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Short Note

A short note on double Roman domination in graphs

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Abstract: In this short note, we report an erroneous result of Mojdeh, Parsian and Masoumi relating the double Roman domination number to the enclaveless number and the differential of a graph.

Keywords: double Roman domination number, trees, differential.

AMS Subject classification: 05C69

1. Introduction

For a graph $G = (V, E)$, let $\gamma(G)$, $\gamma_R(G)$, $\gamma_{dR}(G)$, $\Psi(G)$ and $\partial(G)$ denote the domination number, the Roman domination number, the double Roman domination number, the enclaveless number and the differential of G, respectively.
It has been shown by Mojdeh, Parsian and Masoumi [4] that for every graph G of order n having no isolated vertices,

$$\gamma_{dR}(G) \leq 2n - \Psi(G) - \partial(G) \tag{1.1}$$

It is worth noting that this result, whose invalidity will be shown, is presented in two separate papers by the same authors. The following Gallai theorems have been established in [1] and [2] for the differential of a graph and the enclaveless number, respectively.

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Theorem 1. *[1] If G is a graph of order n, then $\partial(G) = n - \gamma_R(G)$.*

Theorem 2. *[2] For any graph G of order n, then $\Psi(G) = n - \gamma(G)$.*

Note that according to Theorems 1 and 2, the inequality (1.1) becomes $\gamma_{dR}(G) \leq \gamma_R(G) + \gamma(G)$. In the next section, we will provide an infinite family of graphs showing that inequality (1.1) is erroneous.

2. Counterexamples

Recall that a double star $S(r, s)$ with $r \geq s \geq 1$, is a tree with exactly two vertices which are not leaves, one of which is adjacent to r leaves and the other one to s leaves. Let $\mathcal{G}$ be the family of trees T obtained from a double star $S(r, s)$ with $r \geq s \geq 2$, by subdividing twice the central edge and once any other edge of the double star. Figure 1 shows the smallest example of a tree belonging to $\mathcal{G}$. We can easily see that any tree T in $\mathcal{G}$ has order $n = 2(r + s) + 4$, $\gamma(T) = r + s + 1$, $\gamma_R(T) = r + s + 4$ and thus leading to $\Psi(T) = r + s + 3$ and $\partial(T) = r + s$. Now since $\gamma_{dR}(T) = 2(r + s) + 6$, we consequently have $\gamma_{dR}(T) > 2n - \Psi(T) - \partial(T)$.

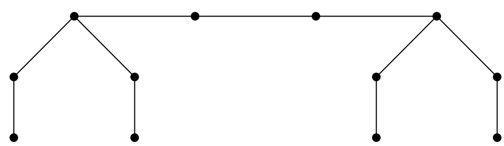

Figure 1. **The tree T in $\mathcal{G}$.**

In the following, we define another class of graphs different from trees for which (1.1) is not also valid. Let $\mathcal{H}$ be the family of graphs G obtained from a star $K_{1,p}$, with $p \geq 3$, by first subdividing once each edge of the star and then adding a new vertex attached to the center vertex and one of its neighbors. Figure 2 shows the smallest example of a graph belonging to $\mathcal{H}$. One can easily see that any graph G in $\mathcal{H}$ has order $n = 2p + 2$, $\gamma(G) = p$, $\gamma_R(G) = p + 2$ and thus leading to $\Psi(G) = p + 2$ and $\partial(G) = p$. Now since $\gamma_{dR}(G) = 2p + 3$, we consequently have $\gamma_{dR}(G) > 2n - \Psi(G) - \partial(G)$.

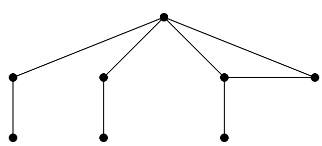

Figure 2. **The graph G in $\mathcal{H}$.**

We conclude by mentioning that inequality (1.1) is used in [3], which therefore calls into question the validity of certain results.

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Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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