, $m \geq 1$, $n \geq 3$, there is no e-super $(a,0)$-edge antimagic total labeling. □

Lemma 4.6. *For every disjoint union of m copies of cycles C_n, $m \geq 1$, $n \geq 3$, the graph $G = T[mC_n]$, has an e-super $(a,1)$-edge antimagic total labeling.*

Proof. Let us define a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, q+p\}$ as follows:

(i) $f(v^i_j v^i_{j+1}) = 2ni + 2 - 2j$; for $1 \leq i \leq m$, $1 \leq j \leq n$

(ii) $f(v^i_j e^i_j) = 2mn + 2ni + 1 - 2j$; for $1 \leq i \leq m$, $1 \leq j \leq n$

(iii) $f(v^i_j e^i_{j+1}) = 2mn + 2ni - 2j$; for $1 \leq i \leq m$, $1 \leq j \leq n-1$
$f(v^i_n e^i_1) = 2mn + 2ni$; for $1 \leq i \leq m$

(iv) $f(e^i_j e^i_{j+1}) = 2n(i-1) + 2j - 1$; for $1 \leq i \leq m$, $1 \leq j \leq n$

(v) $f(v^i_j) = 6mn - 2ni - 1 + 2j$; for $1 \leq i \leq m$, $1 \leq j \leq n$

(vi) $f(e^i_j) = 6mn - 2n(i-1) + 4 - 2j$; for $1 \leq i \leq m$, $2 \leq j \leq n$
$f(e^i_1) = 6mn - 2ni + 2$; for $1 \leq i \leq m$.

One can easily observe that the edge labels form the set

$$\{1, 2, \ldots, 4mn\} = \{1, 2, \ldots, q\}$$

and the vertex labels form the set

$$\{4mn+1, 4mn+2, \ldots, 6mn\} = \{q+1, q+2, \ldots, q+p\}.$$

To complete the proof, we have to prove that the edge weights $\Lambda(uv)$ form an arithmetic sequence $\{a, a+1, \ldots, a+(q-1)\}$.
For $1 \leq i \leq m$, $1 \leq j \leq n-1$,

$$\begin{aligned}\Lambda(v^i_j v^i_{j+1}) &= f(v^i_j) + f(v^i_j v^i_{j+1}) + f(v^i_{j+1}) \\ &= (6mn - 2ni + 2j - 1) + (2ni + 2 - 2j) + (6mn - 2ni + 2(j+1) - 1) \\ &= (12mn - 2ni + 2) + 2j.\end{aligned}$$

For $1 \leq i \leq m$,

$$\begin{aligned}\Lambda(v_n^i v_1^i) &= f(v_n^i) + f(v_n^i v_1^i) + f(v_1^i)\\ &= (6mn - 2ni + 2n - 1) + 2n(i-1) + 2 + (6mn - 2ni + 2 - 1)\\ &= (12mn - 2ni + 2).\end{aligned}$$

For $1 \leq i \leq m, 2 \leq j \leq n$,

$$\begin{aligned}\Lambda(v_j^i e_j^i) &= f(v_j^i) + f(v_j^i e_j^i) + f(e_j^i)\\ &= (6mn - 2ni + 2j - 1) + (2mn + 2ni + 1 - 2j)\\ &\quad + (6mn - 2n(i-1) + 4 - 2j)\\ &= (12mn - 2ni + 2) + (2mn + 2n + 2 - 2j).\end{aligned}$$

For $1 \leq i \leq m$,

$$\begin{aligned}\Lambda(v_1^i e_1^i) &= f(v_1^i) + f(v_1^i e_1^i) + f(e_1^i)\\ &= (6mn - 2ni + 2 - 1) + (2mn + 2ni + 1 - 2) + (6mn - 2ni + 2)\\ &= (12mn - 2ni + 2) + (2mn).\end{aligned}$$

For $1 \leq i \leq m, 1 \leq j \leq n-1$,

$$\begin{aligned}\Lambda(v_j^i e_{j+1}^i) &= f(v_j^i) + f(v_j^i e_{j+1}^i) + f(e_{j+1}^i)\\ &= (6mn - 2ni + 2j - 1) + (2mn + 2ni - 2j)\\ &\quad + (6mn - 2n(i-1) + 4 - 2(j+1))\\ &= (12mn - 2ni + 2) + (2mn + 2n - 2j - 1).\end{aligned}$$

For $1 \leq i \leq m$,

$$\begin{aligned}\Lambda(v_n^i e_1^i) &= f(v_n^i) + f(v_n^i e_1^i) + f(e_1^i)\\ &= (6mn - 2ni + 2n - 1) + (2mn + 2ni) + (6mn - 2ni + 2)\\ &= (12mn - 2ni + 2) + (2mn + 2n - 1).\end{aligned}$$

