

## On connected bipartite $Q$ -integral graphs

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**Abstract:** A graph  $G$  is said to be  $H$ -free if  $G$  does not contain  $H$  as an induced subgraph. Let  $\mathcal{S}_n^2(m)$  be a variation of double star  $\mathcal{S}_n^2$  obtained by adding  $m(\leq n)$  disjoint edges between the pendant vertices which are at distance 3 in  $\mathcal{S}_n^2$ . A graph having integer eigenvalues for its signless Laplacian matrix is known as a  $Q$ -integral graph. The  $Q$ -spectral radius of a graph is the largest eigenvalue of its signless Laplacian. Any connected  $Q$ -integral graph  $G$  with  $Q$ -spectral radius 7 and maximum edge-degree 8 is either  $K_{1,4} \square K_2$  or contains  $\mathcal{S}_4^2(0)$  as an induced subgraph or is a bipartite graph having at least one of the induced subgraphs  $\mathcal{S}_4^2(m)$ , ( $m = 1, 2, 3$ ). In this article, we improve this result by showing that every connected  $Q$ -integral graph  $G$  having  $Q$ -spectral radius 7, maximum edge-degree 8 is always bipartite and  $\mathcal{S}_4^2(3)$ -free.

**Keywords:** edge-degree,  $H$ -free graph, signless Laplacian matrix,  $Q$ -integral graph.

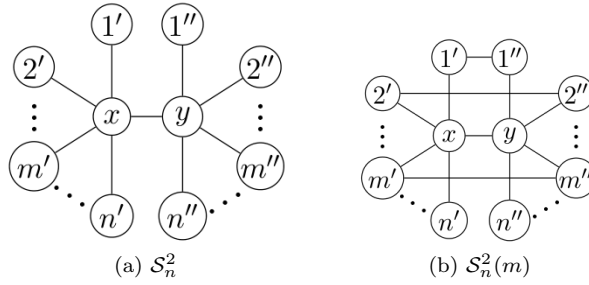
**AMS Subject classification:** 05C50, 05C07

### 1. Introduction

All the graphs considered in this article are simple and undirected. Let  $G = (V(G), E(G))$  be a graph, where  $V(G)$  and  $E(G)$  denote the set of vertices and edges, respectively. Let  $d(v)$  be the *degree* of a vertex  $v$  and  $N(v)$  be the *neighborhood* of  $v \in V(G)$ . The Cartesian product  $G_1 \square G_2$  obtained from the graphs  $G_1$  and  $G_2$  is a graph with vertex set  $V(G_1) \times V(G_2)$ , and two vertices  $(v_1, v_2)$  and  $(u_1, u_2)$  are adjacent in  $G_1 \square G_2$  if and only if either  $v_1 = u_1$  and  $v_2$  is adjacent to  $u_2$  in  $G_2$  or  $v_2 = u_2$  and  $v_1$  is adjacent to  $u_1$  in  $G_1$ . A graph  $G$  is said to be  $H$ -free if  $H$  is not an induced subgraph of  $G$ .

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**Figure 1.**

Let  $D(G)$  be the diagonal matrix with  $D(G)_{vv} = d(v)$ , for  $v \in V(G)$ . The *signless Laplacian*  $Q(G)$  of  $G$  is defined by the matrix  $D(G) + A(G)$ . The matrix  $Q(G)$  is positive semidefinite and irreducible. The  $Q$ -eigenvalues and  $Q$ -spectral radius  $q(G)$  of  $G$  are the eigenvalues and largest eigenvalue of  $Q(G)$ , respectively. A graph  $G$  is called  $Q$ -integral if all the  $Q$ -eigenvalues of  $G$  are integral. For some more studies related to signless Laplacian and  $Q$ -integral graphs, see [2–4, 6, 10–14, 16].

We define a *double star*  $\mathcal{S}_n^2$  by taking two disjoint copies of star graph  $K_{1,n}$  and adding an edge between the vertices of degree  $n$ . Let  $\mathcal{S}_n^2(m)$  be a *variation of double star*  $\mathcal{S}_n^2$  obtained by adding  $m (\leq n)$  disjoint edges between the pendant vertices which are at distance 3 in  $\mathcal{S}_n^2$  (see Figure 1).

We use  $e' = uv \in E(G)$  to denote an edge having  $u, v$  as incident vertices and the *edge-degree*  $e\text{-deg}_G(e')$  is given by  $|N(u) \cup N(v)| - 2$ . The *maximum edge-degree* of  $G$  is denoted by  $e\text{-deg}_G^{max}$ . Simić and Stanić [15] studied connected  $Q$ -integral graphs with  $e\text{-deg}_G^{max} \leq 4$  and also gave partial results about the spectra of  $Q(G)$  for  $e\text{-deg}_G = 5$ . Park and Sano [8] investigated on connected  $Q$ -integral graphs  $G$  having  $e\text{-deg}_G^{max} \leq 6$  and gave a structural theorem when  $q(G) = 6$ . An improvement of this result can be found in [7].

In 2022 [9], we studied connected  $Q$ -integral graph having  $e\text{-deg}_G^{max} \leq 8$  and gave a structural characterization under the restriction  $q(G) = 7$ . We have shown that  $G (\neq K_{1,4} \square K_2)$  must contain one of the four special subgraphs  $\mathcal{S}_4^2(m)$  for  $m = 0, \dots, 3$  (as shown in Figure 2) as an induced subgraph.

