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Research Article

On distance Laplacian spectral invariants of brooms and their complements

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Abstract: For a connected graph G of order n, the distance Laplacian matrix $D^L(G)$ is defined as $D^L(G) = Tr(G) - D(G)$, where $Tr(G)$ is the diagonal matrix of vertex transmissions and $D(G)$ is the distance matrix of G. The largest eigenvalue of $D^L(G)$ is the distance Laplacian spectral radius of G and the quantity $DLE(G) = \sum_{i=1}^{n} |\rho_i^L(G) - \frac{2W(G)}{n}|$, where $W(G)$ is the Wiener index of G, is the distance Laplacian energy of G. Brooms of diameter 4 are the trees obtained from the path P_5 by appending pendent vertices at some vertex of P_5. One of the interesting and important problems in spectral graph theory is to find extremal graphs for a spectral graph invariant and ordering them according to this graph invariant. This problem has been considered for many families of graphs with respect to different graph matrices. In the present article, we consider this problem for brooms of diameter 4 and their complements with respect to their distance Laplacian matrix. Formally, we discuss the distance Laplacian spectrum and the distance Laplacian energy of brooms of diameter 4. We will prove that these families of trees can be ordered in terms of their distance Laplacian energy and the distance Laplacian spectral radius. Further, we obtain the distance Laplacian spectrum and the distance Laplacian energy of complement of the family of double brooms and order them in terms of the smallest non-zero distance Laplacian eigenvalue and the distance Laplacian energy.

Keywords: Laplacian matrix, distance Laplacian matrix, distance Laplacian energy, broom trees, ordering

AMS Subject classification: 05C50, 05C12, 15A18.

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1. Introduction

The present article considers only connected, simple and undirected graphs. A graph G is as usual denoted by $G = G(V,E)$, where V and E are its vertex and edge set. The number of elements in V and E is the *order* n and the *size* m of G, respectively. The complement of G is denoted by $\overline{G}$. For other undefined notations, see [8].
The adjacency matrix $A(G)$ associated to G is a $(0,1)$-square matrix indexed by order n, where (i,j) term is 1, if i is adjacent to j and taken 0, otherwise. Let $Deg(G)$ be the diagonal matrix of vertex degrees . The real symmetric matrix $L(G) = Deg(G)-A(G$ is called the Laplacian matrix. $L(G)$ is positive semi-definite matrix, so its eigenvalues are non-negative real numbers and can be ordered as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$. The Laplacian spectral ordering and the ordering of complements of trees on the basis of μ_{n-1} are given [1, 16, 21].

In a connected graph G, the *distance* between two vertices $u \neq v \in V$, denoted by $d(u,v)$, is defined as the length of the smallest path between them. The *diameter* of G is the largest distance among any pair of vertices of G. The *distance matrix* $D(G)$ of G, is defined as $D(G) = (d(u,v))_{u,v\in V}$. The *transmission degree* $Tr_G(u)$ of $u \in V$ is the sum of the distances from u to every other vertex of G, mathematically, $Tr_G(u) = \sum_{v\in V(G)} d(u,v)$. We observe that $Tr_G(u)$ is same as the u-th row sum of $D(G)$. The *Weiner index* $W(G)$ of G is the sum of distances between all unordered pairs of vertices. Let $Tr(G)$ be the diagonal matrix with entries as the row sums of $D(G)$. The distance Laplacian matrix of G is defined by $D^L(G) = Tr(G) - D(G)$ and is shortly denoted by D^L. It immediately follows that $D^L(G)$ is the real symmetric and positive semi-definite matrix. Besides, every row sum of $D^L(G)$ is 0, so 0 must be the smallest eigenvalue of $D^L(G)$. The collection of all eigenvalues of $D^L(G)$ is called the distance Laplacian spectrum (D^L-spectrum) of G and we index them from the largest to the smallest as follows

$$\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} \geq \rho_n = 0,$$

where ρ_1 and ρ_{n-1} are known as the distance Laplacian spectral radius and the second smallest distance Laplacian eigenvalue of graph G. More about D^L matrix can be found in [4–6, 19].
The distance Laplacian energy $DLE(G)$ of a connected graph G is defined as

$$DLE(G) = \sum_{i=1}^{n} \left| \rho_i - \frac{2W}{n} \right|.$$

Let σ be the positive integer such that $\rho_\sigma \geq \frac{2W(G)}{n}$ and let $S_k(G) = \sum_{i=1}^{k} \rho_i$ be the sum of k largest D^L-eigenvalues of G. Then using the fact that $\sum_{i=1}^{n} \rho_i = 2W$, it follows that [9]

$$DLE(G) = 2\left(S_\sigma(G) - \frac{2\sigma W}{n}\right) = 2\max_{1\le j\le n}\left(\sum_{i=1}^{j}\rho_i(G) - \frac{2jW}{n}\right) = 2\max_{1\le i\le n}\left(S_i(G) - \frac{2iW}{n}\right).$$

For some recent progress on $DLE(G)$, we refer to [9, 11, 12]. The D^L spectral ordering of trees on the basis of $DLE(G)$ and the D^L spectral radius ρ_1 can be seen in [13, 17, 18]. The distance based spectral ordering of graph invariants can be seen in [2, 3, 7, 20].

One of the interesting and important problems in spectral graph theory is to find extremal graphs among all graphs of order n or among a class of graphs for a spectral graph invariant. Another related problem is to order the graphs with respect a spectral graph invariant. These problems has been considered for many families of graphs with respect to different graph matrices(like adjacency matrix, (signless)Laplacian matrix, distance matrix, distance (signless)Laplacian matrix) and is one of the hot topics of the present research in spectral graph theory. The importance of these problems is manifold. In matrix theory it is a well known problem to find the extremal values for the spectral and trace norms of a class of matrices. Also ordering the matrices in a given class with respect to spectral and trace norms is an interesting and hard problem in matrix theory. Since for a symmetric non-negative matrix the spectral norm is same as the largest eigenvalue and the trace norm is the sum of the absolute values of the eigenvalues of the matrix. It follows that distance Laplacian spectral radius is the spectral norm of the matrix $D^L(G)$ and the distance Laplacian energy is the trace norm of the matrix $D^L(G) - \frac{2W(G)}{n}I$, where I is the identity matrix. These problems are also important from application point of view in different branches of science and social science. Applications of graph energies in the chemistry of unsaturated conjugated molecules are well known. Somewhat related are applications in crystallography, theory of macromolecules, as well as analysis and comparison of protein sequences. Also not particularly unexpected are attempts to apply graph energies in network analysis, including problems of air transportation, satellite communication, and biology, see [14].

Here, in this paper our motive is to study these problems for brooms of diameter 4 and their complements with respect to their distance Laplacian matrix.

The manuscript is organized as follows: In Section 2, we find the D^L spectrum and the D^L energy of broom trees of diameter 4 and discuss their ordering. We show these trees can be arranged in order on the basis of both these spectral invariants. In Section 3, we obtain D^L eigenvalues and DLE of the complement of the family of double broom of diameter 4 and order them in terms of the second smallest D^L eigenvalue and DLE.

2. Distance Laplacian eigenvalues and energy of brooms of diameter 4

For a connected graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, a real valued vector $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ can be regarded as a mapping with domain $V(G)$ which maps v_i to x_i, for every $i = 1, 2, 3, \ldots, n$. Also, it is known that

$$X^T D^L(G) X = \sum_{\{u,v\} \subseteq V(G)} d(u,v)\big(x_u - x_v\big)^2,$$

and ρ is the D^L eigenvalue with the corresponding eigenvector $0 \neq X$ if and only if for every $v \in V(G)$,

$$\rho x_v = \sum_{u \in V(G)} d(u,v)\big(x_v - x_u\big),$$

or equivalently,

$$\rho x_v = Tr(v) x_v - \sum_{u \in V(G)} d(u,v) x_u. \tag{2.1}$$

Equation (2.1) is known as (ρ, X)-eigenequation of $D^L(G)$ at v.
The following results can be found in [6] and is helpful in finding some D^L-eigenvalues of G.

Lemma 1. *[6] Let G be a graph of order n. If $\mathcal{S} = \{v_1, v_2, \ldots, v_s\}$ is the independent set of G satisfying $N(v_i) = N(v_j)$ for every $i, j \in \{1, 2, \ldots, s\}$. Then $\partial = Tr(v_i) = Tr(v_j)$ for each $i, j \in \{1, 2, \ldots, s\}$ and $\partial + 2$ is the D^L eigenvalue of G with multiplicity at least $s - 1$.*

Lemma 2. *[6] Let G be a graph of order n. If $\mathcal{W} = \{v_1, v_2, \ldots, v_\omega\}$ is the clique of G satisfying $N(v_i) - \mathcal{W} = N(v_j) - \mathcal{W}$ for every $i, j \in \{1, 2, \ldots, \omega\}$, then $\partial = Tr(v_i) = Tr(v_j)$ for each $i, j \in \{1, 2, \ldots, \omega\}$ and $\partial + 1$ is the D^L eigenvalue of G with multiplicity at least $\omega - 1$.*

The next result gives connects the eigenvalues of real symmetric matrix with the eigenvalues of its principal submatrix.

Theorem 1 (Interlacing Theorem, [15]). *Let $M \in \mathbb{M}_n$ be a real symmetric matrix. Let A be a principal submatrix of M of order m, $(m \leq n)$. Then the eigenvalues of M and A satisfy the following inequalities*

$$\lambda_{i+n-m}(M) \leq \lambda_i(A) \leq \lambda_i(M), \qquad \textit{with} \ \ 1 \leq i \leq m.$$

The *broom* $P_5(2, a)$ (see [10]) of diameter of 4 is obtained by attaching a pendent vertices to the vertex v_2 of P_5, the *generalized broom* $P_5(3, a)$ is the obtained by attaching $a \geq 2$ pendent vertices to the vertex v_3 of P_5. Similarly, the double broom $P_5(a, b)$, (or $P_5(1, a|n, b)$) has degree 2 at the central vertex, while the other two non pendent vertices may have arbitrary degree (say a and b with $(a < b)$). For reference, we will denote these graphs by $B(a)$, $B(a')$ and $B(a, b)$ and are shown in Figure 1.

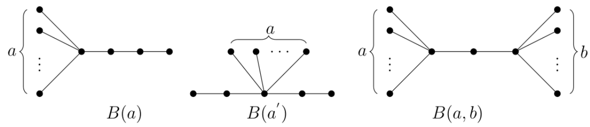

Figure 1. **Brooms of diameter 4.**

In Spectral graph theory one of the most attractive and difficult problems is " *to find the extremal graphs for a spectral invariant and to order the graphs on the basis of this spectral invariant*". This problem has been considered for various families of graphs with respect to different graph matrices and as such many articles have been published in this direction. In general for a given graph matrix these types of problems are not so easy. However, if one picks a class of graphs with some symmetry then such type of problems can be solved up to some extent. For example, the D^L spectral ordering of trees on the basis of $DLE(G)$ and the D^L spectral radius can be seen in [13, 17, 18]. The distance based spectral ordering of graph invariants can be seen in [2, 3, 7, 20]. In this work we consider two classes of graphs namely, the brooms of diameter 4 and their complements. We ask the following problems.

