Communications in Combinatorics and Optimization
Vol. 10, No. 3 (2025), pp. 617-625
https://doi.org/10.22049/cco.2024.28703.1678

CCO
Commun. Comb. Optim.

Research Article

Restrained double Roman domatic number

Lutz Volkmann

Lehrstuhl II fur Mathematik, RWTH Aachen University, 52056 Aachen, Germany
volkm@math2.rwth-aachen.de

Received: 1 June 2023; Accepted: 12 January 2024
Published Online: 23 January 2024

Abstract: Let G be a graph with vertex set $V(G)$. A double Roman dominating function (DRDF) on a graph G is a function $f : V(G) \rightarrow \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then the vertex v mus have at least one neighbor u with $f(u) \geq 2$. If f is a DRDF on G, then let $V_0 = \{v \in V(G) : f(v) = 0\}$. A restrained double Roman dominating function is a DRDF f having the property that the subgraph induced by V_0 does not have an isolated vertex. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct restrained double Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 3$ for each $v \in V(G)$ is called a restrained double Roman dominating family (of functions) on G. The maximum number of functions in a restrained double Roman dominating family on G is the restrained double Roman domatic number of G, denoted by $d_{rdR}(G)$. We initiate the study of the restrained double Roman domatic number, and we present different sharp bounds on $d_{rdR}(G)$. In addition, we determine this parameter for some classes of graphs.

Keywords: Restrained double Roman domination, restrained double Roman domatic number.

AMS Subject classification: 05C69.

1. Introduction

For definitions and notations not given here we refer to [6]. We consider simple and finite graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V|$. The *neighborhood* of a vertex v is the set $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E\}$. The *degree* of vertex $v \in V$ is $d(v) = d_G(v) = |N(v)|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. The *complement* of a graph G is denoted by $\overline{G}$. For a subset D of vertices in a graph G, we denote by $G[D]$ the subgraph of G induced by D. A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching* in G. A *leaf* is a vertex of degree one, and its neighbor is called a *support vertex*. We write P_n for the *path* of order n, C_n for the *cycle* of length n, K_n for the *complete graph* of order n. Also, let $K_{n_1,n_2,\ldots,n_p}$ denote the *complete p-partite graph* with vertex set $S_1 \cup S_2 \cup \ldots \cup S_p$ where $|S_i| = n_i$ for $1 \le i \le p$. For $n \ge 2$, the *star* $K_{1,n-1}$ has one vertex of degree $n-1$ and $n-1$ leaves.

A set $S \subseteq V(G)$ is called a *dominating set* if every vertex is either an element of S or is adjacent to an element of S. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. A *minimal dominating set* in a graph G is a dominating set that contains no dominating set as a proper subset.

In this paper we continue the study of Roman dominating functions and Roman domatic numbers in graphs (see, for example, the survey articles [2–5]). If $f : V(G) \longrightarrow \{0,1,2,3\}$ is a function, then let (V_0, V_1, V_2, V_3) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0,1,2,3\}$. There is a 1-1 correspondence between the function f and the ordered partition (V_0, V_1, V_2, V_3). So we also write $f = (V_0, V_1, V_2, V_3)$. A *double Roman dominating function* (DRDF) on a graph G is defined in [1] as a function $f : V(G) \longrightarrow \{0,1,2,3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two neighbors in V_2 or one neighbor in V_3, and if $f(v) = 1$, then the vertex v must have at least one neighbor in $V_2 \cup V_3$. The weight of a DRDF f is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *double Roman domination number* $\gamma_{dR}(G)$ is the minimum weight of a DRDF on G, and a double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$-*function* of G.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct double Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \le 3$ for each $v \in V(G)$ is called in [10] a *double Roman dominating family* (of functions) on G. The maximum number of functions in a double Roman dominating family on G is the *double Roman domatic number* of G, denoted by $d_{dR}(G)$.