## 2. Preliminaries

Let  $n \in \mathbb{N}$  be any number. We use  $\mathbf{0}$  and  $\mathbf{1}$  to denote the matrices of appropriate orders whose entries are all equal to 0 and 1, respectively. The multiset of all the eigenvalues of  $N$  together with their multiplicities is called as the *spectrum*  $\text{Spec}(N)$  of the matrix  $N_{n \times n}$ . The *spectral radius* of  $N$  is  $\rho(N) = \max\{|\beta| \mid \beta \in \text{Spec}(N)\}$ . The *least*, *second smallest*, and the *second largest eigenvalues* of a matrix  $N$  are denoted by  $\lambda_{\min}(N)$ ,  $\lambda_{s2}(N)$ , and  $\lambda_{l2}(N)$ , respectively.

The principal submatrix of  $Q(G)$  corresponding to the vertices of  $H \subseteq V(G)$  is de-

noted by  $Q_p(H)$ . For any two distinct vertices  $u, v \in V(G)$ , the  $(u, v)$ -th entry of  $Q_p(H)$  is denoted by  $a_{uv}$ . We use  $a_{..}$  and  $d(\cdot)$  in place of  $a_{uv}$  and  $d(z)$  when  $u, v$  and  $z$  are suitable vertices within the context.

**Proposition 1 ([1], Proposition 1.3.9).** *The number of connected bipartite components of  $G$  is equal to the multiplicity of the  $Q$ -eigenvalue 0 in  $G$ .*

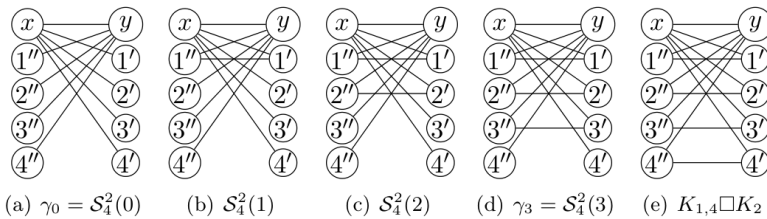
**Proposition 2 ([8], Proposition 2.7).** *A connected graph  $G$  has  $d(v) \leq \lceil q(G) - 1 \rceil$  for any  $v \in V(G)$ , where  $q(G)$  is the  $Q$ -spectral radius of  $G$ . If  $G$  has a vertex  $v$  having  $d(v) = q(G) - 1$  and  $q(G) \in \mathbb{Z}^+$ , then  $G = K_{1, q(G)-1}$ .*

The following results give bounds for the maximum edge-degree of a graph  $G$ .

**Remark 1 ([9], Remark 3.2).** For a connected edge-regular graph  $G$ ,  $e\text{-deg}_G = q(G) - 2$ .

**Lemma 1 ([9], Lemma 3.3).** *Let  $G$  be a connected edge-non-regular  $Q$ -integral graph. Then  $q(G) - 1 \leq e\text{-deg}_G^{max} \leq 2q(G) - 6$ .*

**Remark 2 ([9], Remark 3.4).** There does not exist any connected edge-non-regular  $Q$ -integral graph with  $q(G) \leq 4$ . Moreover, if  $q(G) = 5$ , then  $e\text{-deg}_G^{max} = 4$ .



**Figure 2.**

The following remark is a consequence of the results (Remark 1, Lemma 1, Remark 2) stated above.

**Remark 3.** A connected  $Q$ -integral graph  $G$  is edge-regular if and only if  $q(G) = 4$ .

Now, suppose  $G$  is any connected  $Q$ -integral graph with  $e\text{-deg}_G^{max} = 2q(G) - 6$ .

- $q(G) \leq 3$ : There does not exist any such graph  $G$ .
- $q(G) = 4$ :  $G$  must be one of the graphs  $C_3, C_4, C_6, K_{1,3}$ . Note that  $G(\neq C_3)$  is bipartite and  $G(\neq C_6)$  is  $S_1^2(0)$ -free.

- $q(G) = 5$ :  $G$  must be one of the graphs  $K_{1,2} \square K_2, \overline{K_{3,3} - e}$  (the complement of the graph  $K_{3,3} - e$ , where  $e$  is an edge). Note that both of them are  $\mathcal{S}_2^2(1)$ -free.
- $q(G) = 6$ :  $G$  is bipartite and  $\mathcal{S}_3^2(2)$ -free.

Recently in [9], it was proved that  $Q$ -integral graph contains  $\mathcal{S}_4^2(m)$  ( $0 \leq m \leq 3$ ) as an induced subgraph when  $q(G) = 7$ .

**Theorem 1 ([9], Theorem 4.1).** *Suppose  $G$  is a connected  $Q$ -integral graph having  $q(G) = 7$ . If  $e\text{-deg}_G^{max} = 8$ , then one of the following hold.*

- (a)  $G = K_{1,4} \square K_2$ .
- (b)  $G$  is bipartite with at least one of  $\mathcal{S}_4^2(m)$  ( $m = 1, 2, 3$ ), given in Figure 2, as induced subgraph, and  $1, 6 \in \text{Spec}(Q(G))$ .
- (c)  $G$  has  $\mathcal{S}_4^2(0)$  as induced subgraph, and  $1, 6 \in \text{Spec}(Q(G))$ .

In this article, we show that a connected  $Q$ -integral graph  $G$  with  $q(G) = 7$  and  $e\text{-deg}_G^{max} = 8$  is bipartite and  $\mathcal{S}_4^2(3)$ -free.

