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Research Article

On the zero forcing number of complementary prism graphs

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Abstract: The zero forcing number of a graph is the minimum cardinality among all the zero forcing sets of a graph G. The aim of this article is to compute the zero forcing number of complementary prism graphs. Some bounds on the zero forcing number of complementary prism graphs are presented. The remainder of this article discusses the following result. Let G and $\overline{G}$ be connected graphs. Then $Z(G\bar{G}) \leq n-1$ if and only if there exists two vertices $v_i, v_j \in V(G)$ and $i \neq j$ such that, either $N(v_i) \subseteq N(v_j)$ or $N[v_i] \subseteq N[v_j]$ in G.

Keywords: zero forcing set, zero forcing number, complementary prism graph.

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1. Introduction

All the graphs considered here are simple, finite, and undirected graphs, where $G = (V(G), E(G))$ represents a graph. If $x, y \in V(G)$ and $xy \in E(G)$, then x and y are said to be adjacent to each other and x is the neighbour of y (vice-versa). The order (number of vertices) of the graph G is denoted by $|V(G)|$ and the size (number of edges) of the graph is denoted by $|E(G)|$ respectively. All other definition that is not defined here is referred from [14]. Zero forcing is a type of dynamic coloring process, where given a set of initially black colored vertices, a black vertex with a single white neighbour (or uncolored vertex) changes the color of that white neighbour to become black. The color change rule states that a black colored vertex can force a white

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neighbour black if and only if, it is the only white neighbour of that black colored vertex. A zero forcing set is a set of initially black colored vertices, which causes the entire graph to eventually become black after iteratively applying the color change rule. The zero forcing number of a graph G, is the minimum cardinality of the zero forcing set in the graph G, and the zero forcing number is denoted by $Z(G)$.

The concept of zero forcing was first introduced in [9] where zero forcing number is used to bound maximum nullity or minimum rank of a graph G. Independently it was introduced in [2] and in [1] used to study the quantum controlability of the system. Since its introduction zero forcing number has been a topic of interest and many research have been carried out in this regard [4, 7, 8, 11–13]. Zero forcing number of graph and its complement is studied in [6], it is used to study the logic circuit as well in [3].

Complementary prism graph was first introduced in [10]. Complementary prism graph denoted by $G\bar{G}$, is a graph obtained by taking a copy of G, its complement $\bar{G}$ and edges connecting each vertex of G to its unique copy in $\bar{G}$. Throughout this paper we denote the vertex set of the graph G in $G\bar{G}$ part as $v_1, v_2, \ldots, v_n$ and the vertex set of $\bar{G}$ in $G\bar{G}$ part as $\bar{v_1}, \bar{v_2}, \ldots, \bar{v_n}$. The degree of each vertex of G in $G\bar{G}$ will be 1+ the degree of the vertex in G. Similarly the degree of each vertex of $\bar{G}$ in $G\bar{G}$ will be 1+ the degree of the vertex in $\bar{G}$.

Some bounds on the zero forcing number of complementary prism graphs are presented. The remainder of this article discusses the following result. Let G and $\overline{G}$ be connected graphs. Then $Z(G\bar{G}) \leq n-1$ if and only if there exists two vertices $v_i, v_j \in V(G)$ and $i \neq j$ such that, either $N(v_i) \subseteq N(v_j)$ or $N[v_i] \subseteq N[v_j]$ in G.

2. Some bounds of $Z(G\bar{G})$

Complementary prism graph has either G or $\bar{G}$ connected and hence complementary prism itself is connected. For any simple graph G the diameter of $G\bar{G}$ is at most 3. For any graph G the vertex set, $V(G) = \{v_1, v_2, \ldots, v_n\}$ then its complement $\bar{G}$ will have vertex set, $V(\bar{G}) = \{\bar{v_1}, \bar{v_2}, \ldots \bar{v_n}\}$. Throughout this paper the vertices v_1 and $\bar{v_1}$, v_2 and $\bar{v_2}, \ldots, v_n$ and $\bar{v_n}$ are respectively referred to as corresponding vertices in the graph $G\bar{G}$.