For $1 \leq i \leq m, 2 \leq j \leq n-1$,

$$\begin{aligned}\Lambda(e_j^i e_{j+1}^i) &= f(e_j^i) + f(e_j^i e_{j+1}^i) + f(e_{j+1}^i)\\ &= (6mn - 2n(i-1) + 4 - 2j) + (2n(i-1) + 2j - 1)\\ &\quad + (6mn - 2n(i-1) + 4 - 2(j+1))\\ &= (12mn - 2ni + 2) + (2n - 2j + 3).\end{aligned}$$

For $1 \leq i \leq m$,

$$\begin{aligned}\Lambda(e_n^i e_1^i) &= f(e_n^i) + f(e_n^i e_1^i) + f(e_1^i)\\ &= (6mn - 2n(i-1) + 4 - 2n) + (2ni - 1) + (6mn - 2ni + 2)\\ &= (12mn - 2ni + 2) + 3.\end{aligned}$$

For $1 \leq i \leq m$,

$$\begin{aligned}\Lambda(e_1^i e_2^i) &= f(e_1^i) + f(e_1^i e_2^i) + f(e_2^i)\\ &= (6mn - 2ni + 2) + (2n(i-1) + 1) + (6mn - 2n(i-1))\\ &= (12mn - 2ni + 2) + 1.\end{aligned}$$

Thus, the edge weights are

$$(10mn + 2), (10mn + 2) + 1, \ldots, (10mn + 2) + (4mn - 1).$$

Hence, f is an e-super $(a,1)$-edge antimagic total labeling of $T[mC_n]$ where $a = 10mn + 2$. □

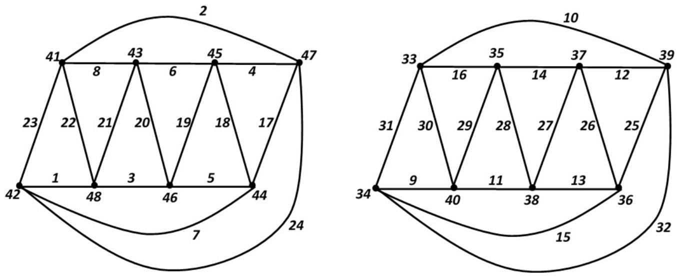

Figure 2. **e-Super (82, 1)-edge antimagic total labeling of** $T[2C_4]$

By Lemmas 4.4, 4.5 and 4.6, we have the following theorem:

Theorem 4.4 . *The graph* $T[mC_n]$, $m \geq 1$, $n \geq 3$, *has an e-super* (a, d)*-edge antimagic total labeling if and only if* $d = 1$.

As a particular case to the above theorem, when $m = 1$, we have the following corollary.

Corollary 4.1. *The graph* $T[C_n]$, $n \geq 3$, *has an e-super* (a, d)*-edge antimagic total labeling if and only if* $d = 1$.

5. Total graph of disjoint union of cycles

Let $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ and $\{e_i = u_iu_{i+1} : 1 \leq i \leq m\} \cup \{h_j = v_jv_{j+1} : 1 \leq j \leq n\}$ (where the subscripts i and j are taken modulo m and modulo n respectively) be the set of vertices and edges of the disjoint union of cycles $C_m \cup C_n$, $m \neq n$. Then we have,

$$V(T[C_m \cup C_n]) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\} \cup \{e_i : 1 \leq i \leq m\} \cup \{h_j : 1 \leq j \leq n\}$$

and $E(T[C_m \cup C_n]) = E_1 \cup E_2 \cup E_3 \cup E_4$ where

$$E_1 = \{u_iu_{i+1}, v_jv_{j+1} : 1 \leq i \leq m-1,\ 1 \leq j \leq n\}$$

$$E_2 = \{u_ie_i, v_jh_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

$$E_3 = \{u_ie_{i+1}, v_jh_{j+1} : 1 \leq i \leq m-1,\ 1 \leq j \leq n\}$$

$$E_4 = \{e_ie_{i+1}, h_jh_{j+1} : 1 \leq i \leq m-1,\ 1 \leq j \leq n\}.$$

It is clear that, for the graph $T[C_m \cup C_n]$, $p = 2(m + n)$ and $q = 4(m + n)$.
By Theorem 2.1 , the following lemma is immediate.

Lemma 5.7. *If the graph $T[C_m \cup C_n]$, $m \neq n$, $m, n \geq 3$, has an e-super (a, d)-edge antimagic total labeling, then $d < 2$.*

Similar to the proof of Lemma 4.6, we have the following lemma.