### 3. Main Result

For the rest of the article,  $G$  denotes a connected  $Q$ -integral graph having  $q(G) = 7$  and maximum edge-degree  $e\text{-deg}_G^{max} = 8$ . As a main result, we improve Theorem 1 by showing that  $G$  is bipartite, and if  $G \neq K_{1,4} \square K_2$  then it contains at least one of  $\mathcal{S}_4^2(m)$ , for  $m = 0, 1, 2$  as an induced subgraph.

**Theorem 2. (Main Result)** *Suppose  $G$  is a connected  $Q$ -integral graph having  $q(G) = 7$ . If  $e\text{-deg}_G^{max} = 8$ , then  $G$  is bipartite and  $\mathcal{S}_4^2(3)$ -free, where  $\mathcal{S}_4^2(3)$  is given in Figure 2.*

Before we prove the theorem, we require the following notations.

For any  $S_1, S_2 \subseteq V(G)$  and  $S_1 \cap S_2 = \phi$ , let  $A_{S_1, S_2}$  be a matrix of order  $|S_1| \times |S_2|$  whose rows and columns corresponds to the vertices of  $S_1$  and  $S_2$ , respectively. Let the  $(u, v)$ -th element,  $a_{uv}$ , for  $u \in S_1, v \in S_2$  be such that

$$a_{uv} = \begin{cases} 1, & \text{if } uv \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Thus,  $A_{S_1, S_2}^t = A_{S_2, S_1}$ . For brevity and clarity, we use  $\gamma_0$  (resp.  $\gamma_3$ ) to denote  $\mathcal{S}_4^2(0)$  (resp.  $\mathcal{S}_4^2(3)$ ) in the rest of the article. Let  $\Gamma_0 = V(\gamma_0) = V(\gamma_3) = \{x, y, 1', 2', 3', 4', 1'', 2'', 3'', 4''\}$  be the subset of  $V(G)$ . In the proof, we iteratively

define  $\Gamma_{i+1} = \Gamma_i \cup S_i$  where  $S_i \subseteq V(G) \setminus \Gamma_i$ ,  $i \geq 0$ . For  $\Gamma_{i+1}$  obtained from the pair  $(\Gamma_i, S_i)$ , we have the principal submatrix

$$Q_p(\Gamma_{i+1}) = \begin{pmatrix} Q_p(\Gamma_i) & A_{\Gamma_i, S_i} \\ A_{S_i, \Gamma_i} & Q_p(S_i) \end{pmatrix}. \tag{3.1}$$

We will use these notations repeatedly in rest of the paper with appropriate definitions of  $\Gamma_i$  and  $S_i$ , respectively. We use MATLAB to calculate the eigenvalues of matrices.

**Proof of Theorem 2**

Let  $G$  be as stated in the theorem with  $e\text{-deg}_G^{max} = 8$ . By Proposition 2, the maximum vertex-degree of  $G$  must be less than or equal to 5. In order to prove this theorem, we observe that it is sufficient to show that

- (a)  $G$  is  $\gamma_3$ -free, where  $\gamma_3$  is given in Figure 2.
- (b)  $G$  is bipartite whenever it contains the induced subgraph  $\gamma_0$ .

We prove (a) by contradiction, that is, suppose  $\gamma_3$  is an induced subgraph of  $G$ . By Theorem 1,  $G$  is bipartite. Thus, 0, 7 are simple  $Q$ -eigenvalues of  $G$  by Proposition 1 and Perron-Frobenius Theorem [[5], Theorem 8.4.4]. Since  $Spec(Q(G)) \subseteq \mathbb{Z}$ , we have  $\lambda_{s2}(Q(G)) = 1$  and  $\lambda_{l2}(Q(G)) = 6$ . By Interlacing Theorem [[5], Theorem 4.3.17] on eigenvalues, we have the following remark which will be used repeatedly to prove our main theorem.

**Remark 4.** If  $G$  is a bipartite connected  $Q$ -integral graph having  $e\text{-deg}_G^{max} = 8$  and  $q(G) = 7$ , then every principal submatrix  $Q_p(H)$  of  $Q(G)$  corresponding to a set of vertices  $H \subseteq V(G)$  have  $\lambda_{s2}(Q_p(H)) \geq 1$  and  $\lambda_{l2}(Q_p(H)) \leq 6$ .

Recall  $\Gamma_0 = V(\gamma_3) = \{x, y, 1', 2', 3', 4', 1'', 2'', 3'', 4''\} \subseteq V(G)$ . Note that the subgraph  $\gamma_3$  is given by Figure 2(d) and its principal submatrix  $Q_p(\Gamma_0)$  of  $Q(G)$  is given by

$$Q_p(\Gamma_0) = \begin{pmatrix} 5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d(1') & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & d(2') & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & d(3') & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & d(4') & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & d(1'') & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & d(2'') & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & d(3'') & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d(4'') \end{pmatrix} \tag{3.2}$$

where  $d(\cdot) \in \{1, \dots, 5\}$ . We have the following 4 non-isomorphic choices for  $d(\cdot)$  as  $G$  has  $Q$ -spectral radius 7:

