

On e-super (a, d) -edge antimagic total labeling of total graphs of paths and cycles

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Received: 20 April 2023; Accepted: 1 February 2024

Published Online: 10 February 2024

Abstract: A (p, q) -graph G is (a, d) -edge antimagic total if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \dots, p + q\}$ such that for each edge $uv \in E(G)$, the edge weight $\Lambda(uv) = f(u) + f(uv) + f(v)$ forms an arithmetic progression with first term $a > 0$ and common difference $d \geq 0$. An (a, d) -edge antimagic total labeling in which the vertex labels are $1, 2, \dots, p$ and edge labels are $p + 1, p + 2, \dots, p + q$ is called a *super (a, d) -edge antimagic total labeling*. Another variant of (a, d) -edge antimagic total labeling called as e-super (a, d) -edge antimagic total labeling in which the edge labels are $1, 2, \dots, q$ and vertex labels are $q + 1, q + 2, \dots, q + p$. In this paper, we investigate the existence of e-super (a, d) -edge antimagic total labeling for total graphs of paths, copies of cycles and disjoint union of cycles.

Keywords: graph labeling, magic labeling, antimagic labeling.

AMS Subject classification: 05C78, 05C76

1. Introduction

All graphs G considered in this paper are finite, undirected, connected without any loops or multiple edges. Let $V(G)$ and $E(G)$ be the set of vertices and edges of a graph G respectively. The *order* and *size* of a graph G is denoted as $p = |V(G)|$ and $q = |E(G)|$ respectively. For general graph theoretic notions we refer to Harary [8]. A *labeling* of a graph G is a one-to-one mapping that carries the set of graph elements onto a set of numbers (usually positive or non-negative integers), called *labels*. There

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are several types of labeling and a detailed survey of many of them can be found in the dynamic survey of graph labeling by Gallian [7].

Kotzig and Rosa [10] introduced the concept of *magic labeling*. They defined an *edge-magic total labeling* of a (p, q) -graph G as a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \dots, p + q\}$ such that for all edges uv , the edge weight $f(u) + f(uv) + f(v)$ is constant.

As a natural extension of the notion of edge-magic total labeling, Hartsfield and Ringel [9] introduced the concept of an *antimagic labeling* and they defined an *antimagic labeling* of a (p, q) -graph G as a bijection from $E(G)$ to the set $\{1, 2, \dots, q\}$ such that the sums of label of the edges incident with each vertex $v \in V(G)$ are distinct.

In 1993, Bodendiek and Walther [6] introduced the concept of an (a, d) -*antimagic labelings* and they defined a (p, q) -graph G as (a, d) -antimagic if there exist a bijection f from $E(G)$ to $\{1, 2, \dots, q\}$ such that for each vertex $v \in V(G)$, the vertex weight $\Lambda(v) = \sum_{u \in N(v)} f(uv)$ forms an arithmetic progression with first term $a > 0$ and common difference $d \geq 0$. In [11] Lin, Miller, Simanjuntak and Slamim called this labeling as (a, d) -*vertex antimagic edge labeling*.

In 2000, Baca et al. [4] introduced the notion of (a, d) -*vertex antimagic total labeling* of a graph G as a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \dots, p + q\}$ such that for each vertex $v \in V(G)$, the vertex weight $\Lambda(v) = f(v) + \sum_{u \in N(v)} f(uv)$ forms an arithmetic progression with first term $a > 0$ and common difference $d \geq 0$. In the case where the vertices are labeled with the smallest possible integers $1, 2, \dots, p$, the (a, d) -vertex antimagic total labeling is called a *super (a, d) -vertex antimagic total labeling*.

In [4] Baca et al. have proved that every super magic graph has an $(a, 1)$ -vertex antimagic total labeling. They also proved that every (a, d) -antimagic graph has an $(a + q + 1, d + 1)$ -vertex antimagic total labeling and an $(a + p + q, d - 1)$ -vertex antimagic total labeling for $d > 1$. In the same paper they have presented labeling schemes for paths P_n , cycles C_n . They also investigated (a, d) -vertex antimagic total labeling for prisms, antiprisms and generalised Petersen graphs.

As a variation of (a, d) -vertex antimagic edge labeling, Simanjuntak et al. [12] introduced (a, d) -*edge antimagic vertex labeling* and they defined an (a, d) -*edge antimagic vertex* ((a, d) -*EAV*) *labeling* of a (p, q) -graph G as a bijection f from $V(G)$ to $\{1, 2, \dots, p\}$ such that for each edge $uv \in E(G)$, the edge weight $\Lambda(uv) = f(u) + f(v)$ forms an arithmetic progression with first term $a > 0$ and common difference $d \geq 0$. They have also defined an (a, d) -*edge antimagic total labeling* and a *super (a, d) -edge antimagic total labeling* of a graph G as follows: An (a, d) -*edge antimagic total labeling* of a graph G is defined as a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \dots, p + q\}$ such that for each edge $uv \in E(G)$, the edge weight $\Lambda(uv) = f(u) + f(uv) + f(v)$ forms an arithmetic progression with first term $a > 0$ and common difference $d \geq 0$. An (a, d) -edge antimagic total labeling in which the vertex labels are $1, 2, \dots, p$ and the edge labels are $p + 1, p + 2, \dots, p + q$ is called a *super (a, d) -edge antimagic total* ((a, d) -*SEAT*) *labeling*.