Lemma 5.8. *For every disjoint union of cycles $C_m \cup C_n$, $m \neq n$, $m, n \geq 3$, the graph $G = T[C_m \cup C_n]$, has no e-super $(a, 0)$-edge antimagic total labeling.*

Lemma 5.9. *For every disjoint union of cycles $C_m \cup C_n$, $m \neq n$, $m, n \geq 3$, the graph $G = T[C_m \cup C_n]$, has an e-super $(a, 1)$-edge antimagic total labeling.*

Proof. Let us define a bijection $f : V(G) \cup E(G) \to \{1, 2, \dots, q + p\}$ as follows:

(i) $f(u_i u_{i+1}) = 2m + 2 - 2i$; for $1 \leq i \leq m$

$f(v_j v_{j+1}) = 2m + 2n + 2 - 2j$; for $1 \leq j \leq n$

(ii) $f(u_i e_i) = 4m + 2n + 1 - 2i$; for $1 \leq i \leq m$

$f(v_j h_j) = 4m + 4n + 1 - 2j$; for $1 \leq j \leq n$

(iii) $f(u_i e_{i+1}) = 4m + 2n - 2i$; for $1 \leq i \leq m - 1$, $f(u_m e_1) = 4m + 2n$

$f(v_j h_{j+1}) = 4m + 4n - 2j$; for $1 \leq j \leq n - 1$, $f(v_n h_1) = 4m + 4n$

(iv) $f(e_i e_{i+1}) = 2i - 1$; for $1 \leq i \leq m$

$f(h_j h_{j+1}) = 2m - 1 + 2j$; for $1 \leq j \leq n$

(v) $f(u_i) = 4m + 6n - 1 + 2i$; for $1 \leq i \leq m$

$f(v_j) = 4m + 4n - 1 + 2j$; for $1 \leq j \leq n$

(vi) $f(e_i) = 6m + 6n + 4 - 2i$; for $2 \leq i \leq m$, $f(e_1) = 4m + 6n + 2$

$f(h_j) = 4m + 6n + 4 - 2j$; for $2 \leq j \leq n$, $f(h_1) = 4m + 4n + 2$.

One can easily observe that the edge labels form the set

$$\{1, 2, \dots, 4(m + n)\} = \{1, 2, \dots, q\}$$

and the vertex labels form the set

$$\{4(m + n) + 1, 4(m + n) + 2, \dots, 6(m + n)\} = \{q + 1, q + 2, \dots, q + p\}.$$

To complete the proof, we have to prove that the edge weights $\Lambda(uv)$ form an arithmetic sequence $\{a, a + 1, \dots, a + (q - 1)$.

For $1 \leq i \leq m - 1$,

$$\begin{aligned}\Lambda(u_i u_{i+1}) &= f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) \\ &= (4m + 6n - 1 + 2i) + (2m + 2 - 2i) + (4m + 6n - 1 + 2(i + 1)) \\ &= (10(m + n) + 2) + (2n + 2i)\end{aligned}$$

and

$$\begin{aligned}\Lambda(u_m u_1) &= f(u_m) + f(u_m u_1) + f(u_1)\\ &= (4m + 6n - 1 + 2m) + 2 + (4m + 6n - 1 + 2)\\ &= (10(m + n) + 2) + 2n.\end{aligned}$$

For $1 \leq j \leq n - 1$,

$$\begin{aligned}\Lambda(v_j v_{j+1}) &= f(v_j) + f(v_j v_{j+1}) + f(v_{j+1})\\ &= (4m + 4n - 1 + 2j) + (2m + 2n + 2 - 2j) + (4m + 4n - 1 + 2(j + 1))\\ &= (10(m + n) + 2) + 2j\end{aligned}$$

and

$$\begin{aligned}\Lambda(v_n v_1) &= f(v_n) + f(v_n v_1) + f(v_1)\\ &= (4m + 4n - 1 + 2n) + (2m + 2) + (4m + 4n - 1 + 2)\\ &= (10(m + n) + 2).\end{aligned}$$

For $2 \leq i \leq m$,

$$\begin{aligned}\Lambda(u_i e_i) &= f(u_i) + f(u_i e_i) + f(e_i)\\ &= (4m + 6n - 1 + 2i) + (4m + 2n + 1 - 2i) + (6m + 6n + 4 - 2i)\\ &= (10(m + n) + 2) + (4m + 4n + 2 - 2i)\end{aligned}$$

and

$$\begin{aligned}\Lambda(u_1 e_1) &= f(u_1) + f(u_1 e_1) + f(e_1)\\ &= (4m + 6n - 1 + 2) + (4m + 2n + 1 - 2) + (4m + 6n + 2)\\ &= (10(m + n) + 2) + (2m + 4n).\end{aligned}$$