**Case 1.**  $d(i') = 3, d(3') = d(j'') = 2; i = 1, 2; j = 1, 2, 3;$



**Claim 2.1.**  $d(4') = d(4'') = 2$ .

The degree of the vertices  $4'$  and  $4''$  can be at most 3, otherwise  $\rho(Q_p(\Gamma_0))$  exceeds 7. Let  $d(4') = 3$  such that the distinct vertices  $5'', 6'' \in V(G) \setminus \Gamma_0$  are the other neighbors of  $4'$ . Consider  $S_0 = \{5'', 6''\}$ , and define  $\Gamma_1 = \Gamma_0 \cup S_0$ . Here  $5'', 6''$  can not be adjacent to any vertices of  $\Gamma_1$  except  $4'$  as  $G$  is bipartite and  $d(i') = 2(i = 1, 2, 3)$ . Thus, we have  $a_{5''j} = a_{6''j} = a_{5''6''} = 0(j \in \Gamma_0 \setminus \{4'\})$ . Now, the matrix  $Q_p(\Gamma_1)$  is given by

$$Q_p(\Gamma_1) = \begin{pmatrix} Q_p(\Gamma_0) & A_{\Gamma_0, S_0} \\ A_{S_0, \Gamma_0} & Q_p(S_0) \end{pmatrix} \tag{3.4}$$

where  $Q_p(\Gamma_0)$  is given in (3.3),  $A_{\Gamma_0, S_0} = A_{S_0, \Gamma_0}^t$ ,

$$A_{S_0, \Gamma_0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, Q_p(S_0) = \begin{pmatrix} d(5'') & 0 \\ 0 & d(6'') \end{pmatrix},$$

and  $d(4') = 3, 1 \leq d(4'') \leq 3$ . Also,  $d(4'') = 2$ , otherwise we get a contradiction to  $\lambda_{s2}(Q(G)) \geq 1$  and  $q(G) = 7$  by interlacing theorem. Therefore  $(d(4'), d(4'')) = (3, 2)$ . Let  $4''$  be adjacent to a vertex  $5' \in V(G) \setminus \Gamma_1$  and  $S_1 = \{5'\}$  and  $\Gamma_2 = \Gamma_1 \cup S_1$ . In this case,  $5'$  is not adjacent to  $5'', 6''$ , otherwise either  $\lambda_{s2}(Q_p(\Gamma_2)) < 1$  or  $\lambda_{l2}(Q_p(\Gamma_2)) > 6$  or  $\rho(Q_p(\Gamma_2)) > 7$ . The principal submatrix corresponding to  $\Gamma_2$  is

$$Q_p(\Gamma_2) = \begin{pmatrix} Q_p(\Gamma_1) & A_{\Gamma_1, S_1} \\ A_{S_1, \Gamma_1} & Q_p(S_1) \end{pmatrix} \tag{3.5}$$

where  $Q_p(\Gamma_1)$  is given in (3.4) with  $d(4') = 3, d(4'') = 2$ , and

$$A_{S_1, \Gamma_1} = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0), Q_p(S_1) = (d(5')).$$

We have the following three choices for  $(d(5''), d(6''), d(5'))$  so that  $\lambda_{s2}(Q_p(\Gamma_2)) \geq 1$ , and  $\lambda_{l2}(Q_p(\Gamma_2)) \leq 6, \rho(Q_p(\Gamma_2)) \leq 7$ : (i)  $(3, 2, 4)$ , (ii)  $(3, 3, 3)$ , (iii)  $(3, 2, 3)$ .

Suppose that  $5''$  is adjacent to the vertices of  $S_2 = \{6', 7'\} \subseteq V(G) \setminus \Gamma_2$ , where  $6' \neq 7'$  and  $\Gamma_3 = \Gamma_2 \cup S_2$ . Therefore, we conclude the following to have  $\lambda_{s2}(Q_p(\Gamma_3)) \geq 1$  and  $\rho(Q_p(\Gamma_3)) \leq 7$ :

- $(d(5''), d(6''), d(5')) = (3, 2, 3)$ ;
- $a_{6' i'} = a_{6' j''} = a_{7' i'} = a_{7' j''} = 0(i = 1, \dots, 5; j = 1, \dots, 4, 6)$  in  $Q_p(\Gamma_3)$ .

Hence,  $d(6'') = 2$  and let  $S_3 = \{8'\} \subset V(G) \cap N(6'') \setminus \{\Gamma_3\}$ , and define  $\Gamma_4 = \Gamma_3 \cup S_3$ . Clearly,  $8'$  is not adjacent to any vertices of  $\Gamma_4$  except  $6''$  as  $G$  is bipartite,  $d(j'') = 2(j = 1, \dots, 4)$  and  $d(5'') = 3$ ; see Figure 4(a). For every possible choices of  $(d(6'), d(7'), d(8'))$ , we get either  $\lambda_{s2}(Q_p(\Gamma_4)) < 1$  or  $\rho(Q_p(\Gamma_4)) > 7$ , which is a contradiction by Remark 4. Hence,  $d(4') \leq 2$  and by symmetric structure given in Figure 3(d), we have  $d(4'') \leq 2$ .

However, if any one of  $d(4'), d(4'') < 2$ , then  $\lambda_{s2}(Q_p(\Gamma_0))$  given in (3.3) becomes less than 1 which contradicts to Remark 4. Therefore, we conclude that  $(d(4'), d(4'')) = (2, 2)$  which completes the proof of Claim 2.1.