A collection of graphs have been studied in the past that admit (a, d) -SEAT labeling. Bača et al. [1–3] have discussed the existence of (a, d) -SEAT labeling for paths, cycles, friendship graphs, fan graphs, wheel graphs, complete graphs, generalized Petersen

graphs and trees. Sugeng et al. [13, 15, 16] have studied various properties of (a, d) -SEAT labeling and proved several results on ladders, prisms and caterpillars. For a detailed survey about super edge antimagic graphs one can refer to [5].

Another variant of (a, d) -edge antimagic total labeling called as e-super (a, d) -edge antimagic total labeling was introduced by Sugeng et al. [14]. Similar to (a, d) -edge antimagic total labeling, they defined an *e-super (a, d) -edge antimagic total labeling* of a graph G as a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \dots, q + p\}$ such that for each edge $uv \in E(G)$, the edge weight $\Lambda(uv) = f(u) + f(uv) + f(v)$ forms an arithmetic progression $a, a + d, \dots, a + (q - 1)d$ with an additional property that the edge labels are $1, 2, \dots, q$ and the vertex labels are $q + 1, q + 2, \dots, q + p$.

Sugeng et al. [14] have proved that the generalized Petersen graph $P(m, n)$ has an e-super (a, d) -edge antimagic total labeling for odd $n \geq 3, m \in \{1, 2, \frac{n-1}{2}\}$ and $d \in \{0, 1, 2\}$. They also proved that every caterpillar has an e-super $(a, 0)$ -edge antimagic total labeling and an e-super $(a, 2)$ -edge antimagic total labeling for any number of vertices $p \geq 3$ and has an e-super $(a, 1)$ -edge antimagic total labeling for even number of vertices $p \geq 4$. Further the relationship between (a, d) -EAV labeling and e-super (a, d) -edge antimagic total labeling are also obtained in [14].

The *total graph* of a graph G denoted by $T(G)$ is defined as a graph in which the set of vertices is both the set of vertices and edges of G and any two vertices in $T(G)$ are adjacent if and only if their corresponding elements are either adjacent or incident in G .

In this paper, we investigate the existence of e-super (a, d) -edge antimagic total labeling for total graphs of paths, copies of cycles and disjoint union of cycles.

2. Properties of e-super (a, d) -edge antimagic total labeling

The following theorem gives an upper bound for d of an e-super (a, d) -edge antimagic total labeling.

Theorem 2.1 . *If a graph G has an e-super (a, d) -edge antimagic total labeling, then $d \leq \frac{2p+q-5}{q-1}$.*

Proof. Let us assume that the graph G has an e-super (a, d) -edge antimagic total labeling. Then by definition, there exist a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, q + p\}$ such that

(i) $f(E(G)) = \{1, 2, \dots, q\}$

(ii) $f(V(G)) = \{q + 1, q + 2, \dots, q + p\}$ and

(iii) for any edge $uv \in E(G)$, the set of edge weight

$$\Lambda(uv) = \{a, a + d, a + 2d, \dots, a + (q - 1)d\}.$$

Clearly the minimum possible edge weight is $(q + 1) + 1 + (q + 2) = 2q + 4$. Thus, we have

$$a \geq 2q + 4. \quad (2.1)$$

Also, the maximum possible edge weight is $(q + p - 1) + q + (q + p) = 3q + 2p - 1$. Thus, we have

$$a + (q - 1)d \leq 3q + 2p - 1 \Rightarrow a \leq 3q + 2p - 1 - (q - 1)d. \quad (2.2)$$

From (2.1) and (2.2) we get, $2q + 4 \leq 3q + 2p - 1 - (q - 1)d$ implying that $(q - 1)d \leq 3q + 2p - 1 - 2q - 4$. Hence, $d \leq \frac{2p+q-5}{(q-1)}$. \square

The following theorem provides a relationship between e-super $(a, 0)$ -edge antimagic total labeling and e-super $(b, 2)$ -edge antimagic total labeling of a graph G .

Theorem 2.2 . *If a graph G has an e-super $(a_1, 0)$ -edge antimagic total labeling then it has an e-super $(a_2, 2)$ -edge antimagic total labeling where $a_2 = a_1 + 1 - q$.*

Proof. Let us assume that the graph G has an e-super $(a_1, 0)$ -edge antimagic total labeling. Then by definition, there exist a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, q+p\}$ such that

$$(i) \ f(E(G)) = \{1, 2, \dots, q\}$$

$$(ii) \ f(V(G)) = \{q + 1, q + 2, \dots, q + p\} \text{ and}$$

$$(iii) \ \text{for every edge } uv \in E(G), \ f(u) + f(uv) + f(v) = a_1.$$

Let us define an induced function $g : V(G) \cup E(G) \rightarrow \{1, 2, \dots, q + p\}$ as follows:

$$(i) \ \text{for every vertex } v \in V(G), \ g(v) = f(v)$$

$$(ii) \ \text{for every edge } uv \in E(G), \ g(uv) = q + 1 - f(uv).$$

Then, we have

$$(i) \ g(E(G)) = \{1, 2, \dots, q\}$$

$$(ii) \ g(V(G)) = \{q + 1, q + 2, \dots, q + p\}$$

and for any edge $uv \in E(G)$,

$$\begin{aligned} g(u) + g(uv) + g(v) &= f(u) + q + 1 - f(uv) + f(v) \\ &= q + 1 + f(u) + f(uv) + f(v) - 2f(uv) \\ &= q + 1 + a_1 - 2f(uv) \\ &= (a_1 + 1 - q) + 2(q - f(uv)). \end{aligned}$$