Mojdeh, Masoumi and Volkmann [7] defined the *restrained double Roman dominating function* (RDRDF) as a double Roman dominating function f with the property that the subgraph induced by V_0 does not have an isolated vertex. The *restrained double Roman domination number* $\gamma_{rdR}(G)$ equals the minimum weight of an RDRDF on G. An RDRDF on G with weight $\gamma_{rdR}(G)$ is called a $\gamma_{rdR}(G)$-function.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct restrained double Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \le 3$ for each $v \in V(G)$ is called a *restrained double Roman dominating family* (of functions) on G. The maximum number of functions in a restrained double Roman dominating family on G is the *restrained double Roman domatic number* of G, denoted by $d_{rdR}(G)$. The definitions lead to $\gamma_{dR}(G) \le \gamma_{rdR}(G)$ and $d_{rdR}(G) \le d_{dR}(G)$.

We initiate the study of the restrained double Roman domatic number, and we present different sharp bounds on $d_{rdR}(G)$. In addition, we determine this parameter for some classes of graphs. Furthermore, if G is a connected graph of order $n \ge 3$, then we show that $6 \le \gamma_{rdR}(G) + d_{rdR}(G) \le \frac{3n}{2} + 2$.

We make use of the following results.

Proposition 1. *[10] If G is a graph, then $d_{dR}(G) \leq \delta(G) + 1$.*

Since $d_{rdR}(G) \leq d_{dR}(G)$, the next corollary is immediate.

Corollary 1. *If G is a graph of order n, then $d_{rdR}(G) \leq \delta(G) + 1 \leq n$.*

Proposition 2. *[10] Let C_n be a cycle of order $n \geq 3$. Then $d_{dR}(C_n) = 3$, when $n \equiv 0 \pmod 3$ and $d_{dR}(C_n) = 2$, when $n \equiv 1, 2 \pmod 3$.*

Proposition 3. *[10] Let G be a graph of order $n \geq 2$. If $\Delta(G) \leq n-2$, then $d_{dR}(G) \leq \frac{n}{2}$.*

Proposition 4. *[10] If G is a graph of order n, then $d_{dR}(G) + d_{dR}(\overline{G}) \leq n + 1$, with equality if and only if $G = K_n$ or $\overline{G} = K_n$.*

Proposition 5. *[7] If G is a connected graph of order $n \geq 2$, then $\gamma_{rdR}(G) \leq \frac{3n}{2}$.*

Proposition 6. *If G is a graph of order $n \geq 3$, then $\gamma_{rdR}(G) \geq 3$, with equality if and only if $\Delta(G) = n-1$ and G contains a vertex w of maximum degree such that $\delta(G[N_G(w)]) \geq 1$.*

Proof. Since $n \geq 3$, it is easy to see that $\gamma_{rdR}(G) \geq 3$. Assume that G contains a vertex w with $d_G(w) = n-1$ such that $\delta(G[N_G(w)]) \geq 1$. Define the function f by $f(w) = 3$ and $f(x) = 0$ for $x \in V(G) \setminus \{w\}$. Since $G[N_G(w)]$ does not contain an isolated vertex, we observe that f is an RDRDF on G of weight 3 and so $\gamma_{rdR}(G) = 3$. Conversely, assume that $\gamma_{rdR}(G) = 3$. Let f be a $\gamma_{rdR}(G)$-function. Since $n \geq 3$, there exists a vertex w with $f(w) = 3$ such that the remaining $n-1$ vertices with value 0 are adjacent to w and $\delta(G[N_G(w)]) \geq 1$. □

Proposition 7. *[8] If G is a graph without isolated vertices and S is a minimal dominating set of G, then $V(G) \setminus S$ is a dominating set of G.*

Proposition 8. *[7] If $p, q \geq 2$ are integers, then $\gamma_{rdR}(K_{p,q}) = 6$.*

Proposition 9. *[9] Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with $p \geq 2$ and $n_1 \leq n_2 \leq \ldots \leq n_p$. If $n = n_1 + n_2 + \ldots + n_p$ and M is a maximum matching, then $|M| = \min\{n - n_p, \lfloor \frac{n}{2} \rfloor\}$.*

2. Properties and bounds

In this section we present basic properties and bounds on the restrained double Roman domatic number.