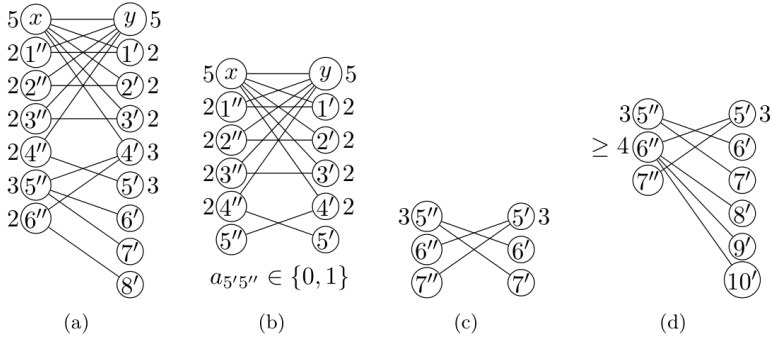


Figure 4.

Next, we look into the neighbors of  $4'$  and  $4''$  in the subgraph induced by  $\Gamma_0 = V(\gamma_3)$ . Let  $5', 5'' \in V(G) \setminus \Gamma_0$  be the neighbors of  $4'', 4'$ , respectively. Since  $G$  is bipartite,  $5' \neq 5''$ . Let  $\Gamma_1 = \Gamma_0 \cup S_0$ , where  $S_0 = \{5', 5''\}$ . So, we get the subgraph given in Figure 4(b) corresponding to the vertex set  $\Gamma_1$ . The principal submatrix  $Q_p(\Gamma_1)$  is given by

$$Q_p(\Gamma_1) = \begin{pmatrix} 5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & d(5') & a_{5'5''} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & a_{5'5''} & d(5'') & 0 & 0 \end{pmatrix}. \tag{3.6}$$

Now, we claim the following about vertices  $5'$  and  $5''$ :

**Claim 2.2.**  $5'$  is not adjacent to  $5''$ .

Assume on the contrary that  $5'$  is adjacent to  $5''$ . Since  $G$  is connected,  $d(5') \geq 3$  or  $d(5'') \geq 3$ . Now, we look into all possible degree pairs for  $5'$  and  $5''$ .

(i)  $(d(5'), d(5'')) \in \{(2, 3), (3, 3)\}$ : Let  $6' \in V(G) \setminus \Gamma_1$  be the remaining neighbor of  $5''$ . Obviously,  $5'$  is not adjacent to  $6'$  as  $G$  is bipartite. Define  $\Gamma_2 = \Gamma_1 \cup S_1$ , where  $S_1 = \{5''\}$ . Thus, the matrix  $Q_p(\Gamma_2)$  is

$$Q_p(\Gamma_2) = \begin{pmatrix} Q_p(\Gamma_1) & A_{\Gamma_1, S_1} \\ A_{S_1, \Gamma_1} & Q_p(S_1) \end{pmatrix}, \tag{3.7}$$

where  $Q_p(\Gamma_1)$  is given in (3.6) with  $d(5') \in \{2, 3\}, d(5'') = 3, a_{5'5''} = 1$ , and

$$A_{S_1, \Gamma_1} = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1), \quad Q_p(S_1) = (d(6')).$$

Now  $(d(5'), d(5'')) \neq (2, 3)$ , otherwise  $\lambda_{s_2}(Q_p(\Gamma_2)) \leq 0.0669 < 1$ . Let  $d(5') = 3$  and  $6'' \in V(G) \setminus \Gamma_2$  be the remaining neighbor of  $5'$ . Define  $\Gamma_3 = \Gamma_2 \cup S_2$ , where  $S_2 = \{6''\}$ . For each admissible choices of  $d(\cdot), a_{..}$ , we get a contradiction to  $\lambda_{s_2}(Q_p(\Gamma_3)) \geq 1$ . Hence,  $(d(5'), d(5'')) \neq (3, 3)$ . Therefore, (i) is not possible.



(ii)  $((d(5'), d(5'')) = (4, 4)$ : Let  $S_1 = \{6', 7', 6'', 7''\} \subseteq V(G) \setminus \Gamma_1$  be a set of distinct vertices in  $G$  and define  $\Gamma_2 = \Gamma_1 \cup S_1$ . Assume that  $5''$  (resp.  $5'$ ) is adjacent to  $6', 7'$  (resp.  $6'', 7''$ ). The matrix  $A_{S_1, \Gamma_1}$  and  $Q_p(S_1)$  are given by

$$A_{S_1, \Gamma_1} = \left( \mathbf{0} \mid \begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{array} \right), \quad Q_p(S_1) = \begin{pmatrix} d(6') & 0 & a_{6'6''} & a_{6'7''} \\ 0 & d(7') & a_{7'6''} & a_{7'7''} \\ a_{6'6''} & a_{7'6''} & d(6'') & 0 \\ a_{6'7''} & a_{7'7''} & 0 & d(7'') \end{pmatrix}.$$

Here,  $a_{i'j''} = 0, \forall i, j = \{6, 7\}$  and  $d(7'') = 3$  in  $G$  to have  $\lambda_{s_2}(Q_p(\Gamma_2)) \geq 1$  or  $\lambda_{t_2}(Q_p(\Gamma_2)) \leq 6$  or  $\rho(Q_p(\Gamma_2)) = 7$ . Let  $S_2 = \{8', 9'\} \subseteq V(G) \setminus \Gamma_2$  be a subset of  $N(7'')$  containing distinct vertices and define  $\Gamma_3 = \Gamma_2 \cup S_2$ . However,  $\lambda_{s_2}(Q_p(\Gamma_3)) < 1$ , which contradicts to Remark 4. Therefore,  $(d(5'), d(5'')) \neq (4, 4)$ .

(iii)  $(d(5'), d(5'')) \in \{(3, 4), (3, 5), (4, 5), (5, 5)\}$ : Similar to (ii), it can be verified using Remark 4 and  $q(G) = 7$  that this subcase is not possible.