Theorem 1. *For any simple graph G, $\frac{n+1}{4} < Z(G\overline{G}) \leq n$, where n is the order of the graph G.*

Proof. We have the following result from [4]

$$Z(G) > \frac{m}{n}, \tag{1}$$

where m is the number of edges $|E(G)|$ and n is the number of vertices $|V(G)|$. In complementary prism graph the total number of edges is given by the formula [5]

$$E(G\overline{G}) = \frac{n(n+1)}{2}. \tag{2}$$

Total number of vertices in $G\overline{G}$ is $2n$ then according to (1) we have

$$Z(G\overline{G}) > \frac{\frac{n(n+1)}{2}}{2n} = \frac{n+1}{4}.$$

The upper bound can be obtained by considering all n vertices of G or $\overline{G}$ in the forcing set, we can force the entire graph black by forcing its corresponding vertices black. □

Lemma 1. *Let G be a graph of order n. If $\Delta(G)$ or $\Delta(\overline{G}) = n-1$, then $Z(G\overline{G}) \leq n-1$. Where $\Delta(G)$ denotes maximum degree of the graph.*

Proof. Without loss of generality let us assume that $\Delta(G) = n-1$, then there exist an isolated vertex in $\overline{G}$, say $\overline{v}$. Also assume that, v is the vertex corresponding to $\overline{v}$ in $G\overline{G}$. Now let us consider all $n-1$ vertices other than $\overline{v}$ among $\overline{v_1}, \overline{v_2}, \ldots, \overline{v}, \ldots, \overline{v_n}$ as the elements of the zero forcing set S. All these $n-1$ vertices can force their corresponding white neighbours in $G\overline{G}$ black, since there exist exactly one white neighbour in the open neighbourhood of these black vertices. Again if we consider the vertex v, then we can see that any black vertex in $G\overline{G}$ has no white neighbour or v as its only white neighbour. Hence any one of these vertices can force v black. Finally v can force its corresponding neighbour $\overline{v}$ black. Thereby forcing the entire graph $G\overline{G}$ black. Hence $Z(G\overline{G}) \leq n-1$. □

Lemma 2. *Let G be a graph of order n. If $\Delta(G)$ or $\Delta(\overline{G}) = n-2$, then $Z(G\overline{G}) \leq n-1$.*

Proof. Without loss of generality let u be the vertex in G whose degree is $\Delta = n-2$ and let $\overline{u}$ be the vertex corresponding to u in $G\overline{G}$. Since the degree of u is $n-2$, there is one vertex v in G that is not adjacent to the vertex u and $\overline{v}$ be the vertex corresponding to the vertex v in $G\overline{G}$. Clearly in the complement $\overline{G}$ the vertex $\overline{u}$ is adjacent only to the vertex $\overline{v}$. Let S be the set of all initially black colored vertices required to force the entire graph $G\overline{G}$ black.

Case 1. Assume that the vertex v is also adjacent to all the neighbors of the vertex u except $\overline{u}$. Then by taking these $n-2$ vertices $(N(u) \setminus \overline{u})$ and vertex v as the initially colored black vertex in S, we can force the entire graph $G\overline{G}$ black. We can observe that now the vertex v can force its corresponding vertex $\overline{v}$ black, then the vertex $\overline{v}$ can force the vertex $\overline{u}$ black further the vertex $\overline{u}$ can force the vertex u black. Again in $G\overline{G}$ graph the vertices of the subgraph G is black. Now these black vertices can force the entire graph $G\overline{G}$ black. Therefore, we obtain a derived coloring by using $\mid S \mid= n-1$ black vertices. Hence in this case $Z(G\overline{G}) \leq n-1$.

Case 2. Assume that the vertex v is not adjacent to any of the neighbors of the vertex u. Then G is disconnected implying that $\overline{G}$ contains a vertex $\overline{v}$ whose degree is $n-1$, and by Lemma 1 we get the required result.

Case 3. Assume that the vertex v is adjacent to some $1 \leq k < n-2$ neighbours of vertex u except $\overline{u}$. Then by taking u and $N(u) \setminus \overline{u}$ in S, u can force $\overline{u}$ black. Which further can force $\overline{v}$. Since v is adjacent to k neighbours of u except $\overline{u}$ in $G\overline{G}$, $N(u) \setminus \{N(v), \overline{u}\}$ can force their corresponding neighbours black (in other words $N(\overline{v}) \setminus \overline{u}$ black). Now we observe that $N(\overline{v})$ has exactly one white neighbour v. Hence $\overline{v}$ can force v black. In $G\overline{G}$ all the vertices of $N(u)$ are black so these vertices have at-most one white neighbour or no white neighbours in $G\overline{G}$.Therefore all the black vertices of $N(u)$ in $G\overline{G}$ can force the remaining white vertices in $G\overline{G}$ black. □

2.1. Zero forcing number of the complementary prism graph of few basic graph classes

In this section we start with the complete graph class K_n and its complimentary prism $K_n\overline{K_n}$