Problem 1. Characterize the brooms of diameter four in terms of spectral graph invariants associated to the D^L matrix?

Problem 2. Order the complement of family of brooms in terms of the D^L spectral parameters?

In the rest of present section, we will answer Problem 1 with respect to D^L spectral radius and the D^L energy. In fact, we will prove that the brooms of diameter 4 can be ordered on the basis of D^L spectral radius and D^L energy.
Next result completely gives the $D^L(G)$-eigenvalues of the family $B(a)$.

Proposition 1. *The D^L spectrum of $B(a)$ with $a = n-4$, consists of the eigenvalue $2a+10$ with multiplicity $a-1$, the simple eigenvalue 0 and the four zeros $y_1 \geq y_2 \geq y_3 \geq y_4$ of the following polynomial*

$$x^4 - x^3(10a+30) + x^2(35a^2+220a+328) - x(50a^3+504a^2+1586a+1544) \\ + 24a^4+356a^3+1804a^2+3716a+2640,$$

with $y_4 \in (a, a+8)$, $y_3 \in (2a+5, 2a+6)$, $y_2 \in (3a+6.5, 3a+7)$, $y_1 \in (4a+10, 4a+11)$ for $a \geq 4$.

Proof. Clearly, $B(a)$ has a pendent vertices sharing the same vertex with transmission $T = 2(a-1)+1+2+3+4 = 2a+8$, so by Lemma 1, $T+2 = 2a+10$ is the D^L eigenvalue of $B(a)$ with multiplicity $a-1$. In order to find the remaining

D^L-eigenvalues of $B(a)$, we use eigenequations (2.1). Let X be the eigenvector of $B(a)$ with $x_i = X(v_i)$ for $i = 1, 2, 3, \dots, n$. Then every component of X corresponding to the pendent vertices $\{v_1, v_2, \dots, v_a\}$ is x_1 and every components of X related to the vertices $v_{a+1}, v_{a+2}, v_{a+3}$ and v_{a+4} are x_2, x_3, x_4 and x_5, respectively. By using equation (2.1), the (ρ, X)-eigenequations of $B(a)$ are given by

$$\begin{aligned}
\rho x_1 =& 10x_1 - x_2 - 2x_3 - 3x_4 - 4x_5,\\
\rho x_2 =& -ax_1 + (a+6)x_2 - x_3 - 2x_4 - 3x_5,\\
\rho x_3 =& -2ax_1 - x_2 + (2a+4)x_3 - x_4 - 2x_5,\\
\rho x_4 =& -3ax_1 - 2x_2 - x_3 + (3a+4)x_4 - x_5,\\
\rho x_5 =& -4ax_1 - 3x_2 - 2x_3 - x_4 + (4a+6)x_5.
\end{aligned}$$

The other D^L eigenvalues of $B(a)$ are the solutions of the above system of equations and matrix of coefficients of right side is given below

$$\begin{pmatrix}
10 & -1 & -2 & -3 & -4\\
-a & a+6 & -1 & -2 & -3\\
-2a & -1 & 2a+4 & -1 & -2\\
-3a & -2 & -1 & 3a+4 & -1\\
-4a & -3 & -2 & -1 & 4a+6
\end{pmatrix}. \tag{2.2}$$

The characteristic polynomial of (2.2) is

$$\begin{aligned}
f(x) = -x\big(&x^4 - x^3(10a+30) + x^2(35a^2+220a+328) - x(50a^3+504a^2+1586a+1544)\\
&+ 24a^4 + 356a^3 + 1804a^2 + 3716a + 2640\big).
\end{aligned}$$

Next, by using the intermediate value theorem, we approximate the zeros of the polynomial

$$\begin{aligned}
p(x) = x^4 - &x^3(10a+30) + x^2(35a^2+220a+328) - x(50a^3+504a^2\\
&+ 1586a + 1544) + 24a^4 + 356a^3 + 1804a^2 + 3716a + 2640.
\end{aligned} \tag{2.3}$$

Since, $p(a) = 6(7a^3 + 91a^2 + 362a + 440)$, which is increasing function of a and is clearly positive. Also, $p(a+8) = -2(a-2)\left(3a^2 - 3a + 4\right) < 0$, for each $a \geq 3$. Similarly, we can verify that

$$\begin{aligned}
p(2a+5) &= -(2a^3 + a^2 - 8a + 5) < 0,\\
p(2a+6) &= 4a(a-2) > 0,\\
p(3a+6.5) &= a^3 - \frac{21a^2}{4} + \frac{15a}{4} + 8.3125 > 0, \text{ for } a \geq 4\\
p(3a+7) &= -3(a-1)(3a+5) < 0,\\
p(4a+10) &= -16a(2a+5) < 0,\\
p(4a+11) &= 6a^3 + 15a^2 + 4a + 55 > 0.
\end{aligned}$$

If $y_1 \geq y_2 \geq y_3 \geq y_4$ are the zeros of (2.3), then by above observation, it follows that

$$y_4 \in (a, a+8),\ y_3 \in (2a+5, 2a+6),\ y_2 \in (3a+6.5, 3a+7),\ y_1 \in (4a+10, 4a+11).$$

□

Proceeding similar to Proposition 1, we obtain the D^L-spectrum of the family $B(a')$, given in the next result.

Proposition 2. *The D^L spectrum of $B(a')$ consists of the eigenvalue $2a+11$ with multiplicity $a-1$, the simple eigenvalue 0, the eigenvalues $\frac{5a+23\pm\sqrt{a^2+10a+41}}{2}$ and the three zeros $x_1' \geq x_2' \geq x_3'$ of the following polynomial*

$$x^3 - x^2(6a+28) + x(11a^2+107a+257) - 6a^3 - 93a^2 - 469a - 770,$$

where $a = n-5, a \geq 4$ with $x_3' \in (a, a+8),\ x_2' \in (2a+9, 2a+9.1),\ x_1' \in (3a+11.8, 3a+12)$.

Proof. The proof follows similar to the proof of Proposition 1. For the polynomial $g(x) = x^3 - x^2(6a+28) + x(11a^2+107a+257) - 6a^3 - 93a^2 - 469a - 770$, it can easily seen that $g(a) = -(14a^2+212a+770) < 0, g(a+8) = 2a^2+4a+6 > 0, g(2a+9) = 4 > 0, g(2a+9.1) = -\frac{a^2}{10} - \frac{a}{2} + 3.591 < 0, g(3a+11.8) = -\frac{2a^2}{5} - \frac{2a}{25} + 6.912 < 0$ and $g(3a+12) = 2a+10 > 0$. The result now follows. □

The next result shows that for $a \geq 5$ the D^L spectral radius ρ_1 of the family $B(a)$ is bigger than the D^L spectral radius of the family $B(a')$.

Proposition 3. *For $a \geq 5$, $\rho_1(B(a)) > \rho_1(B(a'))$.*

Proof. By Proposition 1, the D^L spectral radius of $B(a)$ is bounded below by $4a+10$ and by Proposition 2, ρ_1 of $B(a')$ is $\frac{5a+23+\sqrt{a^2+10a+41}}{2}$. Therefore, $\frac{5a+23+\sqrt{a^2+10a+41}}{2} < 4a+10$ implies that

$$\sqrt{a^2+10a+41} < 3a-3,$$

which gives $4(2a^2-7a-8) > 0$, and this quadratic inequality is true for $a \geq 5$. □

It is clear that ρ_1 of the family $B(a')$ is an increasing function of a while as ρ_1 of the family $B(a)$ is $\rho_1(B(a)) = 4a+10+\varepsilon_a$, where $0 < \varepsilon_a < 1$. Since, $0 < \varepsilon_a < 1$, it is clear that $\rho_1(B(a))$ is also an increasing function of a. Thus, we note the following observation which gives the ordering of the trees belonging to the class $B(a) \cup B(a')$ on the basis of their D^L spectral radius.

Corollary 1. *For $a \geq 5$, we have*

$$\rho_1(B(5')) \leq \rho_1(B(6')) \leq \cdots \leq \rho_1(B((n-5)'))$$
$$;\rho_1(B(5)) \leq \rho_1(B(6)) \leq \cdots \leq \rho_1(B(n-5)) \leq \rho_1(B(n-4)).$$

The following result presents the D^L energy of the families $B(a)$ and $B(a')$ for $a \geq 5$.

Theorem 2. *Let $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_{n-1}(G) \geq \rho_n(G) = 0$ be the D^L eigenvalues of G. Then the following holds.*

(i) *The D^L energy of $B(a)$ with $n-4=a, a \geq 5$ is*

$$DLE(B(a)) = 2\left(\rho_1(B(a)) + \rho_2(B(a)) - \frac{4a^2+16a+60}{a+4}\right).$$

(ii) *If $a \geq 6$, then the D^L energy of $B(a')$ is $DLE(B(a')) = 2\rho_2(B(a')) + \sqrt{a^2+10a+41} - \frac{a^2+4a+75}{a+5}$ and if $a=5$, then the D^L energy of $B(a')$ is $DLE(B(a')) = 53.57962961$.*