Since $f(E(G)) = \{1, 2, \dots, q\}$, for any edge $uv \in E(G)$, we have the set of edge weights as

$$g(u) + g(uv) + g(v) = \left\{ \begin{array}{c} (a_1 + 1 - q) + 2(q - 1), (a_1 + 1 - q) + 2(q - 2), \\ \dots, (a_1 + 1 - q) + 2(q - q) \end{array} \right\}$$

$$= \{a_2, a_2 + 2(1), \dots, a_2 + 2(q - 1)\}, \text{ where } a_2 = a_1 + 1 - q.$$

Thus, g is an e-super $(a_2, 2)$ -edge antimagic total labeling of G .

Hence, if G has an e-super $(a_1, 0)$ -edge antimagic total labeling then it has an e-super $(a_2, 2)$ -edge antimagic total labeling where $a_2 = a_1 + 1 - q$. □

3. Total graph of paths P_n

In this section we establish the e-super (a, d) -edge antimagic total labeling for the total graph of paths P_n .

Let $\{v_1, v_2, \dots, v_n\}$ and $\{e_i = v_i v_{i+1} : 1 \leq i \leq n - 1\}$ be the set of vertices and edges respectively of a path P_n . Then we have,

$V(T[P_n]) = \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n - 1\}$ and $E(T[P_n]) = E_1 \cup E_2 \cup E_3 \cup E_4$ where

$$E_1 = \{v_i v_{i+1} : 1 \leq i \leq n - 1\}$$

$$E_2 = \{v_i e_i : 1 \leq i \leq n - 1\}$$

$$E_3 = \{v_i e_{i-1} : 2 \leq i \leq n\}$$

$$E_4 = \{e_i e_{i+1} : 1 \leq i \leq n - 2\}.$$

It is clear that, for the graph $T[P_n]$, $p = 2n - 1$ and $q = 4n - 5$.

By Theorem 2.1, the following lemma is immediate.

Lemma 3.1. *If the graph $T[P_n]$, $n \geq 3$, has an e-super (a, d) -edge antimagic total labeling, then $d \leq 2$.* □

Lemma 3.2. *For every path P_n , $n \geq 3$, the graph $G = T[P_n]$ has an e-super $(a, 0)$ -edge antimagic total labeling.*

Proof. Let us define a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, q + p\}$ as follows:

- (i) $f(v_i v_{i+1}) = 4n - 6 - 4(i - 1)$; for $1 \leq i \leq n - 1$
- (ii) $f(v_i e_i) = 4n - 5 - 4(i - 1)$; for $1 \leq i \leq n - 1$
- (iii) $f(v_i e_{i-1}) = 4n - 7 - 4(i - 2)$; for $2 \leq i \leq n$
- (iv) $f(e_i e_{i+1}) = 4n - 8 - 4(i - 1)$; for $1 \leq i \leq n - 2$

(v) $f(v_i) = 4n - 6 + 2i$; for $1 \leq i \leq n$

(vi) $f(e_i) = 4n - 5 + 2i$; for $1 \leq i \leq n - 1$.

One can easily observe that the edge labels form the set

$$\{1, 2, \dots, 4n - 5\} = \{1, 2, \dots, q\}$$

and the vertex labels form the set

$$\{(4n - 5) + 1, (4n - 5) + 2, \dots, (4n - 5) + (2n - 1)\} = \{q + 1, q + 2, \dots, q + p\}.$$

To complete the proof, we have to prove that for any edge $uv \in E(G)$, $\Lambda(uv)$ is a constant.

For $1 \leq i \leq n - 1$,

$$\begin{aligned} \Lambda(v_i v_{i+1}) &= f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) \\ &= (4n - 6 + 2i) + (4n - 6 - 4(i - 1)) + (4n - 6 + 2(i + 1)) \\ &= 12n - 12 = 12(n - 1). \end{aligned}$$

For $1 \leq i \leq n - 1$,

$$\begin{aligned} \Lambda(v_i e_i) &= f(v_i) + f(v_i e_i) + f(e_i) \\ &= (4n - 6 + 2i) + (4n - 5 - 4(i - 1)) + (4n - 5 + 2i) \\ &= 12n - 12 = 12(n - 1). \end{aligned}$$

For $2 \leq i \leq n$,

$$\begin{aligned} \Lambda(v_i e_{i-1}) &= f(v_i) + f(v_i e_{i-1}) + f(e_{i-1}) \\ &= (4n - 6 + 2i) + (4n - 7 - 4(i - 2)) + (4n - 5 + 2(i - 1)) \\ &= 12n - 12 = 12(n - 1). \end{aligned}$$

For $1 \leq i \leq n - 2$,

$$\begin{aligned} \Lambda(e_i e_{i+1}) &= f(e_i) + f(e_i e_{i+1}) + f(e_{i+1}) \\ &= (4n - 5 + 2i) + (4n - 8 - 4(i - 1)) + (4n - 5 + 2(i + 1)) \\ &= 12n - 12 = 12(n - 1). \end{aligned}$$

Thus, for any edge $uv \in E(G)$, we have $\Lambda(uv) = 12(n - 1)$.