Theorem 1. *If G is a graph without isolated vertices, then $d_{rdR}(G) \geq 2$.*

Proof. Let T be a spanning forest of G without isolated vertices, and let X and Y be a bipartion of T. Define the functions f and g by $f(x) = 1$, $f(y) = 2$ and $g(x) = 2$, $g(y) = 1$ for $x \in X$ and $y \in Y$. Since T has no isolated vertices, f and g are distinct restrained double Roman dominating functions on T and also on G such that $f(u) + g(u) = 3$ for each $u \in V(G)$. Therefore $\{f, g\}$ is a restrained double Roman dominating family on G and thus $d_{rdR}(G) \geq 2$. □

We deduce from Corollary 1 and Theorem 1 the next resutl immediately.

Corollary 2. *Let G be a graph without isolated vertices. If G has a leaf, then $d_{rdR}(G) = 2$. In particular, if T is a nontrivial tree, then $d_{rdR}(T) = 2$.*

Corollary 3. *Let C_n be a cycle of order $n \geq 3$. Then $d_{rdR}(C_n) = 3$, when $n \equiv 0 \pmod 3$ and $d_{rdR}(C_n) = 2$, when $n \equiv 1, 2 \pmod 3$.*

Proof. If $n \equiv 1, 2 \pmod 3$, then $d_{rdR}(C_n) \geq 2$ by Theorem 1, and Proposition 2 implies $d_{rdR}(C_n) \leq d_{dR}(C_n) \leq 2$. This leads to $d_{rdR}(C_n) = 2$ in this case.
Let now $n = 3t$ for an integer $t \geq 1$, and let $C_n = v_1 v_2 \ldots v_n v_1$. We deduce from Corollary 1 that $d_{rdR}(C_n) \leq 3$. Now define f_1, f_2 and f_3 by $f_1(v_{3i-2}) = 3$ for $1 \leq i \leq t$ and $f_1(x) = 0$ otherwise, $f_2(v_{3i-1}) = 3$ for $1 \leq i \leq t$ and $f_2(x) = 0$ otherwise and $f_3(v_{3i}) = 3$ for $1 \leq i \leq t$ and $f_3(x) = 0$ otherwise. Then $\{f_1, f_2, f_3\}$ is a restrained double Roman dominating family on C_{3t} and thus $d_{rdR}(C_{3t}) \geq 3$. Therefore $d_{rdR}(C_n) = 3$, when $n \equiv 0 \pmod 3$. □

Theorem 2. *If G is a graph, then $\gamma_{rdR}(G) \cdot d_{rdR}(G) \leq 3n$. Moreover, if we have the equality $\gamma_{rdR}(G) \cdot d_{rdR}(G) = 3n$, then for each restrained double Roman dominating family $\{f_1, f_2, \ldots, f_d\}$ on G with $d = d_{rdR}(G)$, each f_i is a $\gamma_{rdR}(G)$-function and $\sum_{i=1}^{d} f_i(v) = 3$ for all $v \in V(G)$.*

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a restrained double Roman dominating family on G with $d = d_{rdR}(G)$. Then

$$
\begin{aligned}
d \cdot \gamma_{rdR}(G) &= \sum_{i=1}^{d} \gamma_{rdR}(G) \leq \sum_{i=1}^{d} \sum_{v \in V(G)} f_i(v) \\
&= \sum_{v \in V(G)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(G)} 3 = 3n.
\end{aligned}
$$

If $\gamma_{rdR}(G) \cdot d_{rdR}(G) = 3n$, then the two inequalities occuring in the proof become equalities. Hence for the restrained double Roman dominating family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{v \in V(G)} f_i(v) = \gamma_{rdR}(G)$. Thus each f_i is a $\gamma_{rdR}(G)$-function, and $\sum_{i=1}^{d} f_i(v) = 3$ for each $v \in V(G)$. □