From above, we conclude that Claim 2.2 holds, i.e.,  $a_{5'5''} = 0$  in  $Q_p(\Gamma_1)$  given by (3.6).

From Claim 2.2, the induced subgraph  $G[\Gamma_1]$  is as shown in Figure 4(b). Now, we look into the degrees of  $d(5')$  and  $d(5'')$ .

**Claim 2.3.**  $d(5') \geq 3$  and  $d(5'') \geq 3$ .

The claim holds otherwise  $\lambda_{s_2}(Q_p(\Gamma_1)) \leq 0.9194 < 1$ , where  $Q_p(\Gamma_1)$  given in (3.6).

**Claim 2.4.**  $(d(5'), d(5'')) \neq (3, 3)$ .

Assume on the contrary that  $d(5') = d(5'') = 3$ . Let  $S_1 = \{6', 7', 6'', 7''\} \subseteq V(G) \setminus \Gamma_1$  be a set of distinct vertices in  $G$  such that  $5''$  (resp.  $5'$ ) is adjacent to  $6', 7'$  (resp.  $6'', 7''$ ), see Figure 4(c). The corresponding submatrix  $Q_p(\Gamma_1)$  of the principal submatrix  $Q_p(\Gamma_2)$  in (3.1), where  $\Gamma_2 = \Gamma_1 \cup S_1$  is given in (3.6) with  $a_{5'5''} = 0$ , and  $A_{S_1, \Gamma_1}, Q_p(S_1)$  are

$$A_{S_1, \Gamma_1} = \left( \mathbf{0} \mid \begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{array} \right), \quad Q_p(S_1) = \begin{pmatrix} d(6') & 0 & a_{6'6''} & a_{6'7''} \\ 0 & d(7') & a_{7'6''} & a_{7'7''} \\ a_{6'6''} & a_{7'6''} & d(6'') & 0 \\ a_{6'7''} & a_{7'7''} & 0 & d(7'') \end{pmatrix}. \tag{3.8}$$

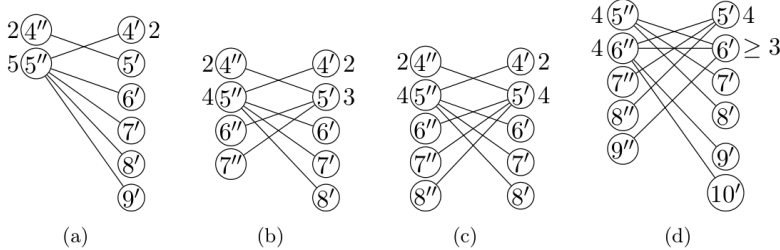
From  $q(G) = 7$ , Remark 4 on  $Q_p(\Gamma_2)$ , we have  $a_{i'j''} = 0; i, j \in \{6, 7\}$  and  $d(6'') \geq 4$ . Let  $S_2 = \{8', 9', 10'\} \subseteq N(6'') \setminus \Gamma_2$  be set of distinct vertices and  $\Gamma_3 = \Gamma_2 \cup S_2$ , see Figure 4(d). However, for each choices of  $a_{..}, d(\cdot)$ , we get a contradiction to Remark 4. Hence,  $(d(5'), d(5'')) \neq (3, 3)$ .

**Claim 2.5.**  $d(i) \neq 5$  for  $i = 5', 5''$ .

Let  $d(5'') = 5$  and  $S_1 = \{6', 7', 8', 9'\} \subseteq (V(G) \setminus \Gamma_1) \cap N(5'')$  be a set of distinct vertices in  $G$  and define  $\Gamma_2 = \Gamma_1 \cup S_1$ , see Figure 5(a). We get a contradiction to Remark 4 for  $Q_p(\Gamma_2)$ . Therefore, neither  $d(5') = 5$  nor  $d(5'') = 5$ .

**Claim 2.6.**  $d(5'), d(5'') \neq 4$ .

Assume on the contrary that the claim holds. From Claim 2.3-2.5, we have the non-isomorphic choices for  $(d(5'), d(5''))$  as  $(3, 4)$ ,  $(4, 4)$ .



**Figure 5.**

**Case 2.6.1.**  $(d(5'), d(5'')) = (3, 4)$ .

Let  $S_1 = \{6', 7', 8', 6'', 7''\} \subseteq V(G) \setminus \Gamma_1$  be a set of distinct vertices such that  $\{6', 7', 8'\} \subseteq N(5'')$  and  $\{6'', 7''\} \subseteq N(5')$ , see Figure 5(b). Define  $\Gamma_2 = \Gamma_1 \cup S_1$ , and the principal submatrix  $Q_p(\Gamma_2)$  given by (3.1) contains  $Q_p(\Gamma_1)$  as given in (3.6). Hence, each neighbor of  $5'$  can be adjacent to at most one vertex of  $\{6', 7', 8'\}$  in  $S_1$ , otherwise either  $\lambda_{s_2}(Q_p(\Gamma_2)) < 1$  or  $\lambda_{l_2}(Q_p(\Gamma_2)) > 6$ . Since  $G$  is bipartite, we have the following non-isomorphic cases: (2.6.1.1)  $E(S_1) = \{6'6'', 6'7''\}$ , (2.6.1.2)  $E(S_1) = \{6'6'', 7'7''\}$ , (2.6.1.3)  $E(S_1) = \{6'6''\}$ , (2.6.1.4)  $E(S_1) = \phi$ .