Proof. By Proposition 1, we can order the D^L-spectrum of $B(a)$ from the least D^L-eigenvalue to the largest D^L-eigenvalues as:

$$\rho_n(B(a)) = 0,\ \rho_{n-1}(B(a)) = y_4,\ \rho_{n-2}(B(a)) = y_3,$$
$$\rho_i(B(a)) = 2a+10, \text{ for } i = 3,4,\ldots,n-3,$$
$$\rho_2(B(a)) = y_2,\ \rho_1(B(a)) = y_1,$$

where $a < y_4 < a+8,\ 2a+5 < y_3 < 2a+6,\ 3a+6.5 < y_2 < 3a+7$ and $4a+10 < y_1 < 4a+11$. Since, $\sum_{i=1}^{n-1} \rho_i(B(a)) = 2W(B(a)) = 2a^2+18a+20$, so we obtain $\frac{2W(B(a))}{n} = \frac{2a^2+18a+20}{a+4}$. Let $\sigma = \sigma(B(a))$ be the greatest positive integer such that $\rho_\sigma(B(a)) \geq \frac{2W(B(a))}{n}$, then it is always true that $\rho_1(B(a)) \geq \frac{2W}{n}$. Further $\rho_2 \geq \rho_i = 2a+10$, for $i = 3,4,\ldots,n-3$ and by direct calculation we have $2a+10 \geq \frac{2W}{n}$. Again, $2a+5 < \rho_{n-2}(B(a)) < 2a+6$ gives that $\rho_{n-2}(B(a)) = 2a+5+t$, where $t \in (0,1)$. We have $\frac{2W(B(a))}{n} = \frac{2a^2+18a+20}{a+4} = 2(a+4) - \frac{12}{a+4} = 2a+5+\frac{3a}{a+4} > 2a+6 > 2a+5+t = \rho_{n-2}(B(a)) \geq \rho_{n-1}(B(a))$, for all t. From this discussion, we obtain $\sigma = a+1$. Therefore, the D^L energy of $B(a)$ is

$$\begin{aligned}
DLE(B(a)) &= 2\left(\sum_{i=1}^{\sigma} \rho_i(B(a)) - \frac{2\sigma W(B(a))}{n}\right) = \left(\sum_{i=1}^{a+1} \rho_i(B(a)) - \frac{2(a+1)W(B(a))}{n}\right) \\
&= 2\left(\rho_1(B(a)) + \rho_2(B(a)) + (a-1)(2a+10) - \frac{2(a+1)W(B(a))}{a+4}\right) \\
&= 2\left(\rho_1(B(a)) + \rho_2(B(a)) - \frac{4a^2+16a+60}{a+4}\right).
\end{aligned}$$

(ii). Let $\rho_1(B(a')) \geq \rho_2(B(a')) \geq \cdots \geq \rho_{n-1}(B(a')) \geq \rho_n(B(a')) = 0$ be the D^L eigenvalues of $B(a')$. From the Proposition 2, we have $a < x_3' < a+8$,

$2a+9 < x_2' < 2a+9.1$ and $3a+11.8 < x_1' < 3a+12$. Since, $\frac{5a+23-\sqrt{a^2+10a+41}}{2} > a+8$ gives that $8a^2+32a+8 > 0$, which is always true. It follows that $\rho_{n-1}(B(a')) = x_3'$. Also, $\frac{5a+23-\sqrt{a^2+10a+41}}{2} < 2a+9$ gives that $41 > 25$, which is always true. It follows that $\rho_{n-2}(B(a')) = \frac{5a+23-\sqrt{a^2+10a+41}}{2}$ and $\rho_{n-3}(B(a')) = x_2'$. Further, we have $\rho_1(B(a')) = \frac{5a+23-\sqrt{a^2+10a+41}}{2}$, $\rho_2(B(a^{'})) = x_1^{'}$, $\rho_i(B(a')) = 2a+11$ for $i = 3,\dots,a+1$. Therefore, the D^L eigenvalues of $B(a^{'})$ can be ordered as: $\rho_n(B(a')) = 0$, $\rho_{n-1}(B(a')) = x_3'$, $\rho_{n-2}(B(a')) = \frac{5a+23-\sqrt{a^2+10a+41}}{2}$, $\rho_{n-3}(B(a^{'})) = x_2^{'}$, $\rho_i(B(a')) = 2a+11$, for $i = 3,4,\dots,n-4$, $\rho_2(B(a')) = x_1'$, $\rho_1(B(a')) = \frac{5a+23+\sqrt{a^2+10a+41}}{2}$. Also, the average transmission of $B(a^{'})$ is $\frac{2W(B(a^{'}))}{n} = \frac{2a^2+20a+40}{a+5}$. Let $\sigma = \sigma(B(a'))$ be the greatest positive integer such that $\rho_\sigma(B(a')) \geq \frac{2W(B(a'))}{n}$, then it is always true that $\rho_1(B(a')) \geq \frac{2W(B(a'))}{n}$. Now, $2a+11 \geq \frac{2W(B(a'))}{n}$ implies that $a+15 \geq 0$, which is always true. It follows that $\rho_2(B(a')) \geq \rho_i(B(a')) = 2a+11 \geq \frac{2W(B(a'))}{n}$, for all $i = 3,4,\dots,n-4$. For the eigenvalue $\rho_{n-2}(B(a')) = \frac{5a+23-\sqrt{a^2+10a+41}}{2}$, we have $\rho_{n-2}(B(a')) < \frac{2W(B(a'))}{n}$ giving that $-4(a^3+8a^2+25a-50) < 0$, which is always true. This shows that $\rho_{n-3}(B(a')) \leq \rho_{n-2}(B(a')) < \frac{2W(B(a'))}{n}$. Since $2a+9 < \rho_{n-3}(B(a')) < 2a+9.1$ and $2a+9.1 < \frac{2W(B(a'))}{n}$ gives that $55-9a < 0$, which is true for $a \geq 7$. For $a = 6$, the polynomial $g(x)$ of the Proposition 2 becomes $g(x) = x^3 - 64x^2 + 1295x - 8228$. By direct calculation it can be seen that $x_2' = 21.0570 < 21.0909 = \frac{2W(B(a'))}{n}$. Lastly, for $a = 5$, the polynomial $g(x)$ of the Proposition 2 becomes $g(x) = x^3 - 58x^2 + 1067x - 6190$. By manual calculation it can be seen that $x_2' = 19.0739 > 19 = \frac{2W(B(a'))}{n}$. Thus, it follows that $\sigma = a+1$, for all $a \geq 6$ and $\sigma = a+2$, for $a = 5$. So, for $a \geq 6$, the D^L energy is

$$\begin{aligned} DLE(B(a')) &= 2\left(\sum_{i=1}^{a+1} \rho_i(B(a')) - \frac{2(a+1)W(B(a'))}{n}\right) \\ &= 2\Big(\frac{5a+23+\sqrt{a^2+10a+41}}{2} + \rho_2(B(a')) + (a-1)(2a+11) \\ &\qquad - \frac{2(a+1)W(B(a'))}{a+5}\Big) \\ &= 2\rho_2(B(a')) + \sqrt{a^2+10a+41} - \frac{a^2+4a+75}{a+5}. \end{aligned}$$

For $a = 5$, we have $\sigma = a+2$, and the D^L energy of $B(a')$ is

$$DLE(B(a')) = 2(\rho_2(B(a')) + \rho_{n-3}(B(a'))) + \sqrt{a^2+10a+41} - \frac{5a^2+44a+155}{a+5}.$$

By direct calculation it can be verified that for $a = 5$, we have $\rho_2(B(a')) = x_1' = 26.8307$, $\rho_{n-3}(B(a')) = x_2' = 19.0739$ and $\rho_{n-2}(B(a')) = x_3' = 12.0953$. Thus, for $a = 5$, we have $DLE(B(a')) = 53.57962961$. This completes the proof. $\square$

Using the fact that $4a+10 < \rho_1(B(a)) < 4a+11, 3a+6.5 < \rho_2(B(a)) < 3a+7$ and $3a+11.8 < \rho_2(B(a')) < 3a+12$, we have the following observation from Theorem 2.

Corollary 2. *The D^L energy of $B(a)$ and $B(a')$ lies in the following open intervals.*

(i)

$$6a+33-\frac{120}{a+4} < DLE(B(a)) < 6a+36-\frac{120}{a+4}.$$

(ii)

$$5a+24.6+\sqrt{a^2+10a+41}-\frac{80}{a+5} < DLE(B(a')) < 5a+25+\sqrt{a^2+10a+41}-\frac{80}{a+5}.$$

From Corollary 2, we see that the D^L energy of $B(a)$ lies in the interval of length 3 while as the D^L energy of $B(a')$ lies in the interval of length 0.4. Thus, the D^L energy bounds for $B(a)$ and $B(a')$ given in Corollary 2 are best approximated.
Since $4a+10 < \rho_1(B(a)) < 4a+11$ and $3a+6.5 < \rho_2(B(a)) < 3a+7$, it follows that $\rho_1(B(a)) = 4a+10+\varepsilon_1(a)$ and $\rho_2(B(a)) = 3a+6.5+\varepsilon_2(a)$, where $0 < \varepsilon_1(a) < 1$ and $0 < \varepsilon_2(a) < 0.5$. With this it follows from Theorem 2 that

$$DLE(B(a)) = 2\left(7a+16.5+\varepsilon(a)-\frac{4a^2+16a+60}{a+4}\right) = 6a+33+2\varepsilon(a)-\frac{120}{a+4}, \tag{2.4}$$

where $\varepsilon(a) = \varepsilon_1(a)+\varepsilon_2(a)$, $0 < \varepsilon(a) < 1.5$. It is clear from (2.4) that $DLE(B(a))$ is an increasing function of $a, a \geq 5$.
Again, since $3a+11.8 < \rho_2(B(a')) < 3a+12$, it follows that $\rho_2(B(a')) = 3a+11.8+\varepsilon_3(a)$, where $0 < \varepsilon_3(a) < 0.2$. With this it follows from Theorem 2 that

$$\begin{aligned} DLE(B(a')) &= 6a+23.6+2\varepsilon_3(a)+\sqrt{a^2+10a+41}-\frac{a^2+4a+75}{a+4} \\ &= 5a+24.6+2\varepsilon_3(a)+\sqrt{a^2+10a+41}-\frac{80}{a+5}. \end{aligned} \tag{2.5}$$

It is clear from (2.5) that $DLE(B(a'))$ is an increasing function of $a, a \geq 5$.

Theorem 3. *For $a \geq 14$, we have $DLE(B(a)) > DLE(B(a'))$ and for $5 \leq a \leq 13$, we have $DLE(B(a)) \geq DLE(B(a'))$, provided that $a+8.4+2(\varepsilon(a)-\varepsilon_3(a))+\frac{80}{a+5} \geq \sqrt{a^2+10a+41}+\frac{120}{a+4}$, where $\varepsilon(a), \varepsilon_3(a)$ are defined above.*

Proof. From the Corollary 2, we have

$$DLE(B(a)) - DLE(B(a')) = a+8-\sqrt{a^2+10a+41}+\frac{80}{a+5}-\frac{120}{a+4} = f(a). \tag{2.6}$$

By direct calculation, we have $f(14) = \frac{1114}{57}-\sqrt{377} > 0$, $f(15) = \frac{393}{19}-4\sqrt{26} > 0$, $f(16) = \frac{458}{21}-\sqrt{457} > 0$ and $f(17) = \frac{1765}{77}-10\sqrt{5} > 0$. So, suppose that $a \geq 18$.