Theorem 3. *Let G be a graph of order $n \geq 3$. If G has $1 \leq p \leq n-1$ vertices of degree $n-1$, then $d_{rdR}(G) \geq p+1$.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of G and let $v_1, v_2, \ldots, v_p$ be the vertices of degree $n-1$. If $p=1$, then Theorem 1 implies $d_{rdR}(G) \geq 2 = p+1$. Let now $p \geq 2$. Define the functions f_i by $f_i(v_i) = 3$ and $f_i(x) = 0$ for $x \neq v_i$ for $1 \leq i \leq p$ and f_{p+1} by $f_{p+1}(v_n) = f_{p+1}(v_{n-1}) = \ldots = f_{p+1}(v_{p+1}) = 3$ and $f_{p+1}(v_i) = 0$ for $1 \leq i \leq p$. Since $p \geq 2$, $f_1, f_2, \ldots, f_{p+1}$ are disdinct RDRD functions on G such that $f_1(x) + f_2(x) + \ldots + f_{p+1}(x) = 3$ for each $x \in V(G)$. Therefore $\{f_1, f_2, \ldots, f_{p+1}\}$ is a restrained double Roman dominating family on G and so $d_{rdR}(G) \geq p+1$. □

Corollary 4. *Let G be a graph of order n. Then $d_{rdR}(G) \leq n$ with equality if and only if G is complete.*

Proof. Corollary 1 implies $d_{rdR}(G) \leq n$. Let now G be complete. If $n=1$, then obviously $d_{rdR}(G) = 1 = n$. If $n=2$, then it follows from Corollary 2 that $d_{rdR}(G) = 2 = n$. Let now $n \geq 3$. Then Theorem 3 with $p = n-1$ leads to $d_{rdR}(G) \geq n$ and so $d_{rdR}(G) = n$.
Conversely assume that $d_{rdR}(G) = n$. If G is not complete, then $\delta(G) \leq n-2$ and Corollary 1 leads to the contradiction $n = d_{rdR}(G) \leq \delta(G) + 1 \leq n-1$. □

Example 1. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of the complete graph K_n ($n \geq 3$), and let k be an integer with $1 \leq k \leq n-2$. Define the graph $G = K_n - \{v_1v_n, v_2v_n \ldots, v_kv_n\}$. Then $\delta(G) = n-k-1$, and it follows from Corollary 1 that $d_{rdR}(G) \leq n-k$. Since $v_{k+1}, v_{k+2}, \ldots, v_{n-1}$ are vertices of degree $n-1$, we deduce from Theorem 3 that $d_{rdR}(G) \geq n-k$ and thus $d_{rdR}(G) = n-k = \delta(G)+1$.

This example shows that Corollary 1 is sharp. Since $d_{rdR}(G) \leq d_{dR}(G)$, Proposition 3 implies the next bound.

Corollary 5. *Let G be a graph of order $n \geq 2$. If $\Delta(G) \leq n-2$, then $d_{rdR}(G) \leq \frac{n}{2}$.*

Corollary 6. *If G is a graph of order n, then $d_{rdR}(G) + d_{rdR}(\overline{G}) \leq n+1$, with equality if and only if $G = K_n$ or $\overline{G} = K_n$.*

Proof. Proposition 4 implies $d_{rdR}(G) + d_{rdR}(\overline{G}) \leq n+1$ and $d_{rdR}(G) + d_{rdR}(\overline{G}) \leq n$ when $G \neq K_n$ and $\overline{G} \neq K_n$. If, without loss of generality, $G = K_n$, then we deduce from Corollary 4 that $d_{rdR}(G) + d_{rdR}(\overline{G}) = n+1$. □