Hence, f is an e-super $(a, 0)$ -edge antimagic total labeling of $T[P_n]$ where $a = 12(n - 1)$. □

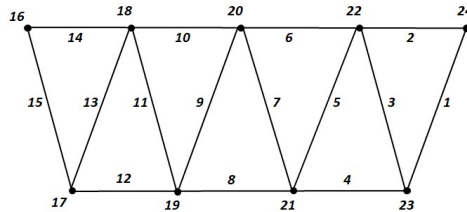


Figure 1. e-Super $(48, 0)$ -edge antimagic total labeling of $T[P_5]$

Lemma 3.3. *For every path P_n , $n \geq 3$, the graph $G = T[P_n]$ has an e -super $(a, 1)$ -edge antimagic total labeling.*

Proof. Let us define a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, q + p\}$ as follows:

- (i) $f(v_i v_{i+1}) = 4n - 3 - 2i$; for $1 \leq i \leq n - 1$
- (ii) $f(v_i e_i) = 2n - 2i$; for $1 \leq i \leq n - 1$
- (iii) $f(v_i e_{i-1}) = 2n + 1 - 2i$; for $2 \leq i \leq n$
- (iv) $f(e_i e_{i+1}) = 4n - 4 - 2i$; for $1 \leq i \leq n - 2$
- (v) $f(v_i) = 4n - 6 + 2i$; for $1 \leq i \leq n$
- (vi) $f(e_i) = 4n - 5 + 2i$; for $1 \leq i \leq n - 1$.

One can easily observe that the edge labels form the set

$$\{1, 2, \dots, 4n - 5\} = \{1, 2, \dots, q\}$$

and the vertex labels form the set

$$\{(4n - 5) + 1, (4n - 5) + 2, \dots, (4n - 5) + (2n - 1)\} = \{q + 1, q + 2, \dots, q + p\}.$$

To complete the proof, we have to prove that the edge weights $\Lambda(uv)$ form an arithmetic sequence $\{a, a + 1, \dots, a + (q - 1)\}$.

For $1 \leq i \leq n - 1$,

$$\begin{aligned} \Lambda(v_i v_{i+1}) &= f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) \\ &= (4n - 6 + 2i) + (4n - 3 - 2i) + (4n - 6 + 2(i + 1)) \\ &= 12n - 13 + 2i = (10n - 9) + 2(n + i) - 4. \end{aligned}$$

For $1 \leq i \leq n - 1$,

$$\begin{aligned} \Lambda(v_i e_i) &= f(v_i) + f(v_i e_i) + f(e_i) \\ &= (4n - 6 + 2i) + (2n - 2i) + (4n - 5 + 2i) \\ &= 10n - 11 + 2i = (10n - 9) + 2(i - 1). \end{aligned}$$

For $2 \leq i \leq n$,

$$\begin{aligned} \Lambda(v_i e_{i-1}) &= f(v_i) + f(v_i e_{i-1}) + f(e_{i-1}) \\ &= (4n - 6 + 2i) + (2n + 1 - 2i) + (4n - 5 + 2(i - 1)) \\ &= 10n - 12 + 2i = (10n - 9) + 2(i - 1) - 1. \end{aligned}$$

For $1 \leq i \leq n - 2$,

$$\begin{aligned} \Lambda(e_i e_{i+1}) &= f(e_i) + f(e_i e_{i+1}) + f(e_{i+1}) \\ &= (4n - 5 + 2i) + (4n - 4 - 2i) + (4n - 5 + 2(i + 1)) \\ &= 12n - 12 + 2i = (10n - 9) + 2(n + i) - 3. \end{aligned}$$

Thus, the edge weights are

$$(10n - 9), (10n - 9) + 1, \dots, (10n - 9) + (4n - 6).$$

Hence, f is an e -super $(a, 1)$ -edge antimagic total labeling of $T[P_n]$ where $a = 10n - 9$. □

By Lemmas 3.1, 3.2, 3.3 and Theorem 2.2 , we have the following theorem:

Theorem 3.3 . *The graph $T[P_n]$, $n \geq 3$, has an e-super (a, d) -edge antimagic total labeling if and only if $d \in \{0, 1, 2\}$.*

4. Total graph of copies of cycles C_n

This section deals with the e-super (a, d) -edge antimagic total labeling of total graph of copies of cycles C_n .

Let $\{v_j^i : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $\{e_j^i = v_j^i v_{j+1}^i : 1 \leq i \leq m, 1 \leq j \leq n\}$ (where the subscripts i and j are taken modulo m and modulo n respectively) be the set of vertices and edges of the disjoint union of m copies of cycles C_n . Then for the total graph of m copies of C_n , we have

$V(T[mC_n]) = \{v_j^i : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{e_j^i : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(T[mC_n]) = E_1 \cup E_2 \cup E_3 \cup E_4$ where

$$E_1 = \{v_j^i v_{j+1}^i : 1 \leq i \leq m, 1 \leq j \leq n\}$$

$$E_2 = \{v_j^i e_j^i : 1 \leq i \leq m, 1 \leq j \leq n\}$$

$$E_3 = \{v_j^i e_{j+1}^i : 1 \leq i \leq m, 1 \leq j \leq n\}$$

$$E_4 = \{e_j^i e_{j+1}^i : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

It is clear that, for the graph $T[mC_n]$, $p = 2mn$ and $q = 4mn$.