Now, we analyze these possible cases. Here, (2.6.1.1), (2.6.1.2) are not possible since either  $\lambda_{s_2}(Q_p(\Gamma_2)) < 1$  or  $\lambda_{l_2}(Q_p(\Gamma_2)) > 6$ .

**(2.6.1.3)**  $E(S_1) = \{6'6''\}$ : Degree of  $6''$  is at least 4, otherwise  $\lambda_{s_2}(Q_p(\Gamma_2)) \leq 0.9923 < 1$ . Let  $6''$  be adjacent to the set of distinct vertices  $S_2 = \{9', 10'\} \subseteq V(G) \setminus \Gamma_1$ , and define  $\Gamma_3 = \Gamma_2 \cup S_2$ . However, for all the admissible choices of  $d(\cdot)$  and  $a_{\cdot}$  in  $Q_p(\Gamma_3)$ , either  $\lambda_{s_2}(Q_p(\Gamma_3)) < 1$  or  $\lambda_{l_2}(Q_p(\Gamma_3)) > 6$ . Hence,  $E(S_1) \neq \{6'6''\}$ .

**(2.6.1.4)**  $E(S_1) = \phi$ : Similar to (2.6.1.3), it can be verified that this case is not possible using Remark 4 and  $q(G) = 7$ .

Therefore, Case 2.6.1 is not valid i.e.,  $(d(5'), d(5'')) \neq (3, 4)$ .

**Case 2.6.2.**  $(d(5'), d(5'')) = (4, 4)$ .

Let  $S_1 = \{6', 7', 8', 6'', 7'', 8''\} \subseteq V(G) \setminus \Gamma_1$  be a set of distinct vertices such that  $\{6', 7', 8'\} \subseteq N(5'')$  and  $\{6'', 7'', 8''\} \subseteq N(5')$ , see Figure 5(c). Define  $\Gamma_2 = \Gamma_1 \cup S_1$ . Now,  $i'' \in N(5')(i = 6, 7, 8)$  is adjacent to at most one vertex of  $\{6', 7', 8'\}$ , otherwise either  $\lambda_{s_2}(Q_p(\Gamma_2)) < 1$  or  $\lambda_{l_2}(Q_p(\Gamma_2)) > 6$ . Since  $G$  is bipartite, we have the following non-isomorphic choices for  $E(S_1)$ : (2.6.2.1)  $E(S_1) = \{6'6'', 7'7'', 8'8''\}$ ,

(2.6.2.2)  $E(S_1) = \{6'6'', 7'7''\}$ , (2.6.2.3)  $E(S_1) = \{6'6''\}$ , (2.6.2.4)  $E(S_1) = \phi$ . Now, we analyze these cases.

**(2.6.2.1)**  $E(S_1) = \{6'6'', 7'7'', 8'8''\}$ : This case is not possible, otherwise either we get  $\lambda_{s2}(Q_p(\Gamma_2)) < 1$  or  $\lambda_{l2}(Q_p(\Gamma_2)) > 6$  or  $\rho(Q_p(\Gamma_2)) > 7$ .

**(2.6.2.2)**  $E(S_1) = \{6'6'', 7'7''\}$ : We have  $d(6'') \geq 3$ , otherwise we arrive at a contradiction using Remark 4. Suppose that  $S_2 = \{9''\} \subseteq N(6'')$  be such that  $\Gamma_2 \cap S_2 = \phi$  and define  $\Gamma_3 = \Gamma_2 \cup S_2$ . Then for each possible submatrix  $Q_p(\Gamma_3)$  in (3.1), we get  $\lambda_{l2}(Q_p(\Gamma_3)) > 6$ , which is a contradiction. Therefore,  $E(S_1) \neq \{6'6'', 7'7''\}$ .

**(2.6.2.3)**  $E(S_1) = \{6'6''\}$ : Here we have  $d(6'), d(6'') \geq 3$ , otherwise  $\lambda_{s2}(Q_p(\Gamma_2)) \leq 0.9893$ . Let  $S_2 = \{9', 9''\} \subseteq V(G) \setminus \Gamma_2$  be such that  $9'$  (resp.  $9''$ ) is adjacent to  $6''$  (resp.  $6'$ ). Since  $G$  is bipartite, we have  $9' \neq 9''$ . Define  $\Gamma_3 = \Gamma_2 \cup S_2$ . We have  $N(9') \cap \{7'', 8'', 9''\} = \phi = N(9'') \cap \{7', 8', 9'\}$  and  $d(6'') = 4$ , otherwise we get a contradiction to Remark 4. Let  $S_3 = \{10'\} \subseteq N(6'') \setminus \Gamma_3$  and define  $\Gamma_4 = \Gamma_3 \cup S_3$ , see Figure 5(d). Now for all the possible choices of  $d(\cdot)$ , we arrive at a contradiction to Remark 4 for  $Q_p(\Gamma_4)$ . Hence  $E(S_1) \neq \{6'6''\}$ .

**(2.6.2.4)**  $E(S_1) = \phi$ : Similar to (2.6.2.3), this case is not possible due to Remark 4,  $q(G) = 7$ , and bipartiteness of  $G$ .

Therefore Claim 2.6 holds i.e.,  $d(i) \neq 4$  for all  $i = 5', 5''$ .

Further from Claims 2.1- 2.6, we conclude that Case 4 does not hold. Finally from Claim 1 and 2, we obtain that  $G$  is  $\gamma_3$ -free.