Theorem 4. *If G is a graph of order $n \geq 3$ without isolated vertices, then*

$$6 \leq \gamma_{rdR}(G) + d_{rdR}(G) \leq \frac{3n}{2} + 2.$$

Proof. First we prove the lower bound. Proposition 6 implies $\gamma_{rdR}(G) \geq 3$.
Assume that $\gamma_{rdR}(G) = 3$. Then it follows from Proposition 6 that $\Delta(G) = n-1$, and G contains a vertex w of maximum degree such that $\delta(G[N_G(w)]) \geq 1$. Now let S be a minimal dominating set of $G[N_G(w)]$. According to Proposition 7 $N_G(w) \setminus S$ is also a dominating set of $G[N_G(w)]$. Now define the functions f_1, f_2, f_3 by $f_1(w) = 3$ and $f_1(x) = 0$ otherwise, $f_2(x) = 3$ for $x \in S$ and $f_2(x) = 0$ otherwise and $f_3(x) = 3$ for $x \in N_G(w) \setminus S$ and $f_3(x) = 0$ otherwise. Since w is adjacent to all vertices of S and to all vertices of $N_G(w) \setminus S$, we conclude that $\{f_1, f_2, f_3\}$ is a restrained double Roman dominating family on G and thus $d_{rdR}(G) \geq 3$. This implies $\gamma_{rdR}(G) + d_{rdR}(G) \geq 6$ in this case.
If $\gamma_{rdR}(G) \geq 4$, then Theorem 1 leads to $\gamma_{rdR}(G) + d_{rdR}(G) \geq 6$, and the lower bound is proved.
Now we prove the upper bound. Theorem 2 implies

$$\gamma_{rdR}(G) + d_{rdR}(G) \leq \frac{3n}{d_{rdR}(G)} + d_{rdR}(G).$$

According to Corollary 1 and Theorem 1, we have $2 \leq d_{rdR}(G) \leq n$. Using these bounds and the fact that the function $g(x) = x + \frac{3n}{x}$ is decreasing for $2 \leq x \leq \sqrt{3n}$ and increasing for $\sqrt{3n} \leq x \leq n$, we obtain

$$\gamma_{rdR}(G) + d_{rdR}(G) \leq \frac{3n}{d_{rdR}(G)} + d_{rdR}(G) \leq \max\left\{\frac{3n}{2} + 2, 3 + n\right\} = \frac{3n}{2} + 2,$$

and the upper bound is proved. □

Example 2. Let $H = pK_2$ with an integer $p \geq 2$. Then $n(H) = n = 2p$, $\gamma_{rdR}(H) = 3p = \frac{3n}{2}$ and $d_{rdR}(H) = 2$. Thus $\gamma_{rdR}(H) + d_{rdR}(H) = \frac{3n}{2} + 2$.

This example shows that the upper bound in Theorem 4 is sharp.

Example 3. Let $Wd(2,p)$ be the *windmill graph* consiting of a center vertex z which is adjacent to the vertices of $p \geq 1$ copies of the complete graph K_2. Then we observe that $\gamma_{rdR}(Wd(2,p)) = 3$, $d_{rdR}(Wd(2,p)) = 3$ and so $\gamma_{rdR}(Wd(2,p)) + d_{rdR}(Wd(2,p)) = 6$. Now let W be the the graph obtained form $Wd(2,p)$ by attaching a leaf. Then we note that $\gamma_{rdR}(W) = 4$, $d_{rdR}(W) = 2$ and so $\gamma_{rdR}(W) + d_{rdR}(W) = 6$.

The graphs in Example 3 show that the lower bound in Theorem 4 is sharp.

3. Complete p-partite graphs

Theorem 5. *If $q \geq p \geq 2$ are integers, then $d_{rdR}(K_{p,q}) = p$.*

Proof. Let $X = \{x_1, x_2, \dots, x_p\}$ and $Y = \{y_1, y_2, \dots, y_q\}$ be a bipartition of $K_{p,q}$. First let $|X| \geq 3$. If f is an RDRDF on $K_{p,q}$, then we show that $f(X) = \sum_{x \in X} f(x) \geq 3$. Suppose on the contrary, that $f(X) \leq 2$. Then, since $|X| \geq 3$, there exists a vertex $v \in X$ with $f(v) = 0$ and therefore a vertex $w \in Y$ with $f(w) = 0$. However, now the definition leads to the contradiction $f(X) = f(N(w)) \geq 3$. If $\{f_1, f_2, \dots, f_d\}$ is a restrained double Roman dominating family on $K_{p,q}$ with $d = d_{rdR}(K_{p,q})$, then it follows that