By Theorem 2.1 , the following lemma is immediate.

Lemma 4.4. *If the graph $T[mC_n]$, $m \geq 1, n \geq 3$ has an e-super (a, d) -edge antimagic total labeling, then $d < 2$.*

Lemma 4.5. *For every disjoint union of m copies of cycles C_n , $m \geq 1, n \geq 3$, the graph $G = T[mC_n]$, has no e-super $(a, 0)$ -edge antimagic total labeling.*

Proof. Suppose G has an e-super $(a, 0)$ -edge antimagic total labeling.

Then by definition, there exist a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, q + p\}$ such that

(i) $f(E(G)) = \{1, 2, \dots, q\}$

(ii) $f(V(G)) = \{q + 1, q + 2, \dots, q + p\}$ and

(iii) for all edge $uv \in E(G)$, $\Lambda(uv) = a$.

Since G is a 4-regular graph, we have the sum of all edge weights is equal to

$$4 \sum_{v \in V(G)} f(v) + \sum_{e \in E(G)} f(e) = 4 \sum_{j=1}^{2mn} (4mn + j) + \sum_{i=1}^{4mn} i = 48m^2n^2 + 6mn. \tag{4.1}$$

Also, since G has an e-super $(a, 0)$ -edge antimagic total labeling, the sum of all edge weights is equal to

$$\sum_{i=1}^{4mn} a = 4mna. \tag{4.2}$$

From (4.1) and (4.2) we get, $4mna = 48m^2n^2 + 6mn$ implying that $a = 12mn + \frac{3}{2}$ which is not an integer. Hence, for the graph $T[mC_n]$, $m \geq 1, n \geq 3$, there is no e-super $(a, 0)$ -edge antimagic total labeling. \square

Lemma 4.6. *For every disjoint union of m copies of cycles C_n , $m \geq 1, n \geq 3$, the graph $G = T[mC_n]$, has an e-super $(a, 1)$ -edge antimagic total labeling.*

Proof. Let us define a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, q + p\}$ as follows:

- (i) $f(v_j^i v_{j+1}^i) = 2ni + 2 - 2j$; for $1 \leq i \leq m, 1 \leq j \leq n$
- (ii) $f(v_j^i e_j^i) = 2mn + 2ni + 1 - 2j$; for $1 \leq i \leq m, 1 \leq j \leq n$
- (iii) $f(v_j^i e_{j+1}^i) = 2mn + 2ni - 2j$; for $1 \leq i \leq m, 1 \leq j \leq n - 1$
 $f(v_n^i e_1^i) = 2mn + 2ni$; for $1 \leq i \leq m$
- (iv) $f(e_j^i e_{j+1}^i) = 2n(i - 1) + 2j - 1$; for $1 \leq i \leq m, 1 \leq j \leq n$
- (v) $f(v_j^i) = 6mn - 2ni - 1 + 2j$; for $1 \leq i \leq m, 1 \leq j \leq n$
- (vi) $f(e_j^i) = 6mn - 2n(i - 1) + 4 - 2j$; for $1 \leq i \leq m, 2 \leq j \leq n$
 $f(e_1^i) = 6mn - 2ni + 2$; for $1 \leq i \leq m$.

One can easily observe that the edge labels form the set

$$\{1, 2, \dots, 4mn\} = \{1, 2, \dots, q\}$$

and the vertex labels form the set

$$\{4mn + 1, 4mn + 2, \dots, 6mn\} = \{q + 1, q + 2, \dots, q + p\}.$$

To complete the proof, we have to prove that the edge weights $\Lambda(uv)$ form an arithmetic sequence $\{a, a + 1, \dots, a + (q - 1)\}$.

For $1 \leq i \leq m, 1 \leq j \leq n - 1$,

$$\begin{aligned} \Lambda(v_j^i v_{j+1}^i) &= f(v_j^i) + f(v_j^i v_{j+1}^i) + f(v_{j+1}^i) \\ &= (6mn - 2ni + 2j - 1) + (2ni + 2 - 2j) + (6mn - 2ni + 2(j + 1) - 1) \\ &= (12mn - 2ni + 2) + 2j. \end{aligned}$$

For $1 \leq i \leq m$,

$$\begin{aligned}\Lambda(v_n^i v_1^i) &= f(v_n^i) + f(v_n^i v_1^i) + f(v_1^i) \\ &= (6mn - 2ni + 2n - 1) + 2n(i - 1) + 2 + (6mn - 2ni + 2 - 1) \\ &= (12mn - 2ni + 2).\end{aligned}$$

For $1 \leq i \leq m, 2 \leq j \leq n$,

$$\begin{aligned}\Lambda(v_j^i e_j^i) &= f(v_j^i) + f(v_j^i e_j^i) + f(e_j^i) \\ &= (6mn - 2ni + 2j - 1) + (2mn + 2ni + 1 - 2j) \\ &\quad + (6mn - 2n(i - 1) + 4 - 2j) \\ &= (12mn - 2ni + 2) + (2mn + 2n + 2 - 2j).\end{aligned}$$