We have, $a^2+10a+41 < a^2+12a+36 = (a+6)^2$ giving that $\sqrt{a^2+10a+41} < (a+6)$. With this it follows from (2.6) that

$$f(a) \geq a+8-(a+6)+\frac{80}{a+5}-\frac{120}{a+4} = 2+\frac{80}{a+5}-\frac{120}{a+4} = \frac{2(a^2-11a-120)}{(a+4)(a+5)} > 0,$$

for all $a \geq 18$. This shows that $DLE(B(a)) - DLE(B(a')) > 0$, for $a \geq 18$ and thus the proof is complete in this case.
For $5 \leq a \leq 13$, it follows from (2.4) and (2.5) that

$$DLE(B(a)) - DLE(B(a')) = a+8.4+2(\varepsilon(a)-\varepsilon_3(a)) - \sqrt{a^2+10a+41}+\frac{80}{a+5}-\frac{120}{a+4} \geq 0,$$

if $a+8.4+2(\varepsilon(a)-\varepsilon_3(a))+\frac{80}{a+5} \geq \sqrt{a^2+10a+41}+\frac{120}{a+4}$. □

From the discussion before Theorem 3, we have the following observation which gives the ordering of the trees belong to the class $B(a) \cup B(a')$ on the basis of DLE.

Theorem 4. *For $a \geq 5$, the following holds.*

1. *Among all the trees in $B(a)$, the tree $B(5)$ has the minimal D^L energy while as the tree $B(n-4)$ has the maximal D^L energy.*
2. *Among all the trees in $B(a')$ the tree $B(5')$ has the minimal D^L energy while as the tree $B(n-5')$ has the maximal D^L energy.*

The following result gives the D^L eigenvalues of the family $B(a,b)$.

Theorem 5. *Let $B(a,b)$ be the double broom with $5 \leq a < b$ and $a+b = n-3, a \neq b-3$. Then the D^L spectrum of $B(a,b)$ consists of the eigenvalue $2a+4b+6$ with multiplicity $a-1$, the eigenvalue $4a+2b+6$ with multiplicity $b-1$, the simple eigenvalue 0 and the eigenvalues $y_1 \geq y_2 \geq y_3 \geq y_4$, where $y_1 \in (4a+4b+6, 4a+4b+8)$, $y_2 \in (a+3b, a+3b+7)$, $y_3 \in (2a+2b+3, 2a+2b+6)$ and $y_4 \in (2a+b+4, 2a+2b+3)$.*

Proof. Let $V(B(a,b)) = \{v_1, v_2, \ldots, v_a, w_1, w_2, w_3, u_1, u_2, \ldots, u_b\}$ be the vertex set of $B(a,b)$, where $v_1, v_2, \ldots, v_a$ are the a pendent vertices at the vertex w_1 and $u_1, u_2, \ldots, u_b$ are the b pendent vertices at the vertex w_3 of the path $P_3 : w_1w_2w_3$. It is clear that the vertices $v_1, v_2, \ldots, v_a$ forms an independent set of cardinality of a sharing the same vertex w_1 of degree $a+1$ with $Tr(v_1) = Tr(v_2) = \cdots = Tr(v_a) = 2a-2+1+2+3+4b = 2a+4b+6$. Thus, by Lemma 1, $2a+4b+4$ is the D^L eigenvalue of $B(a,b)$ with multiplicity at least $a-1$. Again, the set of vertices $u_1, u_2, \ldots, u_b$ forms an independent set of cardinality of b sharing the same vertex w_3 of degree $b+1$ with common transmission degree $T = 2b-2+1+2+3+4a = 2b+4a+4$. So, by Lemma 1, it follows that $4a+2b+6$ is the D^L eigenvalue of $B(a,b)$ with multiplicity at least $b-1$. Let X be the eigenvector of $B(a,b)$ with $x_i = X(v_i)$ for $i = 1, 2, 3, \ldots, n$.

Then every component of X corresponding to the pendent vertices $\{v_1, v_2, \dots, v_a\}$ is x_1 and every components of X corresponding to other pendent vertices is x_5. The components of X corresponding to vertices of the degrees $a+1, 2$ and $b+1$ are x_2, x_3 and x_4, respectively. By using Equation (2.1), the (ρ, X)-eigenequation of $B(a,b)$ are given by

$$\begin{aligned}
\rho x_1 =& (4b+6)x_1 - x_2 - 2x_3 - 3x_4 - 4bx_5,\\
\rho x_2 =& -ax_1 + (a+3b+3)x_2 - x_3 - 2x_4 - 3bx_5,\\
\rho x_3 =& -2ax_1 - x_2 + (2a+2b+2)x_3 - x_4 - 2bx_5,\\
\rho x_4 =& -3ax_1 - 2x_2 - x_3 + (3a+b+3)x_4 - bx_5,\\
\rho x_5 =& -4ax_1 - 3x_2 - 2x_3 - x_4 + (4a+6)x_5.
\end{aligned}$$

The remaining non-zero four D^L eigenvalues of $B(a,b)$ are the eigenvalues of the following coefficient matrix corresponding to the right side of above system of equations

$$\begin{pmatrix}
4b+6 & -1 & -2 & -3 & -4b\\
-a & a+3b+3 & -1 & -2 & -3b\\
-2a & -1 & 2a+2b+2 & -1 & -2b\\
-3a & -2 & -1 & 3a+b+3 & -b\\
-4a & -3 & -2 & -1 & 4a+6
\end{pmatrix}. \tag{2.7}$$

It is easy to see that the characteristic polynomial of this matrix is $-x(f(x))$, where

$$\begin{aligned}
f(x) =& x^4 - x^3(10a+10b+20) + x^2\left(35a^2+74ab+146a+35b^2+146b+147\right)\\
&- x\left(50a^3+174a^2b+330a^2+174ab^2+716ab+696a+50b^3+330b^2+696b+468\right)\\
&+ 24a^4+128a^3b+228a^3+208a^2b^2+828a^2b+768a^2+128ab^3+828ab^2+1680ab\\
&+ 1080a+24b^4+228b^3+768b^2+1080b+540.
\end{aligned}$$

Let $y_1 \geq y_2 \geq y_3 \geq y_4$ be the zeros of $f(x)$. By manual calculations it is easy to verify that

$$\begin{aligned}
f(4a+4b+8) =& 12a^3+64a^2+20ab^2+80ab+104a+12b^3+64b^2+20a^2b+104b+60 > 0\\
f(4a+4b+6) =& -32ab^2-48ab-32a^2b < 0.
\end{aligned}$$

Thus, by the intermediate value theorem a zero of $f(x)$ namely y_1 lies in $(4a+4b+6, 4a+4b+8)$. Also,

$$f(a+3b+7) = 8-2b+34ba^2-4b^2-2b^3-46a-68ab-14ab^2+72a^2-18a^3 = \phi(a,b).$$

We will show that $\phi(a,b) < 0$, for all $b > a \geq 5$. By direct computations, we obtain

$$\begin{aligned}
\phi(a,6) &= -18a^3+276a^2-958a-580 < 0, \quad \text{for} \quad a=5\\
\phi(a,7) &= -18a^3+310a^2-1208a-888 < 0, \quad \text{for} \quad a=5,6\\
\phi(a,8) &= -18a^3+344a^2-1486a-1288 < 0, \quad \text{for} \quad a=5,6,7\\
\phi(a,9) &= -18a^3+378a^2-1792a-1792 < 0, \quad \text{for} \quad a=5,6,7,8.
\end{aligned}$$

Therefore, suppose that $b \geq 10$. For $b > a$ and $b \geq 10$, we have $\frac{\partial\phi(a,b)}{\partial a} = 68ab - 46 - 68b - 14b^2 + 144a - 72a^2$. It is easy to verify that $\frac{\partial\phi(a,b)}{\partial a} < 0$, for all $a > \frac{17b}{36} + 1 + \theta$ or $a < \frac{17b}{36} + 1 - \theta$ and $\frac{\partial\phi(a,b)}{\partial a} \geq 00$, for all $a \in \left[\frac{17b}{36} + 1 - \theta, \frac{17b}{36} + 1 + \theta\right]$, where $\theta = \frac{1}{144}\sqrt{592b^2 + 7488}$. This gives that $\phi(a,b)$ is a decreasing function of a, for $a \in \left(\frac{17b}{36} + 1 + \theta, b - 1\right]$ or $a \in \left[5, \frac{17b}{36} + 1 - \theta\right)$ and an increasing function for $a \in \left[\frac{17b}{36} + 1 - \theta, \frac{17b}{36} + 1 + \theta\right]$. So, the sign of $\phi(5,b)$ and $\phi\left(\frac{17b}{36} + 1 + \theta, b\right)$ will decide the sign of $\phi(a,b)$. We have $\phi(5,b) = -2b^3 - 74b^2 + 508b - 672 < 0$, for all $b \geq 10$. Also, $(24.3b+4)^2 < 592b^2 + 7488 < (24.4b+14)^2$ gives that $\frac{923b}{1440} + \frac{37}{36} < \frac{17b}{36} + 1 + \theta < \frac{77b}{120} + \frac{79}{72}$. By direct calculations, we obtain

$$\begin{aligned}
\phi\Big(\frac{923b}{1440} + \frac{37}{36}, b\Big) &= -\frac{964913296^3}{55296000} - \frac{431276236^2}{4147200} - \frac{742009b}{103680} + \frac{44675}{2592} < 0,\\
\phi\Big(\frac{77b}{120} + \frac{79}{72}, b\Big) &= -\frac{501079b^3}{288000} - \frac{1705423b^2}{172800} - \frac{573037b}{103680} + \frac{423665}{20736} < 0,
\end{aligned}$$

for all $b \geq 10$. Thus, it follows that $\phi(a,b) < 0$, for all $b > a \geq 5$. Further,

$$f(3b + a) = 24a^3 + 219a^2 - 222ab + 612a + 12b^3 + 3b^2 - 36a^2b - 324b + 540 = \varphi(a,b).$$

We will show that $\varphi(a,b) \geq 0$, for all $b > a \geq 5$. We have $\frac{\partial\varphi(a,b)}{\partial a} = 72a^2 + 438a - 222b + 612 - 72ab = 72a(a-b) + 222(a-b) + 216a + 612$. It is easy to see that $\frac{\partial\varphi(a,b)}{\partial a} < 0$, for all $a \leq b - 4$, giving that $\varphi(a,b)$ is a decreasing function of a, for $a \in [5, b-4]$. Therefore, the sign of $\varphi(b-4,b)$ will decide the sign of $\varphi(a,b)$ in $[5, b-4]$. We have $\varphi(b-4,b) = 60 > 0$, $\varphi(b-3,b) = 27 - 36b < 0$, $\varphi(b-2,b) = 0$ and $\varphi(b-1,b) = 108b + 123 > 0$ giving that $\varphi(a,b) \geq 0$, for all $b > a \geq 5$ with $a \neq b - 3$. Thus, it follows that for $a \neq b - 3$, we have $y_2 \in [3b + a, 3b + a + 7)$. In a similar way, we have

$$\begin{aligned}
f(2a + 2b + 6) &= 4(a-b)^2(a + b + 3) > 0,\\
f(2a + 2b + 3) &= -(a-b)^2(2a + 2b + 3) < 0,\\
f(2a + b + 4) &= a^2b^2 + 4a^2 + 6ab^3 + 14ab^2 - 12ab + 3b^2 - 2ba^3 - 6ba^2 - 4b - 4\\
&= ab(ab - 12) + 2ab(3b^2 - a^2) + 2ab(7b - 3a) + 4a^2 + 3b^2 - 4b - 4 > 0.
\end{aligned}$$

With the above calculations, it implies that $y_3 \in (2a + 2b + 3, 2a + 2b + 6)$ and $y_4 \in (2a + b + 4, 2a + 2b + 3)$. Thus, the proof is complete. □

Proceeding similar to Theorem 5, we have the following observation which gives the distance Laplacian eigenvalues of $B(a,a)$.