$$3d \leq \sum_{i=1}^{d} \sum_{x \in X} f_i(x) = \sum_{x \in X} \sum_{i=1}^{d} f_i(x) \leq \sum_{x \in X} 3 = 3|X| = 3p$$

and thus $d_{rdR}(K_{p,q}) \leq p$.
Let now $|X| = 2$. Then $d_{rdR}(K_{p,q}) \leq 3$ by Corollary 1. Suppose that $d = d_{rdR}(K_{p,q}) = 3$, and let $\{f_1, f_2, f_3\}$ be a restrained double Roman dominating family on $K_{p,q}$. If $f_i(x_1) = 0$ or $f_i(x_2) = 0$ for an index $i \in \{1, 2, 3\}$ or $f_i(X) \geq 3$ for all $1 \leq i \leq 3$, then we obtain the contradiction $d \leq p = 2$ as above. Therefore assume, without less of generality, that $f_1(x_1) = f_1(x_2) = 1$. This implies $f_1(y) \geq 2$ for $y \in Y$ and thus $f_2(X), f_3(X) \geq 3$. Hence we arrive at the contradiction

$$8 = 3d - 1 \leq \sum_{i=1}^{3} \sum_{x \in X} f_i(x) = \sum_{x \in X} \sum_{i=1}^{3} f_i(x) \leq \sum_{x \in X} 3 = 6.$$

Altogether, we have $d_{rdR}(K_{p,q}) \leq p$.
Conversely, define $f_i(x_i) = f_i(y_i) = 3$ and $f_i(x) = 0$ otherwise for $1 \leq i \leq p$. Then $\{f_1, f_2, \dots, f_p\}$ is a restrained double Roman dominating family on $K_{p,q}$. Hence $d_{rdR}(K_{p,q}) \geq p$ and thus $d_{rdR}(K_{p,q}) = p$. □

If $p \geq 2$ is an integer, then it follows from Proposition 8 and Theorem 5 that $\gamma_{rdR}(K_{p,p}) \cdot d_{rdR}(K_{p,p}) = 6p$. Thus Theorem 2 is sharp.

Theorem 6. *Let $G = K_{n_1,n_2,\dots,n_p}$ be a complete p-partite graph with $p \geq 3$ and $n_1 \leq n_2 \leq \dots \leq n_p$. If $n = n_1 + n_2 + \dots + n_p$, then:*

(i) If $n_{p-1} = 1$, then $d_{rdR}(G) = p$.

(ii) If $n_1 \geq 2$, then

$$d_{rdR}(G) = \min\left\{n - n_p, \left\lfloor \frac{n}{2} \right\rfloor\right\} = \min\left\{\sum_{i=1}^{p-1} n_i, \left\lfloor \frac{1}{2} \sum_{i=1}^{p} n_i \right\rfloor\right\}.$$

(iii) If $n_t = 1$ and $n_{t+1} \geq 2$ for $1 \leq t \leq p-2$, then

$$d_{rdR}(G) = t + \min\left\{\sum_{i=t+1}^{p-1} n_i, \left\lfloor \frac{1}{2} \sum_{i=t+1}^{p} n_i \right\rfloor\right\}.$$

Proof. Let $S_1, S_2, \ldots, S_p$ be the partite sets of G with $|S_i| = n_i$ for $1 \le i \le p$.
(i) Let $n_{p-1} = 1$, and let $S_i = \{s_i\}$ for $1 \le i \le p-1$. Define $f_i(s_i) = 3$ and $f_i(x) = 0$ otherwise for $1 \le i \le p-1$ and $f_p(y) = 3$ for $y \in S_p$ and $f_p(x) = 0$ for $x \in V(G) \setminus S_p$. Then $\{f_1, f_2, \ldots, f_p\}$ is a restrained double Roman dominating family on G and therefore $d_{rdR}(G) \ge p$. Since $\delta(G) = p-1$, it follows from Corollary 1 that $d_{rdR}(G) \le p$ and thus $d_{rdR}(G) = p$ in this case.
(ii) Let $n_1 \ge 2$. Then $\Delta(G) \le n-2$ and thus $d_{rdR}(G) \le \frac{n}{2}$ by Corollary 5. Let now $M = \{u_1v_1, u_2v_2, \ldots, u_mv_m\}$ be a maximum matching of G.
Define f_i by $f_i(u_i) = f_i(v_i) = 3$ and $f_i(x) = 0$ otherwise for $1 \le i \le m = |M|$. Then $\{f_1, f_2, \ldots, f_m\}$ is a restrained double Roman dominating family on G, and therefore we deduce from Proposition 9 that