For $1 \leq i \leq m$,

$$\begin{aligned}\Lambda(v_1^i e_1^i) &= f(v_1^i) + f(v_1^i e_1^i) + f(e_1^i) \\ &= (6mn - 2ni + 2 - 1) + (2mn + 2ni + 1 - 2) + (6mn - 2ni + 2) \\ &= (12mn - 2ni + 2) + (2mn).\end{aligned}$$

For $1 \leq i \leq m, 1 \leq j \leq n - 1$,

$$\begin{aligned}\Lambda(v_j^i e_{j+1}^i) &= f(v_j^i) + f(v_j^i e_{j+1}^i) + f(e_{j+1}^i) \\ &= (6mn - 2ni + 2j - 1) + (2mn + 2ni - 2j) \\ &\quad + (6mn - 2n(i - 1) + 4 - 2(j + 1)) \\ &= (12mn - 2ni + 2) + (2mn + 2n - 2j - 1).\end{aligned}$$

For $1 \leq i \leq m$,

$$\begin{aligned}\Lambda(v_n^i e_1^i) &= f(v_n^i) + f(v_n^i e_1^i) + f(e_1^i) \\ &= (6mn - 2ni + 2n - 1) + (2mn + 2ni) + (6mn - 2ni + 2) \\ &= (12mn - 2ni + 2) + (2mn + 2n - 1).\end{aligned}$$

For $1 \leq i \leq m, 2 \leq j \leq n - 1$,

$$\begin{aligned}\Lambda(e_j^i e_{j+1}^i) &= f(e_j^i) + f(e_j^i e_{j+1}^i) + f(e_{j+1}^i) \\ &= (6mn - 2n(i - 1) + 4 - 2j) + (2n(i - 1) + 2j - 1) \\ &\quad + (6mn - 2n(i - 1) + 4 - 2(j + 1)) \\ &= (12mn - 2ni + 2) + (2n - 2j + 3).\end{aligned}$$

For $1 \leq i \leq m$,

$$\begin{aligned}\Lambda(e_n^i e_1^i) &= f(e_n^i) + f(e_n^i e_1^i) + f(e_1^i) \\ &= (6mn - 2n(i - 1) + 4 - 2n) + (2ni - 1) + (6mn - 2ni + 2) \\ &= (12mn - 2ni + 2) + 3.\end{aligned}$$

For $1 \leq i \leq m$,

$$\begin{aligned}\Lambda(e_1^i e_2^i) &= f(e_1^i) + f(e_1^i e_2^i) + f(e_2^i) \\ &= (6mn - 2ni + 2) + (2n(i - 1) + 1) + (6mn - 2n(i - 1)) \\ &= (12mn - 2ni + 2) + 1.\end{aligned}$$

Thus, the edge weights are

$$(10mn + 2), (10mn + 2) + 1, \dots, (10mn + 2) + (4mn - 1).$$

Hence, f is an e-super $(a, 1)$ -edge antimagic total labeling of $T[mC_n]$ where $a = 10mn + 2$. \square

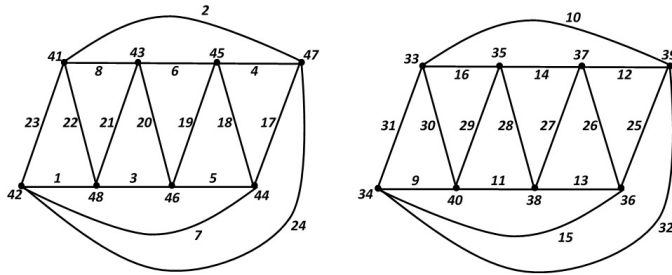


Figure 2. e-Super (82, 1)-edge antimagic total labeling of $T[2C_4]$

By Lemmas 4.4, 4.5 and 4.6, we have the following theorem:

Theorem 4.4 . *The graph $T[mC_n]$, $m \geq 1$, $n \geq 3$, has an e-super (a, d) -edge antimagic total labeling if and only if $d = 1$.*

As a particular case to the above theorem, when $m = 1$, we have the following corollary.

Corollary 4.1. *The graph $T[C_n]$, $n \geq 3$, has an e-super (a, d) -edge antimagic total labeling if and only if $d = 1$.*

5. Total graph of disjoint union of cycles

Let $\{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$ and $\{e_i = u_i u_{i+1} : 1 \leq i \leq m\} \cup \{h_j = v_j v_{j+1} : 1 \leq j \leq n\}$ (where the subscripts i and j are taken modulo m and modulo n respectively) be the set of vertices and edges of the disjoint union of cycles $C_m \cup C_n$, $m \neq n$. Then we have,

$$V(T[C_m \cup C_n]) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\} \cup \{e_i : 1 \leq i \leq m\} \cup \{h_j : 1 \leq j \leq n\}$$

and $E(T[C_m \cup C_n]) = E_1 \cup E_2 \cup E_3 \cup E_4$ where

$$E_1 = \{u_i u_{i+1}, v_j v_{j+1} : 1 \leq i \leq m - 1, 1 \leq j \leq n\}$$

$$E_2 = \{u_i e_i, v_j h_j : 1 \leq i \leq m, 1 \leq j \leq n\}$$

$$E_3 = \{u_i e_{i+1}, v_j h_{j+1} : 1 \leq i \leq m - 1, 1 \leq j \leq n\}$$

$$E_4 = \{e_i e_{i+1}, h_j h_{j+1} : 1 \leq i \leq m - 1, 1 \leq j \leq n\}.$$

It is clear that, for the graph $T[C_m \cup C_n]$, $p = 2(m + n)$ and $q = 4(m + n)$.