Next, we prove the second part of the theorem, namely (b), by contradiction. Let us assume that  $G$  is non-bipartite and has an induced subgraph  $\gamma_0$ . Thus  $\lambda_{\min}(Q(G)) \geq 1$  by Proposition 1, and hence  $\lambda_{\min}(Q_p(H)) \geq 1$  for every  $H \subseteq V(G)$ . The principal submatrix corresponding to  $\Gamma_0$  is

$$Q_p(\Gamma_0) = \begin{pmatrix} 5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d(1') & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & d(2') & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & d(3') & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & d(4') & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & d(1'') & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & d(2'') & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & d(3'') & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d(4'') \end{pmatrix}.$$

There exist at least one  $i, j \in \{1, 2, 3, 4\}$  such that  $d(i'), d(j'') \geq 3$ , otherwise we get a contradiction to the Remark 4 with  $\lambda_{\min}(Q_p(\Gamma_0)) \leq 0.9317 < 1$ . Without any loss of generality, let  $d(1'), d(1'') \geq 3$  and  $S_0 = \{5', 6', 5'', 6''\} \subseteq V(G) \setminus \Gamma_0$  be a set of distinct vertices such that  $5', 6'$  (resp.  $5'', 6''$ ) are adjacent to  $1''$  (resp.  $1'$ ) in  $G$ . Thus the principal submatrix  $Q_p(\Gamma_1)$  where  $\Gamma_1 = \Gamma_0 \cup S_0$ , is given by

$$Q_p(\Gamma_1) = \begin{pmatrix} Q_p(\Gamma_0) & A_{\Gamma_0, S_0} \\ A_{S_0, \Gamma_0} & Q_p(S_0) \end{pmatrix},$$

$$\text{where } A_{S_0, \Gamma_0} = \left( \begin{array}{c|cccc|cccc} \mathbf{0} & a_{1'5'} & a_{2'5'} & a_{3'5'} & a_{4'5'} & 1 & a_{2''5'} & a_{3''5'} & a_{4''5'} & \\ a_{1'6'} & a_{2'6'} & a_{3'6'} & a_{4'6'} & & 1 & a_{2''6'} & a_{3''6'} & a_{4''6'} & \\ \mathbf{0} & 1 & a_{2'5''} & a_{3'5''} & a_{4'5''} & a_{1'5''} & a_{2'5''} & a_{3'5''} & a_{4'5''} & \\ & 1 & a_{2'6''} & a_{3'6''} & a_{4'6''} & a_{1'6''} & a_{2'6''} & a_{3'6''} & a_{4'6''} & \end{array} \right).$$

We have the following claims due to the fact that  $\lambda_{\min}(Q_p(\Gamma_1)) \geq 1$  and  $\rho(Q_p(\Gamma_1)) \leq 7$ :

- $a_{5'i} = a_{6'i} = a_{5''j} = a_{6''j} = 0$ , for  $i \in \Gamma_0 \setminus \{1''\}, j \in \Gamma_0 \setminus \{1'\}$ ;
- $d(1') = d(1'') = 3, a_{5'i''} = a_{6'i''} = 0, i = 5, 6$ ;
- $d(i') \geq 3$ , for some  $i \in \{2', 3', 4'\}$ ; say  $d(2') \geq 3$ .

Thus the matrix  $A_{S_0, \Gamma_0}, Q_p(S_0)$  now becomes

$$A_{S_0, \Gamma_0} = \left( \begin{array}{c|c|c|c|c} \mathbf{0} & 0 & \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & 0 & \mathbf{0} \end{array} \right), \quad Q_p(S_0) = \left( \begin{array}{c|c|cc} d(5') & a_{5'6'} & & \mathbf{0} \\ a_{5'6'} & d(6') & & \\ \mathbf{0} & & d(5'') & a_{5''6''} \\ & & a_{5''6''} & d(6'') \end{array} \right).$$

Let  $7''$  be a neighbor of  $2'$  in  $G$ , where  $S_1 = \{7''\} \subset V(G) \setminus \Gamma_1$ , and thus  $\Gamma_2 = \Gamma_1 \cup S_1$ . Now for all the choices of  $1 \leq d(\cdot) \leq 5$  and  $a_{..} \in \{0, 1\}$ , we get either  $\lambda_{\min}(Q_p(\Gamma_2)) < 1$  or  $\rho(Q_p(\Gamma_2)) > 7$ , which contradicts to the fact that  $\lambda_{s_2}(Q(G)) \geq 1$  and  $q(G) = 7$ . Therefore,  $G$  must be a bipartite graph if it contains  $\gamma_0$  as an induced subgraph.

### 4. Conclusion

We have improved one of our earlier results from [9] on the structural characterization of  $Q$ -integral connected graph  $G$  having  $q(G) = 7$  and maximum edge-degree 8. We have shown that  $G$  must be a bipartite graph. If  $G \neq K_{1,4} \square K_2$ , then  $G$  contains one of the three graphs, namely  $\mathcal{S}_4^2(m) (m = 0, 1, 2)$  as an induced subgraph. Further, 0, 1, 6 and 7 are  $Q$ -eigenvalues of  $G$ .

Thus, we conclude that whenever  $G \notin \{C_3, C_6, \overline{K_{3,3} - e}\}$  is a connected  $Q$ -integral graph with maximum edge-degree  $2q(G) - 6$ , then  $G$  is bipartite and  $\mathcal{S}_{q(G)-3}^2(q(G) - 4)$ -free for  $q(G) \leq 7$ .

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**Data Availability:** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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