$$d_{rdR}(G) \ge |M| = \min\left\{n - n_p, \left\lfloor \frac{n}{2} \right\rfloor\right\}. \tag{3.1}$$

If $n - n_p \ge n_p$, then $\min\left\{n - n_p, \left\lfloor \frac{n}{2} \right\rfloor\right\} = \left\lfloor \frac{n}{2} \right\rfloor$ and hence (3.1) and the bound $d_{rdR}(G) \le \frac{n}{2}$ lead to the desired result.
Next assume that $n_p > n - n_p$. Then $\min\left\{n - n_p, \left\lfloor \frac{n}{2} \right\rfloor\right\} = n - n_p$ and (3.1) implies $d_{rdR}(G) \ge n - n_p$. Let now $\{f_1, f_2, \ldots, f_d\}$ be a restrained double Roman dominating family on G with $d = d_{rdR}(G)$, and let $X = S_1 \cup S_2 \cup \ldots \cup S_{p-1}$.
Assume first that there exists in index i, say $i = 1$, such that $f_1(X) = 0$. Then $f_1(y) \ge 2$ for $y \in S_p$. Since $n_i \ge 2$, we observe in this case that $f_i(X) \ge 4$ for $2 \le i \le d$. Therefore

$$4(d-1) \le \sum_{i=1}^{d} \sum_{x \in X} f_i(x) = \sum_{x \in X} \sum_{i=1}^{d} f_i(x) \le \sum_{x \in X} 3 = 3|X| = 3(n - n_p).$$

Since $p \ge 3$ and $n_i \ge 2$, this leads to $d_{rdR}(G) = d \le n - n_p$.
Assume next that $f_i(X) \ge 1$ for $1 \le i \le p$ and, without loss of generality, that $f_1(X) = 1$. Then $f_1(y) \ge 2$ for $y \in S_p$, and as in the last case, we obtain $d_{rdR}(G) \le n - n_p$.
Now assume that $f_i(X) \ge 2$ for $1 \le i \le p$. We observe that $f_i(X) = 2$ is possible for at most two indices. It follows that

$$3d - 2 \le \sum_{i=1}^{d} \sum_{x \in X} f_i(x) = \sum_{x \in X} \sum_{i=1}^{d} f_i(x) \le \sum_{x \in X} 3 = 3|X| = 3(n - n_p)$$

and so again $d_{rdR}(G) = d \le n - n_p$. As $d_{rdR}(G) \ge n - n_p$, we conclude that $d_{rdR}(G) = n - n_p$ in this case.
(iii) Finally, let $n_t = 1$ and $n_{t+1} \ge 2$ for $1 \le t \le p-2$. Let $S_i = \{s_i\}$ for $1 \le i \le t$. Clearly, $f_i(s_i) = 3$ and $f_i(x) = 0$ for $1 \le i \le t$ are restrained double Roman dominating functions on G. Applying Theorem 5 when $p - t = 2$ and Part (ii) when $p - t \ge 3$ to the complete $(p-t)$-partite graph $G[S_{t+1} \cup S_{t+2} \cup \ldots \cup S_p]$, we obtain the desired result. □

If $n_1 \geq 2$ and $\min\left\{n - n_p, \left\lfloor \frac{n}{2} \right\rfloor\right\} = \left\lfloor \frac{n}{2} \right\rfloor$ in Theorem 6, then $d_{rdR}(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Thus Corollary 5 is sharp.

Conflict of Interest: The author declares that he has no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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