By Theorem 2.1 , the following lemma is immediate.

Lemma 5.7. *If the graph $T[C_m \cup C_n]$, $m \neq n$, $m, n \geq 3$, has an e-super (a, d) -edge antimagic total labeling, then $d < 2$.*

Similar to the proof of Lemma 4.6, we have the following lemma.

Lemma 5.8. *For every disjoint union of cycles $C_m \cup C_n$, $m \neq n$, $m, n \geq 3$, the graph $G = T[C_m \cup C_n]$, has no e-super $(a, 0)$ -edge antimagic total labeling.*

Lemma 5.9. *For every disjoint union of cycles $C_m \cup C_n$, $m \neq n$, $m, n \geq 3$, the graph $G = T[C_m \cup C_n]$, has an e-super $(a, 1)$ -edge antimagic total labeling.*

Proof. Let us define a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, q + p\}$ as follows:

- (i) $f(u_i u_{i+1}) = 2m + 2 - 2i$; for $1 \leq i \leq m$
 $f(v_j v_{j+1}) = 2m + 2n + 2 - 2j$; for $1 \leq j \leq n$
- (ii) $f(u_i e_i) = 4m + 2n + 1 - 2i$; for $1 \leq i \leq m$
 $f(v_j h_j) = 4m + 4n + 1 - 2j$; for $1 \leq j \leq n$
- (iii) $f(u_i e_{i+1}) = 4m + 2n - 2i$; for $1 \leq i \leq m - 1$, $f(u_m e_1) = 4m + 2n$
 $f(v_j h_{j+1}) = 4m + 4n - 2j$; for $1 \leq j \leq n - 1$, $f(v_n h_1) = 4m + 4n$
- (iv) $f(e_i e_{i+1}) = 2i - 1$; for $1 \leq i \leq m$
 $f(h_j h_{j+1}) = 2m - 1 + 2j$; for $1 \leq j \leq n$
- (v) $f(u_i) = 4m + 6n - 1 + 2i$; for $1 \leq i \leq m$
 $f(v_j) = 4m + 4n - 1 + 2j$; for $1 \leq j \leq n$
- (vi) $f(e_i) = 6m + 6n + 4 - 2i$; for $2 \leq i \leq m$, $f(e_1) = 4m + 6n + 2$
 $f(h_j) = 4m + 6n + 4 - 2j$; for $2 \leq j \leq n$, $f(h_1) = 4m + 4n + 2$.

One can easily observe that the edge labels form the set

$$\{1, 2, \dots, 4(m+n)\} = \{1, 2, \dots, q\}$$

and the vertex labels form the set

$$\{4(m+n) + 1, 4(m+n) + 2, \dots, 6(m+n)\} = \{q + 1, q + 2, \dots, q + p\}.$$

To complete the proof, we have to prove that the edge weights $\Lambda(uv)$ form an arithmetic sequence $\{a, a + 1, \dots, a + (q - 1)\}$.

For $1 \leq i \leq m - 1$,

$$\begin{aligned} \Lambda(u_i u_{i+1}) &= f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) \\ &= (4m + 6n - 1 + 2i) + (2m + 2 - 2i) + (4m + 6n - 1 + 2(i + 1)) \\ &= (10(m + n) + 2) + (2n + 2i) \end{aligned}$$

and

$$\begin{aligned}\Lambda(u_m u_1) &= f(u_m) + f(u_m u_1) + f(u_1) \\ &= (4m + 6n - 1 + 2m) + 2 + (4m + 6n - 1 + 2) \\ &= (10(m + n) + 2) + 2n.\end{aligned}$$

For $1 \leq j \leq n - 1$,

$$\begin{aligned}\Lambda(v_j v_{j+1}) &= f(v_j) + f(v_j v_{j+1}) + f(v_{j+1}) \\ &= (4m + 4n - 1 + 2j) + (2m + 2n + 2 - 2j) + (4m + 4n - 1 + 2(j + 1)) \\ &= (10(m + n) + 2) + 2j\end{aligned}$$

and

$$\begin{aligned}\Lambda(v_n v_1) &= f(v_n) + f(v_n v_1) + f(v_1) \\ &= (4m + 4n - 1 + 2n) + (2m + 2) + (4m + 4n - 1 + 2) \\ &= (10(m + n) + 2).\end{aligned}$$

For $2 \leq i \leq m$,

$$\begin{aligned}\Lambda(u_i e_i) &= f(u_i) + f(u_i e_i) + f(e_i) \\ &= (4m + 6n - 1 + 2i) + (4m + 2n + 1 - 2i) + (6m + 6n + 4 - 2i) \\ &= (10(m + n) + 2) + (4m + 4n + 2 - 2i)\end{aligned}$$

and

$$\begin{aligned}\Lambda(u_1 e_1) &= f(u_1) + f(u_1 e_1) + f(e_1) \\ &= (4m + 6n - 1 + 2) + (4m + 2n + 1 - 2) + (4m + 6n + 2) \\ &= (10(m + n) + 2) + (2m + 4n).\end{aligned}$$