Corollary 3. *The D^L spectrum of $B(a,a)$ is*

$$\left\{0, 3 + 4a, 6 + 4a, (6a + 6)^{2a-2}, \frac{1}{2}\left(12a + 11 \pm \sqrt{16a^2 + 24a + 1}\right)\right\}.$$

The next result gives the D^L energy of $B(a,b)$.

Theorem 6. *Let $\rho_1(B(a,b)) \geq \rho_2(B(a,b)) \geq \cdots \geq \rho_{n-1}(B(a,b)) \geq \rho_n(B(a,b)) = 0$ be the D^L eigenvalues of $B(a,b)$ with $b \neq a+3$. Then the following hold.*

(i) *For $a < b \leq a+5+\frac{10}{a-1}$, the D^L energy of $B(a,b)$ is*

$$DLE(B(a,b)) = 2\left(\rho_1(B(a,b)) - 2a - 2b - 4 + \frac{16(ab-1)}{a+b+3}\right).$$

(ii) *For $a+5+\frac{10}{a-1} < b < \frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$, the D^L energy of $B(a,b)$ is*

$$DLE(B(a,b)) = 2\left(\rho_1(B(a,b)) + 2ab - 4b - 6 - \frac{4a(ab-1)}{a+b+3}\right),$$

where $\gamma = 20a^2 + (36-12\varepsilon)a + \varepsilon^2 - 14\varepsilon + 33$ and $0 \leq \varepsilon < 7$ is given by $y_2 = a+3b+\varepsilon$.

(iii) *For $b \geq \frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$, the D^L energy of $B(a,b)$ is*

$$DLE(B(a,b)) = 2\left(\rho_1(B(a,b)) + \rho_{a+1}(B(a,b)) + 2ab - 2a - 6b - 10 - \frac{4(a+1)(ab-1)}{a+b+3}\right).$$

(iv) *For $a = b \geq 5$, the D^L energy of $B(a,a)$ is*

$$DLE(B(a,b)) = 20a - 1 + \sqrt{16a^2+24a+1} + \frac{40}{2a+3}.$$

Proof. **(i)-(iii)** For $5 \leq a < b$, by Theorem 5, the D^L eigenvalues of $B(a,b)$ are the eigenvalue $2a+4b+6$ with multiplicity $a-1$, the eigenvalue $4a+2b+6$ with multiplicity $b-1$, the simple eigenvalue 0 and the eigenvalues $y_1 \geq y_2 \geq y_3 \geq y_4$, where $y_1 \in (4a+4b+6, 4a+4b+8)$, $y_2 \in (a+3b, a+3b+7)$, $y_3 \in (2a+2b+3, 2a+2b+6)$ and $y_4 \in (2a+b+4, 2a+2b+3)$. Since, $y_2 = a+3b+\varepsilon$, where $0 \leq \varepsilon < 7$, it follows that for $b \geq 3a+6-\varepsilon$, we have $4a+4a+6 < \rho_1(B(a,b)) = y_1 < 4a+4b+8, \rho_i(B(a,b)) = 2a+4b+6,\ i = 2,3,\dots,a, 3b+a \leq \rho_{a+1}(B(a,b)) = y_2 < 3b+a+7, \rho_i(B(a,b)) = 2b+4a+6,\ i = a+2, a+2, \dots, a+b, 2a+2b+3 < \rho_{a+b+1}(B(a,b)) = y_3 < 2a+2b+6, 2a+b+3 < \rho_{a+b+2}(B(a,b)) = y_4 < 2a+2b+3$ and $\rho_{a+b+3}(B(a,b)) = 0$. Clearly, the average transmission of $B(a,b)$ is

$$\frac{2W(B(a,b))}{n} = \frac{2(a+b)^2 + 4ab + 10(a+b) + 8}{a+b+3}.$$

Let $\sigma = \sigma(B(a,b))$ be the number of D^L eigenvalues of $B(a,b)$ such that $\rho_\sigma \frac{2W(B(a,b))}{n}$. To obtain the DLE of $B(a,b)$, we need to calculate the value of σ. The D^L spectral radius of $B(a,b)$ always satisfies $\rho_1(B(a,b)) \geq \frac{2W}{n}$. Besides for $2a+4b+6 \geq \frac{2W(B(a,b))}{n}$ gives the inequality $b^2+4b+5 \geq a(b-1)$, which holds for $a \leq \frac{b^2+4b+5}{b-1}$. Clearly,

$\frac{b^2+4b+5}{b-1} > b > a$ and it follows that $2a+4b+6$ is greater or equal to $\frac{2W(B(a,b))}{n}$. Next, $\rho_{a+1}(B(a,b)) = y_2 = a+3b+\varepsilon \geq \frac{2W(B(a,b))}{n}$ gives that

$$(a+b+3)\varepsilon - 4ab + b^2 - a^2 - b - 7a - 8 \geq 0. \tag{2.8}$$

Clearly, the inequality (2.8) holds for $b \geq \frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$, where $\gamma = 20a^2 + (36-12\varepsilon)a + \varepsilon^2 - 14\varepsilon + 33$. This gives that $\rho_{a+1}(B(a,b)) \geq \frac{2W(B(a,b))}{n}$, for $b \geq \frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$ and $\rho_{a+1}(B(a,b)) < \frac{2W(B(a,b))}{n}$ for $b < \frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$. Also, $2b+4a+6 \geq \frac{2W(B(a,b))}{n}$ gives that $b \leq \frac{a^2+4a+5}{a-1} = a+5+\frac{10}{a-1}$. From this it follows that $2b+4a+6 \geq \frac{2W(B(a,b))}{n}$, for $b \leq \frac{a^2+4a+5}{a-1} = a+5+\frac{10}{a-1}$ and $2b+4a+6 < \frac{2W(B(a,b))}{n}$ for $b > \frac{a^2+4a+5}{a-1} = a+5+\frac{10}{a-1}$. Thus, it follows that for $b \geq \frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$, we have $\sigma = a+1$ and for $3a+b-\varepsilon \leq b < \frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$, we have $\sigma = a$. On the other hand for $a+6-\varepsilon \leq b < 3a+6-\varepsilon$, we have $4a+4a+6 < \rho_1(B(a,b)) = y_1 < 4a+4b+8, \rho_i(B(a,b)) = 2a+4b+6,\ i = 2,3,\ldots,a, \rho_i(B(a,b)) = 2b+4a+6,\ i = a+1, a+2,\ldots,a+b-1, 3b+a \leq \rho_{a+1}(B(a,b)) = y_2 < 3b+a+7, 2a+2b+3 < \rho_{a+b+1}(B(a,b)) = y_3 < 2a+2b+6, 2a+b+3 < \rho_{a+b+2}(B(a,b)) = y_4 < 2a+2b+3, \rho_{a+b+3}(B(a,b)) = 0$. Since $2b+4a+6 \geq \frac{2W(B(a,b))}{n}$, for $b \leq \frac{a^2+4a+5}{a-1} = a+5+\frac{10}{a-1}$, it follows that for $a+5+\frac{10}{a-1} < b < 3a+6-\varepsilon$, we have $\sigma = a$. For $b \leq \frac{a^2+4a+5}{a-1} = a+5+\frac{10}{a-1}$, we either have $\rho_{a+1}(B(a,b)) = y_2$ or y_3. In each case we see that $\rho_{a+1}(B(a,b)) < \frac{2W(B(a,b))}{n}$ giving that $\sigma = a+b-1$. From this discussion, it follows that $\sigma = a$, for $a+5+\frac{10}{a-1} < b < 3a+6-\varepsilon$ or $3a+6-\varepsilon \leq b < \frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$; $\sigma = a+1$, for $b \geq \frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$ and $\sigma = a+b-1$, for $a < b \leq a+5+\frac{10}{a-1}$. Therefore, for $a < b \leq a+5+\frac{10}{a-1}$, the D^L energy of $B(a,b)$ is given by

$$\begin{aligned}
DLE(B(a,b)) &= 2\left(\sum_{i=1}^{a+b-1} \rho_i(B(a,b)) - \frac{2(a+b-1)W(B(a,b))}{n}\right)\\
&= 2\Big(\rho_1(B(a,b)) + (a-1)(2a+4b+6) + (b-1)(2b+4a+6)\\
&\qquad - \frac{2(a+b+1)W(B(a,b))}{n}\Big)\\
&= 2\left(\rho_1(B(a,b)) - 2a - 2b - 4 + \frac{16(ab-1)}{a+b+3}\right).
\end{aligned}$$

For $a+5+\frac{10}{a-1} < b < \frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$, the D^L energy of $B(a,b)$ is given by

$$\begin{aligned}
DLE(B(a,b)) &= 2\left(\sum_{i=1}^{a} \rho_i(B(a,b)) - \frac{2aW(B(a,b))}{n}\right)\\
&= 2\left(\rho_1(B(a,b)) + (a-1)(2a+4b+6) - \frac{2aW(B(a,b))}{a+b+3}\right)\\
&= 2\left(\rho_1(B(a,b)) + 2ab - 4b - 6 - \frac{4a(ab-1)}{a+b+3}\right).
\end{aligned}$$

For $b \geq \frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$, the D^L energy of $B(a,b)$ is given by

$$\begin{aligned}
DLE(B(a,b)) &= 2\left(\sum_{i=1}^{a+1}\rho_i(B(a,b)) - \frac{2(a+1)W(B(a,b))}{n}\right) \\
&= 2\left(\rho_1(B(a,b)) + \rho_{a+1}(B(a,b)) + (a-1)(2a+4b+6) - \frac{2(a+1)W(B(a,b))}{a+b+3}\right) \\
&= 2\left(\rho_1(B(a,b)) + \rho_{a+1}(B(a,b)) + 2ab - 2a - 6b - 10 - \frac{4(a+1)(ab-1)}{a+b+3}\right).
\end{aligned}$$