Theorem 2. *Let G be a complete graph K_n of order $n \geq 2$. Then the zero forcing number $Z(K_n\overline{K_n}) = n-1$.*

Proof. From Lemma 2.2 it is clear that $Z(K_n\overline{K_n}) \leq n-1$. It is enough to prove that $Z(K_n\overline{K_n}) \geq n-1$. Clearly, in the graph $K_n\overline{K_n}$ there are two different degrees, a set of vertices having degree 1 and a set of vertices having degree n. Let all the vertices of degree 1 be $\overline{v_i}$ $1 \leq i \leq n$ and let all the vertices of degree n be v_i $1 \leq i \leq n$. On the contrary, let us assume that $n-2$ initial black vertices are sufficient to force the entire graph black. Let H be the zero forcing set consisting of $n-2$ initial black vertices. Suppose we choose $n-k$, where $2 \leq k \leq n$ from the set $\overline{V}$ as the vertices in H, then these vertices can force their corresponding neighbours of $K_n\overline{K_n}$ (that is in set V) black. Now remaining $k-2$ black vertices in H are chosen from V. Clearly, after the first forcing process, there are $n-k+k-2 = n-2$ black vertices in V. Since, all the vertices in V form an induced complete graph, any black vertices in the set V have at least 2 white neighbours. Therefore the forcing process halts. Hence $n-2$ vertices are not sufficient to force the entire graph $K_n\overline{K_n}$ black. □

Again we consider the basic graph class C_n and its complimentary prism graph $C_n\overline{C_n}$ and obtain its zero forcing number.

Theorem 3. *Let G be a cycle graph C_n where n is the order of the cycle C_n. Then $Z(C_n\overline{C_n}) = n$ for $n > 4$.*

Proof. We know that $Z(G\overline{G}) \leq n$ from Theorem 1. To prove equality we need to prove $Z(G\overline{G}) \geq n$. On contrary let us assume that $n-1$ black vertices are enough to force the entire graph black. In $C_n\overline{C_n}$ we have n vertices of degree 3 and n vertices of degree $n-2$. Since $n > 4$, $n-2 \geq 3$. Let S be the set containing vertices which are initially colored black.

Case 1. When $n = 5$ the graph $C_5\overline{C_5}$ is the Petersen graph. It is found that zero forcing number of Petersen graph is 5 [9]. That is $Z(C_5\overline{C_5}) = 5$.

Case 2. Suppose that $n \geq 6$ and $|S| = n-1$.

Subcase 2.1. Assume that all vertices in S have degree 3. Clearly these $n-1$ black vertices force $n-3$ corresponding white vertices black. This process then halts since any black vertices will have either no white vertex or at least 2 white vertices in its open neighbourhood, a contradiction.

Subcase 2.2. When all the vertices in S have degree $n-2$. Then these $n-1$ vertices can force only 2 vertices of C_n part of $C_n\overline{C_n}$.(As the remaining one white vertex of degree $n-2$ is adjacent to all other $n-3$ black vertices.) The process stops as any black vertices will have either no white neighbours or two white neighbours, a contradiction.

Subcase 2.3. When the vertices in S have both the degrees, 3 and $n-2$.

Subcase 2.3.1. Let u_1 be the first vertex in S, consider when $d(u_1) = 3$. In order for u_1 to force, we need two of its three white neighbours in S. Say u_2, u_n (or $\overline{u_1}$,make no difference in forcing pattern), then u_1 will force $\overline{u_1}$ black. Clearly u_2, u_n will have 2 white neighbours and $\overline{u_1}$ has $n-3$ white neighbours. By taking $n-4$ neighbours of $\overline{u_1}$ in S, $\overline{u_1}$ can force its only white neighbour black. After which any black vertices will have either no white neighbour or at least 2 white neighbour. Therefore $|S| = n-1$ and these $n-1$ vertices will not form a zero forcing set, a contradiction. Therefore, $n \leq Z(C_n\overline{C_n})$.

Instead if we take one of the white neighbours of u_2 or u_n in S, say u_3 or u_{n-1} respectively (similarly taking $\overline{u_2}$ or $\overline{u_{n-1}}$ respectively will make no difference in the forcing pattern), that is either $S = \{u_1, u_2, u_n, u_3\}$ or $S = \{u_1, u_2, u_n, u_{n-1}\}$. Now $|S| = 4$, at this stage u_2 or u_n can force $\overline{u_2}$ or $\overline{u_n}$ respectively. Now $\overline{u_2}$ or $\overline{u_n}$ has $n-3$ white neighbour. Hence it is not possible to consider this, as $n-4$ of its neighbours must be in S making $|S| = n$, a contradiction. We have to consider the neighbour of u_3 or u_{n-1}, but this forcing pattern is similar to that of Case 1.