For $2 \leq j \leq n$,

$$\begin{aligned}\Lambda(v_j h_j) &= f(v_j) + f(v_j h_j) + f(h_j)\\ &= (4m + 4n - 1 + 2j) + (4m + 4n + 1 - 2j) + (4m + 6n + 4 - 2j)\\ &= (10(m + n) + 2) + (2m + 4n + 2 - 2j)\end{aligned}$$

and

$$\begin{aligned}\Lambda(v_1 h_1) &= f(v_1) + f(v_1 h_1) + f(h_1)\\ &= (4m + 4n - 1 + 2) + (4m + 4n + 1 - 2) + (4m + 4n + 2)\\ &= (10(m + n) + 2) + (2m + 2n).\end{aligned}$$

For $1 \leq i \leq m - 1$,

$$\begin{aligned}\Lambda(u_i e_{i+1}) &= f(u_i) + f(u_i e_{i+1}) + f(e_{i+1})\\ &= (4m + 6n - 1 + 2i) + (4m + 2n - 2i) + (6m + 6n + 4 - 2(i + 1))\\ &= (10(m + n) + 2) + (4m + 4n - 1 - 2i)\end{aligned}$$

and

$$\begin{aligned}\Lambda(u_m e_1) &= f(u_m) + f(u_m e_1) + f(e_1)\\ &= (4m + 6n - 1 + 2m) + (4m + 2n) + (4m + 6n + 2)\\ &= (10(m + n) + 2) + (4m + 4n - 1).\end{aligned}$$

For $1 \leq j \leq n-1$,

$$\begin{aligned}\Lambda(v_j h_{j+1}) &= f(v_j) + f(v_j h_{j+1}) + f(h_{j+1}) \\ &= (4m+4n-1+2j) + (4m+4n-2j) + (4m+6n+4-2(j+1)) \\ &= (10(m+n)+2) + (2m+4n-1-2j)\end{aligned}$$

and

$$\begin{aligned}\Lambda(v_n h_1) &= f(v_n) + f(v_n h_1) + f(h_1) \\ &= (4m+4n-1+2n) + (4m+4n) + (4m+4n+2) \\ &= (10(m+n)+2) + (2m+4n-1).\end{aligned}$$

For $2 \leq i \leq m$,

$$\begin{aligned}\Lambda(e_i e_{i+1}) &= f(e_i) + f(e_i e_{i+1}) + f(e_{i+1}) \\ &= (6m+6n+4-2i) + (2i-1) + (6m+6n+4-2(i+1)) \\ &= (10(m+n)+2) + (2m+2n+3-2i)\end{aligned}$$

and

$$\begin{aligned}\Lambda(e_1 e_2) &= f(e_1) + f(e_1 e_2) + f(e_2) \\ &= (4m+6n+2) + (2-1) + (6m+6n+4-4) \\ &= (10(m+n)+2) + (2n+1).\end{aligned}$$

For $2 \leq j \leq n$,

$$\begin{aligned}\Lambda(h_j h_{j+1}) &= f(h_j) + f(h_j h_{j+1}) + f(h_{j+1}) \\ &= (4m+6n+4-2j) + (2m+2j-1) + (4m+6n+4-2(j+1)) \\ &= (10(m+n)+2) + (2n+3-2j)\end{aligned}$$

and

$$\begin{aligned}\Lambda(h_1 h_2) &= f(h_1) + f(h_1 h_2) + f(h_2) \\ &= (4m+4n+2) + (2m+2-1) + (4m+6n+4-4) \\ &= (10(m+n)+2) + 1.\end{aligned}$$

Thus, the edge weights are

$$(10(m+n)+2), (10(m+n)+2)+1, \ldots, (10(m+n)+2)+(4m+4n-1).$$

Hence, f is an e-super $(a, 1)$-edge antimagic total labeling of $T[C_m \cup C_n]$ where $a = 10(m+n)+2$. □

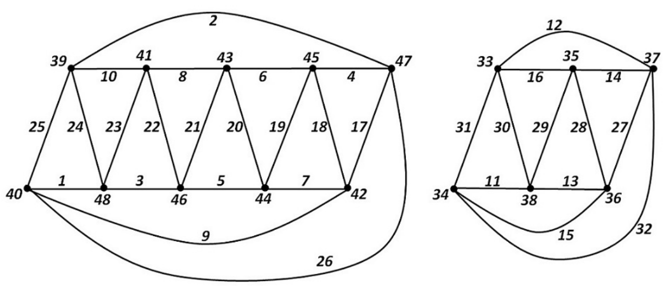

Figure 3. **e-Super (82, 1)-edge antimagic total labeling of** $T[C_5 \cup C_3]$

By Lemmas 5.7, 5.8 and 5.9, we have the following theorem:

Theorem 5.5 . *The graph $T[C_m \cup C_n]$, $m \neq n$, $m, n \geq 3$, has an e-super (a, d)-edge antimagic total labeling if and only if $d = 1$.*

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Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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