(iv). From the Corollary 3, we have $\rho_1(B(a,a)) = \frac{1}{2}(12a+11+\sqrt{16a^2+24a+1})$, $\rho_i(B(a,a)) = 6a+6,\ i = 2,3,\ldots,2a-1$, $\rho_{2a}(B(a,a)) = 4a+6$, $\rho_{2a+1}(B(a,a)) = \frac{1}{2}(12a+11-\sqrt{16a^2+24a+1})$, $\rho_{2a+2}(B(a,a)) = 4a+3$ and $\rho_{2a+3}(B(a,a)) = 0$. Also, the average of the D^L eigenvalues is

$$\frac{2W(B(a,a))}{n} = \frac{1}{n}\sum_{i=1}^{n}\rho_i(B(a,a)) = \frac{12a^2+20a+8}{2a+3} = 6a+1+\frac{5}{2a+3}.$$

Since the D^L spectral radius $\frac{1}{2}\left(12a+11+\sqrt{16a^2+24a+1}\right)$ is always greater or equal to $\frac{2W(B(a,a))}{n}$ and it is clear that $6a+6$ is also greater or equal to $\frac{2W(B(a,a))}{n}$. Further, it is easy to verify that $4a+6 < \frac{2W(B(a,a))}{n}$ giving that $\sigma = 2a-1$. Therefore, the D^L energy, is

$$\begin{aligned}
DLE(B(a,a)) &= \left(\sum_{i=1}^{2a-1}\rho_i(B(a,a)) - \frac{2(2a-1)W(B(a,a))}{n}\right) \\
&= 20a - 1 + \sqrt{16a^2+24a+1} + \frac{40}{2a+3}.
\end{aligned}$$

This completes the proof. □

If $\rho_1 = \rho$ is the D^L spectral radius of $B(a,b)$, then we have following result.

Proposition 4. *For $2 \leq a \leq \lfloor \frac{n-3}{2} \rfloor$, $\rho(B(a,b)) > \rho(B(a-1,b+1))$.*

Proof. From the proof of Theorem 5, $\rho = \rho(B(a,b))$ is the largest zero of

$$\begin{aligned}
f(x) =& x^4 - x^3(10a+10b+20) - x^2\left(-35a^2 - 74ab - 146a - 35b^2 - 146b - 147\right) \\
&- x\left(50a^3 + 174a^2b + 330a^2 + 174ab^2 + 716ab + 696a + 50b^3 + 330b^2 + 696b + 468\right) \\
&+ 24a^4 + 128a^3b + 228a^3 + 208a^2b^2 + 828a^2b + 768a^2 + 128ab^3 + 828ab^2 + 1680ab \\
&+ 1080a + 24b^4 + 228b^3 + 768b^2 + 1080b + 540.
\end{aligned}$$

Replacing $b = n-a-3$, we get $f_a(x) = x^4 - (10n-10)x^3 + (49n + 35n^2 - 4a^2 - 12a - 64n + 24)x^2 - (24an^2 + 50n^3 - 88an - 120n^2 + 16a^2 + 48a + 66n)x + 32an^3 -$

$32a^2n^2+24n^4+48na^2-144an^2-60n^3+144an+12n^2+36n$. It can be easily verified that

$$\begin{aligned}f_a(x)-f_{a-1}(x)&=(4n-8a-8)x^2+(48an-24n^2+64n-32a-32)x\\&\quad+32n^3-64an^2+96an-112n^2+96n\\&=4(n-2a-2)\Big(x^2-6(n-\frac{2}{3})x+8n(n-\frac{3}{2})\Big)\\&=4(n-2a-2)g(x),\end{aligned}$$

where $g(x)=x^2-6(n-\frac{2}{3})x+8n(n-\frac{3}{2})$. For $2\leq i\leq\left\lfloor\frac{n-3}{2}\right\rfloor$, let $\rho_i=\rho(B(i,n-i-3))$. Since, by Theorem 5, we have $4n-6<\rho_i<4n-4$. Therefore, $f_a(\rho_{a-1})=f_a(\rho_{a-1})-f_{a-1}(\rho_{a-1})=4(n-2a-2)g(\rho_{a-1})$. It is easy to see that $g(\rho_{a-1})=\rho_{a-1}^2-6(n-\frac{2}{3})\rho_{a-1}+8n(n-\frac{3}{2})$ is an increasing function of ρ_{a-1} for all $\rho_{a-1}\geq 3n-2$. Since $\rho_{a-1}>4n-6$, it follows that $g(\rho_{a-1})$ is always an increasing function of ρ_{a-1}. So, we have $g(\rho_{a-1})\leq g(4n-4)=-4n<0$. This gives that $f_a(\rho_{a-1})<0$, from which, together with the fact that $f_a(4n-4)>0, f_a(4n-6)<0$ and $\rho_{a-1}>4n-6$, it follows that $\rho_a>\rho_{a-1}$, for $2\leq a\leq\left\lfloor\frac{n-3}{2}\right\rfloor$. This completes the proof. $\square$

For the family of trees $B(a-t,b+t)$, with $t=0,1,\ldots,a-5$. The following result shows that DLE of $B(a-t,b+t)$ is strictly decreasing function of t for some values of a and a strictly increasing function of t for some values of a.

Theorem 7. *For the family $B(a-t,b+t)$ of double brooms with $a+b=n-3$ and $b>a\geq 5$, where $t=0,1,\ldots,a-5$. The following holds.*

1. *If $5\leq a<b\leq a+5+\frac{10}{a-1}$, then $DLE(B(a-t,b+t))$ is a decreasing function of t.*
2. *If $a+5+\frac{10}{a-1}<b<\frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$, then $DLE(B(a-t,b+t))$ is a decreasing function of t, provided that $2n>6a-3l-3k+1+\sqrt{\psi(a,l,k)}$ and an increasing function of t, provided that $2n<6a-3l-3k+\sqrt{\psi(a,l,k)-12a+6k+6l-1}$, where $\psi(a,l,k)=12a^2-12a(k+l+3)+k^2+l^2+10kl+18k+18l-7$.*
3. *If $b\geq\frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$, then $DLE(B(a-t,b+t))$ is a decreasing function of t, for all $a\geq 8$.*

Proof. By Theorem 6, DLE of $B(a,b)$, with $5\leq a<b\leq a+5+\frac{10}{a-1}$, is

$$DLE(B(a,b))=2\Big(\rho_1(B(a,b))-2a-2b-4+\frac{16(ab-1)}{n}\Big),\tag{2.9}$$

where $\rho_1(B(a,b))$ is the D^L spectral radius of $B(a,b)$. As $b=n-a-3$, so the equivalent form of (2.9) is

$$DLE(B(a,b))=2\Big(\rho_1(B(a,b))-2n+2+16a-\frac{16a(a+3)}{n}-\frac{16}{n}\Big).\tag{2.10}$$

Let $B_1 = B(a-l, b+l)$ and $B_2 = B(a-k, b+k)$, with $0 \leq l < k$. We will show that $DLE(B_1) > DLE(B_2)$. Assigning a by $a-l$ in (2.10), we have

$$DLE(B_1) = 2\Big(\rho_1(B_1) - 2n + 2 + 16(a-l) - \frac{16(a-l)(a-l+3)}{n} - \frac{16}{n}\Big). \tag{2.11}$$

Similarly, for $a = a-k$ in (2.10), we have

$$DLE(B_2) = 2\Big(\rho_1(B_2) - 2n + 2 + 16(a-k) - \frac{16(a-k)(a-k+3)}{n} - \frac{16}{n}\Big). \tag{2.12}$$

So, from (2.11) and (2.12), we obtain

$$DLE(B_1) - DLE(B_2) = 2\Big(\rho_1(B_1) - \rho_1(B_2) + 16(k-l) - \frac{16}{n}\Big[(k-l)(k+l-2a-3)\Big]\Big).$$

Since $a \geq 5$, so by Proposition 4 it follows that $\rho_1(B_1) > \rho_1(B_2)$. Besides, $16(k-l) - \frac{16}{n}\Big[(k-l)(k+l-2a-3)\Big] > 0$ implies that $n > 2a+3-(l+k)$, which is holds true as $n > 2a+2$. Thus, $DLE(B_1) - DLE(B_2) > 0$. That completes first part.

2. If $a + 5 + \frac{10}{a-1} < b < \frac{1}{2}(4a + 1 - \varepsilon + \sqrt{\gamma})$, then by Theorem 6, the DLE of $B(a,b)$ is given by

$$DLE(B(a,b)) = 2\Big(\rho_1(B(a,b)) + 2ab - 4b - 6 - \frac{4a(ab-1)}{n}\Big). \tag{2.13}$$

Using $b = n - a - 3$ (2.13), we obtain

$$DLE(B(a,b)) = 2\Big(\rho_1(B(a,b)) + 2an - 6a^2 - 2a - 4n + 6 + \frac{4a}{n}\Big(a^2 + 3a + 1\Big)\Big). \tag{2.14}$$

Let $B_1 = B(a-l, b+l)$ and $B_2 = B(a-k, b+k)$, with $0 \leq l < k$ be the trees defined above. We have

$$\begin{aligned} DLE(B_1) - DLE(B_2) &= 2\Big(\rho_1(B_1) - \rho_1(B_2) + 2(k-l)(n-1) - 6(k-l)(2a-k-l) \\ &+ \frac{4(k-l)}{n}\Big[3a^2 - 3ak - 3al + 6a + k^2 + kl - 3k + l^2 - 3l + 1\Big]\Big). \end{aligned} \tag{2.15}$$

Since $a \geq 5$, it follows by Proposition 4 that $\rho_1(B_1) > \rho_1(B_2)$. Also, $2(k-l)(n-1) - 6(k-l)(2a-k-l) + \frac{4(k-l)}{n}[3a^2 - 3ak - 3al + 6a + k^2 + kl - 3k + l^2 - 3l + 1] > 0$ giving that $2n > 6a - 3l - 3k + 1 + \sqrt{12a^2 - 12a(k+l+3) + k^2 + l^2 + 10kl + 18k + 18l - 7}$. This shows that $DLE(B_1) > DLE(B_2)$, provided that $2n > 6a - 3l - 3k + 1 +$

$\sqrt{\psi(a,l,k)}$. Again using the fact that $4n-6<\rho_1(B_1),\rho_1(B_2)<4n-4$, we get $\rho_1(B_1)-\rho_1(B_2)<2$. From this together with (2.15), we have $DLE(B_1)<DLE(B_2)$, provided $2+2(k-l)(n-1)-6(k-l)(2a-k-l)+\frac{4(k-l)}{n}\Big[3a^2-3ak-3al+6a+k^2+kl-3k+l^2-3l+1\Big]<0$. This last inequality gives $n^2-(6a-3k-3l+1-\frac{1}{k-1})n+2(3a^2-3ak-3al+6a+k^2+kl-3k+l^2-3l+1)<0$. Since $k-l\geq 1$, therefore this inequality will hold provided that the inequality $n^2-(6a-3k-3l)n+2(3a^2-3ak-3al+6a+k^2+kl-3k+l^2-3l+1)<0$ holds. From this inequality we get $2n<6a-3l-3k+\sqrt{12a^2-12a(k+l+4)+k^2+l^2+10kl+24k+24l-8}$. Thus, it follows that $DLE(B_1)<DLE(B_2)$, provided that $2n<6a-3l-3k+\sqrt{\psi(a,l,k)-12a+6k+6l-1}$.