For $2 \leq j \leq n$,

$$\begin{aligned}\Lambda(v_j h_j) &= f(v_j) + f(v_j h_j) + f(h_j) \\ &= (4m + 4n - 1 + 2j) + (4m + 4n + 1 - 2j) + (4m + 6n + 4 - 2j) \\ &= (10(m + n) + 2) + (2m + 4n + 2 - 2j)\end{aligned}$$

and

$$\begin{aligned}\Lambda(v_1 h_1) &= f(v_1) + f(v_1 h_1) + f(h_1) \\ &= (4m + 4n - 1 + 2) + (4m + 4n + 1 - 2) + (4m + 4n + 2) \\ &= (10(m + n) + 2) + (2m + 2n).\end{aligned}$$

For $1 \leq i \leq m - 1$,

$$\begin{aligned}\Lambda(u_i e_{i+1}) &= f(u_i) + f(u_i e_{i+1}) + f(e_{i+1}) \\ &= (4m + 6n - 1 + 2i) + (4m + 2n - 2i) + (6m + 6n + 4 - 2(i + 1)) \\ &= (10(m + n) + 2) + (4m + 4n - 1 - 2i)\end{aligned}$$

and

$$\begin{aligned}\Lambda(u_m e_1) &= f(u_m) + f(u_m e_1) + f(e_1) \\ &= (4m + 6n - 1 + 2m) + (4m + 2n) + (4m + 6n + 2) \\ &= (10(m + n) + 2) + (4m + 4n - 1).\end{aligned}$$

For $1 \leq j \leq n - 1$,

$$\begin{aligned} \Lambda(v_j h_{j+1}) &= f(v_j) + f(v_j h_{j+1}) + f(h_{j+1}) \\ &= (4m + 4n - 1 + 2j) + (4m + 4n - 2j) + (4m + 6n + 4 - 2(j + 1)) \\ &= (10(m + n) + 2) + (2m + 4n - 1 - 2j) \end{aligned}$$

and

$$\begin{aligned} \Lambda(v_n h_1) &= f(v_n) + f(v_n h_1) + f(h_1) \\ &= (4m + 4n - 1 + 2n) + (4m + 4n) + (4m + 4n + 2) \\ &= (10(m + n) + 2) + (2m + 4n - 1). \end{aligned}$$

For $2 \leq i \leq m$,

$$\begin{aligned} \Lambda(e_i e_{i+1}) &= f(e_i) + f(e_i e_{i+1}) + f(e_{i+1}) \\ &= (6m + 6n + 4 - 2i) + (2i - 1) + (6m + 6n + 4 - 2(i + 1)) \\ &= (10(m + n) + 2) + (2m + 2n + 3 - 2i) \end{aligned}$$

and

$$\begin{aligned} \Lambda(e_1 e_2) &= f(e_1) + f(e_1 e_2) + f(e_2) \\ &= (4m + 6n + 2) + (2 - 1) + (6m + 6n + 4 - 4) \\ &= (10(m + n) + 2) + (2n + 1). \end{aligned}$$

For $2 \leq j \leq n$,

$$\begin{aligned} \Lambda(h_j h_{j+1}) &= f(h_j) + f(h_j h_{j+1}) + f(h_{j+1}) \\ &= (4m + 6n + 4 - 2j) + (2m + 2j - 1) + (4m + 6n + 4 - 2(j + 1)) \\ &= (10(m + n) + 2) + (2n + 3 - 2j) \end{aligned}$$

and

$$\begin{aligned} \Lambda(h_1 h_2) &= f(h_1) + f(h_1 h_2) + f(h_2) \\ &= (4m + 4n + 2) + (2m + 2 - 1) + (4m + 6n + 4 - 4) \\ &= (10(m + n) + 2) + 1. \end{aligned}$$

Thus, the edge weights are

$$(10(m + n) + 2), (10(m + n) + 2) + 1, \dots, (10(m + n) + 2) + (4m + 4n - 1).$$

Hence, f is an e-super $(a, 1)$ -edge antimagic total labeling of $T[C_m \cup C_n]$ where $a = 10(m + n) + 2$. □

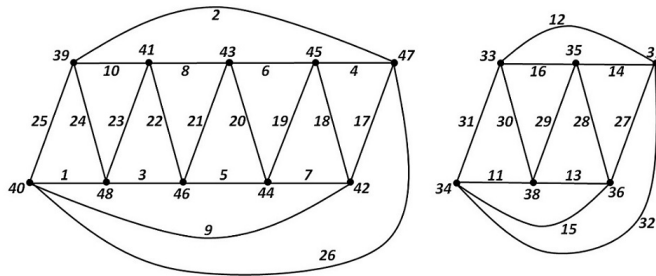


Figure 3. e-Super $(82, 1)$ -edge antimagic total labeling of $T[C_5 \cup C_3]$

By Lemmas 5.7, 5.8 and 5.9, we have the following theorem:

Theorem 5.5 . *The graph $T[C_m \cup C_n]$, $m \neq n$, $m, n \geq 3$, has an e -super (a, d) -edge antimagic total labeling if and only if $d = 1$.*

Acknowledgement: The authors wish to thank the referees for their valuable suggestions which were instrumental in transforming the paper to its present form.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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