3. If $b\geq\frac{1}{2}(4a+1-\varepsilon+\sqrt{\gamma})$, then by Theorem 6, DLE of $B(a,b)$ is

$$\begin{aligned}DLE(B(a,b))=2\Big(&\rho_1(B(a,b))+\rho_{a+1}(B(a,b))+2ab-2a-6b-10\\&-\frac{4(a+1)(ab-1)}{n}\Big).\end{aligned}\tag{2.16}$$

Using $b=n-a-3$ in (2.16), we get

$$\begin{aligned}DLE(B(a,b))=2\Big(&\rho_1(B(a,b))+\rho_{a+1}(B(a,b))+2an-6a^2-6a-6n+8\\&+\frac{4(a+1)}{n}\Big(a^2+3a+1\Big).\end{aligned}\tag{2.17}$$

Let $B_1=B(-l,b+l)$ and $B_2=B(a-k,b+k)$, with $0\leq l<k$ be the trees defined above. We have from (2.17) that,

$$\begin{aligned}DLE(B_1)-DLE(B_2)=2\Big(&\rho_1(B_1)-\rho_1(B_2)+\rho_{a+1}(B_1)-\rho_{a+1}(B_2)+(2n-6)(k-l)\\&-6(k-l)(2a-k-l)+\frac{4(k-l)}{n}\Big[3a^2-3ak-3al+8a\\&+k^2+kl-4k+l^2-4l+4\Big]\Big).\end{aligned}$$

Since $a\geq 5$, it follows by Proposition 4 that $\rho_1(B_1)>\rho_1(B_2)$. Also, $3n-2a-9<\rho_{a+1}(B_1)<3n-2a-2$ and $3n-2a-7<\rho_{a+1}(B_2)<3n-2a$, it follows that $\rho_{a+1}(B_1)-\rho_{a+1}(B_2)>-9$. Therefore, with this observation it follows that $DLE(B_1)-DLE(B_2)>0$, provided that $-9+(k-l)(2n-6)-6(k-l)(2a-k-l)+\frac{4(k-l)}{n}\Big[3a^2-3ak-3al+8a+k^2+kl-4k+l^2-4l+4\Big]>0$. This inequality further gives that $2n^2-(12a-6k-6l+6+\frac{9}{k-l})n+4(3a^2-3ak-3al+8a+k^2+kl-4k+l^2-4l+4)>0$. Since $k-l\geq 1$, therefore this last inequality will follow provided that $2n^2-(12a-6k-6l+15)n+4(3a^2-3ak-3al+8a+k^2+kl-4k+l^2-4l+4)>0$, which is so if

$$4n>12a-6l-6k+15+\sqrt{48a^2-a(48k+48l-104)+4k^2+4l^2+40kl-52k-42l+97}.\tag{2.18}$$

Consider the function

$$\gamma(l,k) = 12a - 6l - 6k + 15 + \sqrt{48a^2 - a(48k + 48l - 104) + 4k^2 + 4l^2 + 40kl - 52k - 42l + 97}.$$

It can be easily verified that for fixed l, $\gamma(l,k)$ is a decreasing function of k, so for $l = s$, (2.18) holds for $k = s+1, s+2, \dots, a-5$, provided it holds for $k = s+1$. Thus, Inequality (2.18) holds for all $l = s, k$, provided that $4n > \gamma(s, s+1)$, where $\gamma(s, s+1) = 12a - 12s + 9 + \sqrt{48a^2 - (96s - 56)a + 48s^2 - 56s + 49}$. Now, by given $b \geq \frac{1}{2}(4a + 1 - \varepsilon + \sqrt{\gamma})$, giving that $4n \geq 12a + 14 - \varepsilon + 2\sqrt{\gamma}$, where $\gamma = 20a^2 + (36 - 12\varepsilon)a + \varepsilon^2 - 14\varepsilon + 33$ and $0 \leq \varepsilon < 7$. We claim that $\gamma(s, s+1) \leq 12a + 14 - \varepsilon + 2\sqrt{\gamma}$. Since $12a + 14 - \varepsilon + \sqrt{\gamma}$ is a decreasing function of ε, therefore to prove our claim it suffices to show $\gamma(s, s+1) \leq 12a + 8 + 2\sqrt{20a^2 - 36a - 15}$. If $s \neq 0$, then using the fact $12a - 12s + 9 < 12a + 8$ together with $4(20a^2 - 36a - 15) > 48a^2 - (96s - 56)a + 48s^2 - 56s + 49$, for all $a \geq 7$, our claim holds. If $s = 0$, then by direct calculation it can be verified that $12a + 8 + 2\sqrt{20a^2 - 36a - 15} \geq 12a + 9 + \sqrt{48a^2 + 56a + 49}$, for all $a \geq 8$. This completes the proof of our claim. Thus, from this discussion we conclude that the inequality (2.18) holds for all $a \geq 8$ and $0 \leq l < k \leq a - 5$. This in turn gives that, $DLE(B_1) > DLE(B_2)$ for all $a \geq 8$ in this case. $\square$

The next consequence is immediate from Theorems 7 and orders the trees in the family $B(a,b)$ in terms of DLE.

Corollary 4. *For all trees in the family $B(a,b)$ with $b > a \geq 5$ and $a + b = n - 3$, the following holds.*

- **(i)** *If $5 \leq a < b \leq a + 5 + \frac{10}{a-1}$ or $b \geq \frac{1}{2}(4a + 1 - \varepsilon + \sqrt{\gamma}), a \geq 8$ or $a + 5 + \frac{10}{a-1} < b < \frac{1}{2}(4a + 1 - \varepsilon + \sqrt{\gamma})$ and $2n > 6a - 3l - 3k + 1 + \sqrt{\psi(a,l,k)}$, where $\psi(a,l,k)$ is as in Theorem 7, then the tree $B(5, n-8)$ has the minimum DLE and the tree $B(\lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n-3}{2} \rceil)$, if n is odd and the tree $B(\lfloor \frac{n-3}{2} \rfloor - 1, \lceil \frac{n-3}{2} \rceil + 1)$, if n is even has the maximum DLE.*
- **(ii)** *If $a + 5 + \frac{10}{a-1} < b < \frac{1}{2}(4a + 1 - \varepsilon + \sqrt{\gamma})$ and $2n < 6a - 3l - 3k + \sqrt{\psi(a,l,k) - 12a + 6k + 6l - 1}$, then the tree $B(\lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n-3}{2} \rceil)$, if n is odd and the tree $B(\lfloor \frac{n-3}{2} \rfloor - 1, \lceil \frac{n-3}{2} \rceil + 1)$, if n is even has the minimum DLE and the tree $B(5, n-8)$ has the maximum DLE.*

3. Distance Laplacian spectrum and energy of complement of $B(a,b)$

This section deals with the solution of the Problem 2. Formally, we give an ordering of complement of $B(a,b)$ on the basis of their smallest non-zero distance Laplacian eigenvalue and their distance Laplacian energy.
The complement $\overline{G}$ of a graph G is the graph with vertex set as in G and an edge set consisting of an edge e if and only if e is not an edge in G.

In our next result, we find the D^L eigenvalues of $\overline{B}(a,b)$.

Proposition 5. *The D^L spectrum of $\overline{B}(a,b)$, $a \leq b$, $a+b=n-3$ consists of the eigenvalue $(n+1)$ with multiplicity $n-5$, the simple eigenvalue 0 and the zeros of polynomial* (3.1)

$$\begin{aligned}
f_{a,b}(x) =& x^4 - x^3(5a+5b+18) + x^2(9a^2+9b^2+19ab+67a+67b+120) - x(7a^3 \\
&+ 7b^3 + 23a^2b + 23ab^2 + 80a^2 + 80b^2 + 168ab + 295a + 295b + 352) + 2a^4 \\
&+ 2b^4 + 9a^3b + 9ab^3 + 14a^2b^2 + 31(a^3+b^3) + 101a^2b + 175(a^2+b^2) \\
&+ 365ab + 428(a+b) + 384.
\end{aligned} \tag{3.1}$$

Proof. Let $V(B(a,b)) = \{v_1, v_2, \ldots, v_a, w_1, w_2, w_3, u_1, u_2, \ldots, u_b\}$ be the vertex set of $B(a,b)$, where $v_1, v_2, \ldots, v_a$ are the a pendent vertices at the vertex w_1 and $u_1, u_2, \ldots, u_b$ are the b pendent vertices at the vertex w_3 of the path $P_3 : w_1w_2w_3$. Clearly, $\overline{B}(a,b)$ consists of clique on $a+b+1$ vertices, where v_i's share the same number of vertices with the transmission $a+b+3$ and u_i's share the same neighbourhood with the transmission $a+b+3$. Thus by Lemma 2, $n+1$ is the D^L-eigenvalue of $\overline{B}(a,b)$ with multiplicity at least $n-5$.
Let X be the eigenvector of $\overline{B}(a,b)$ with $x_i = X(v_i)$ for $i = 1,2,3,\ldots,n$. Clearly, every component of X corresponding to v_i's is x_1 and every components of X corresponding to u_i's is x_5. Also, components of X corresponding to the vertices of the degree $b+1$ is x_2, the degree $a+b$ is x_3 and the degree $a+1$ is x_3. By using Equation (2.1), the (ρ, X)-equation of $\overline{B}(a,b)$ are given by

$$\begin{aligned}
\rho x_1 =& (b+4)x_1 - 2x_2 - x_3 - x_4 - bx_5, \\
\rho x_2 =& -2ax_1 + (2a+b+3)x_2 - 2x_3 - x_4 - bx_5, \\
\rho x_3 =& -ax_1 - 2x_2 + (a+b+4)x_3 - 2x_4 - bx_5, \\
\rho x_4 =& -ax_1 - x_2 - 2x_3 + (a+2b+3)x_4 - 2bx_5, \\
\rho x_5 =& -ax_1 - x_2 - x_3 - 2x_4 + (a+4)x_5.
\end{aligned}$$

The coefficient matrix of the right side of above eigenequations is

$$\begin{pmatrix}
b+4 & -2 & -1 & -1 & -b \\
-2a & 2a+b+3 & -2 & -1 & -b \\
-a & -2 & a+b+4 & -2 & -b \\
-a & -1 & -2 & a+2b+3 & -2b \\
-a & -1 & -1 & -2 & a+4
\end{pmatrix},$$

and its characteristic polynomial is given by

$$\begin{aligned}
-x\big(x^4 - &x^3(5a+5b+18) + x^2(9a^2+9b^2+19ab+67a+67b+120) - x\big(7a^3+7b^3+23a^2b+23ab^2 \\
&+ 80a^2+80b^2+168ab+295a+295b+352\big) + 2a^4+2b^4+9a^3b+9ab^3+14a^2b^2+31a^3 \\
&+ 31b^3+101a^2b+175a^2+175b^2+365ab+428a+428b+384\big).
\end{aligned}$$

This proves the result. $\square$

By manual calculation, we have

$$\begin{aligned} f_{a,b}(n) =& n > 0, \\ f_{a,b}(n+1) =& -ab < 0, \\ f_{a,b}(n+2) =& a+b-1 > 0. \end{aligned}$$

If $z_1 \geq z_2 \geq z_3 \geq z_4$ be the zeros of $f_{a,b}(x)$, then by intermediate value theorem, it follows that $z_4 \in (n, n+1)$ and $z_3 \in (n+1, n+2)$.
The next result orders the graphs in the family $\overline{B}(a,b)$ in terms of ρ_{n-1}.

Theorem 8. *For positive integers a, b $(a \leq b)$ and $a+b = n-3$, we have*

$$\rho_{n-1}\big(\overline{B}(a,b)\big) \leq \rho_{n-1}\big(\overline{B}(a-1,b+1)\big).$$

Proof. By Proposition 5, the second smallest D^L-eigenvalue of $\overline{B}(a,b)$ is the smallest root of $f_{a,b}(x) = 0$. It can be seen that,

$$f_{a,b}(x) - f_{a-1,b+1}(x) = (x-n)(x-n-2)(n-2a-2).$$

By Proposition 5, we have $n < \rho_{n-1}\big(\overline{B}(a-1,b+1)\big) < n+1$. Also, $f_{a-1,b+1}(\rho_{n-1}\big(\overline{B}(a-1,b+1)) = 0$, therefore we obtain

$$\begin{aligned} & f_{a,b}(\rho_{n-1}\big(\overline{B}(a-1,b+1)) \\ & = f_{a,b}(\rho_{n-1}\big(\overline{B}(a-1,b+1)) - f_{a-1,b+1}(\rho_{n-1}\big(\overline{B}(a-1,b+1)) \\ & = (\rho_{n-1}\big(\overline{B}(a-1,b+1) - n)(\rho_{n-1}\big(\overline{B}(a-1,b+1) - n - 2)(n-2a-2) < 0, \end{aligned}$$

for all $a \leq b$. This together with the fact that $f_{a,b}(n) > 0$ and $f_{a,b}(n+1) < 0$, we arrive at $\rho_{n-1}(\overline{B}(a,b)) < \rho_{n-1}(\overline{B}(a-1,b+1))$. $\square$

Similar to Theorem 8, we have the following result, which gives the ordering of the graphs belonging to the family $\overline{B}(a,b)$ on the basis of ρ_1.

Theorem 9. *For positive integers a, b $(a \leq b)$ and $a+b = n-3$, we have*

$$\rho_1\big(\overline{B}(a,b)\big) \leq \rho_1\big(\overline{B}(a-1,b+1)\big).$$

The following result gives the DLE of $\overline{B}(a,b)$.

Theorem 10. *For the graph $\overline{B}(a,b)$ with $n-3=a+b,(a\leq b)$ and let $\rho_{n-1}(\overline{B}(a,b))$ be its second smallest D^L eigenvalue. Then*

$$DLE\big(\overline{B}(a,b)\big)=2\left(2n^2-6-\rho_{n-1}\big(\overline{B}(a,b)\big)+\frac{4}{n}\right).$$

Proof. Let $\rho_1\geq\rho_2\geq\cdots\geq\rho_{n-1}>\rho_n=0$, be the D^L eigenvalues of $\overline{B}(a,b)$. By Proposition 5, the D^L-eigenvalues of $\overline{B}(a,b)$ consists of the simple eigenvalue 0, the eigenvalue $n+1$ with multiplicity $n-5$ and the four zeros of $f_{a,b}(x)$ given by (3.1). Besides, by Proposition 5, the eigenvalue ρ_{n-1} is same as the zero z_4 of $f_{a,b}(x)$. Thus, we order the D^L-eigenvalues of $\overline{B}(a,b)$ from the smallest to the largest as: $\rho_n=0,\ \rho_{n-1}\in(n,n+1),\rho_i=n+1$ for $i=4,5,\dots,n-2,\ \rho_3=z_3\in(n+1,n+2)$ and $\rho_1\geq\rho_2\geq z_3$. In order to validate the ordering, we show that $\rho_2\geq n+2$. For $a=1,2,3$, we see by direct calculation that $\rho_2\geq n+2$ holds. So, suppose that $a\geq 4$. The 2×2 principal submatrix of $D^L(\overline{B}(a,b))$ corresponding to the vertex of maximum transmission and the vertex of second maximum transmission is

$$\begin{pmatrix}2a+b+3 & -2\\ -2 & a+2b+3\end{pmatrix},$$

and its eigenvalues are $\frac{1}{2}\left(3a+3b+6\pm\sqrt{(a-b)^2+16}\right)$. By using the interlacing property of Theorem 1, we have

$$\rho_2\geq\frac{1}{2}\left(3a+3b+6-\sqrt{(a-b)^2+16}\right),$$

and to show that $\rho_2\geq n+2$, it is equivalent to show that

$$b(a-2)\geq 2a. \tag{3.2}$$

It is clear that this inequality holds for $b\geq a\geq 4$. Also, the average transmission of $\overline{B}(a,b)$ is

$$\frac{2W\big(\overline{B}(a,b)\big)}{n}=\frac{(n+1)(n-5)+5a+5b+18}{n}=\frac{n(n+1)-2}{n}=n+1-\frac{2}{n}.$$

Clearly, $\frac{2W\left(\overline{B}(a,b)\right)}{n}$ is function of n and is constant for any values of a and b. Let $\sigma=\sigma(\overline{B}(a,b))$ be the greatest positive integer such that $\rho_\sigma\geq\frac{2W\left(\overline{B}(a,b)\right)}{n}$, then $\rho_1\geq\frac{2W\left(\overline{B}(a,b)\right)}{n}$ is always true and by above D^L-eigenvalue ordering its is clear that $\rho_2\geq\rho_3\geq\rho_4=\cdots=\rho_{n-2}=n+1>\frac{2W\left(\overline{B}(a,b)\right)}{n}=n+1-\frac{2}{n}$. It follows that $\sigma=n-2$, so by the definition of distance Laplacian energy, we have

$$\begin{aligned}DLE(\overline{B}(a,b))&=2\left(S_{n-2}(G)-\frac{2(n-2)W(\overline{B}(a,b))}{n}\right)\\&=2\left(\rho_1+\rho_2+\rho_{n-2}+(n-5)(n+1)-(n-2)(n+1-\frac{2}{n})\right)\\&=2\left(\rho_1+\rho_2+\rho_{n-2}+2n^2-5n-9+\frac{4}{n}\right).\end{aligned}$$

From (3.1), we see that $\rho_1+\rho_2+\rho_3+\rho_{n-2}=5a+5b+18-\rho_{n-1}=5n+3-\rho_{n-1}$. Therefore, DLE of $\overline{B}(a,b),\ a\leq b$ is given by

$$DLE(\overline{B}(a,b))=2\left(2n^2-6-\rho_{n-1}(\overline{B}(a,b))+\frac{4}{n}\right).$$

□

The following consequence of Theorem 10 is immediate from Proposition 5.

Corollary 5. *The D^L energy of $\overline{B}(a,b)$, with $a<b$ and $n-3=a+b$ satisfies the following*

$$2\left(2n^2-n-7+\frac{4}{n}\right)<DLE(\overline{B}(a,b))<2\left(2n^2-n-6+\frac{4}{n}\right).$$

By Corollary 5, we observe that DLE of $\overline{B}(a,b)$ lies in an open interval of length 2. Let $\overline{B}(a-t,b+t)$, with $t=1,\ldots,a-1$ be the family of graphs with $a+b=n-3$. Next result shows that DLE of $\overline{B}(a-t,b+t)$ is strictly decreasing function of t.

Theorem 11. *For $t=1,2,\ldots,a-1$, $a+b=n-3$, and $a\leq b$, the D^L energy of the family $\overline{B}(a-t,b+t)$ is a decreasing function of t.*

Proof. By Theorem 10, DLE of $\overline{B}(a,b)$ is given by

$$DLE(\overline{B}(a,b))=2\left(2n^2-6+\frac{4}{n}-\rho_{n-1}(\overline{B}(a,b))\right).$$

where ρ_{n-1} is the second smallest D^L eigenvalue of $\overline{B}(a,b)$. In order to prove the result, it is enough to show that $DLE(G_1)>DLE(G_2)$, where $G_1=\overline{B}(a-l,b+l)$ and $G_2=\overline{B}(a-k,b+k)$, $1\leq l<k$. Now, $DLE(G_1)-DLE(G_2)=2(-\rho_{n-1}(G_1)+\rho_{n-1}(G_2))$, which is positive, since by Proposition 8, $\rho_{n-1}(\overline{B}(a-k,b+k)))>\rho_{n-1}(\overline{B}(a-l,b+l))$. Therefore, $DLE(\overline{B}(a-t,b+t))$ is a decreasing function of t. □

Remark 1. An important observation from Theorem 10 is that only one non-zero distance Laplacian eigenvalue of $\overline{B}(a,b)$ is strictly less than the graph invariant $\frac{2W(\overline{B}(a,b))}{n}$. Thus $\sigma=n-2$ for the family of graphs $\overline{B}(a-t,b+t)$, $t=0,1,2,3,\ldots,a-1$.

4. Conclusion

The article gives the distance Laplacian spectral invariant ordering of the brooms and their complements. Formally, we determine the extremal graphs with respect to spectral parameters like the largest distance Laplacian eigenvalue, the smallest non-zero distance Laplacian eigenvalue and the distance Laplacian energy of brooms of diameter 4 and their complements. For general graphs these properties are very hard and restriction to special graphs is considered to carry a deep rigorous study of these spectral invariants. In future, a similar type of analysis can be interesting for more general families of graphs.

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