Subcase 2.3.2. Let $\overline{u_1}$ be the first vertex in S, consider when $d(\overline{u_1}) = n-2$ in S then $n-3$ neighbours of $\overline{u_1}$ must be in S so that it can force the remaining white neighbour. After this, the process stops as any black vertex either has no white neighbour or at least two white neighbour, in order to continue we need to consider neighbours of u_1 else the forcing pattern will be same as Case 2. u_1 has 2 white neighbours so one of them must be in S to continue the process. Further, this process stops once u_1 forces its other white neighbour, but any more addition of the vertices will lead to a contradiction as $|S| = 1 + n - 3 + 1 = n - 1$. □

Remark: If $n = 3$ or 4, then we can easily verify that $Z(C_3\overline{C_3}) = 2$ and $Z(C_4\overline{C_4}) = 3$ as depicted in the Figure 1.

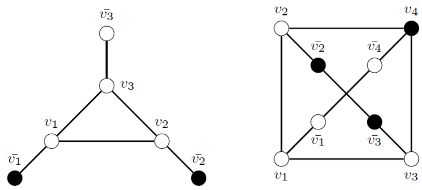

Figure 1. $C_3\overline{C_3}$ **and** $C_4\overline{C_4}$

A wheel graph is the join of cycle C_n with K_1 that is $W_{n+1} = C_n + K_1$. Next we consider the complementary prism of wheel graph $W_{n+1}\overline{W_{n+1}}$.

Theorem 4. *If $G = W_{n+1}$ is a wheel graph of order $n+1$, then the zero forcing number $Z(W_{n+1}\overline{W_{n+1}}) = n$.*

Proof. Consider the set S of black vertices , W be the set containing vertices $\{v_1, v_2, \ldots, v_n, v_{n+1}\}$ with v_{n+1} as the Central vertex of W_{n+1} and $\overline{W}$ be the set containing vertices $\{\overline{v_1}, \overline{v_2} \ldots \overline{v_n}, \overline{v_{n+1}}\}$. From Lemma 1

$$Z(W_{n+1}\overline{W_{n+1}}) \leq n. \tag{3}$$

In Figure 2, the black vertices represents the zero forcing sets of complementary wheel graph when $3 \leq n \leq 4$. For $n > 4$, let us assume that $n-1$ initial black vertices are sufficient to force the entire graph black.

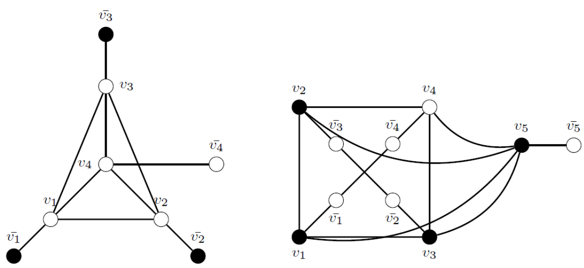

Figure 2. $W_{3+1}\overline{W_{3+1}}$ **and** $W_{4+1}\overline{W_{4+1}}$

Case 1. If all the $n-1$ initially colored black vertices in S are taken from W.
Then it is obvious that the central vertex v_{n+1} must be S, else the forcing process doesn't progress. This clearly implies that there will be exactly two white vertices in W when $n-1$ vertices are taken from W, meaning only $n-4$ vertices can be forced

black in $\overline{W}$ as other two black vertices and the central vertex v_{n+1} in W will have at least two white neighbour, a contradiction. Hence we need at least n initial black vertices to force the entire graph black.

Case 2. If all the $n-1$ initially colored black vertices in S are taken from $\overline{W}$.

Subcase 2.1. If all the $n-1$ black vertices in S are taken from $\{\overline{W} \setminus \{\overline{v_{n+1}}\}$. Then there will be 2 white vertex in $\overline{W}$ (one white vertex from $\{\overline{W} \setminus \{\overline{v_{n+1}}\}$ and other is $\overline{v_{n+1}}$). This white vertex in $\{\overline{W} \setminus \{\overline{v_{n+1}}\}$ is adjacent to $n-3$ other vertices in $\overline{W}$. Hence only 2 vertices among $n-1$ black vertices can force their corresponding neighbour in $W_{n+1}\overline{W_{n+1}}$ black, and the process stops, a contradiction.

Subcase 2.2. If $\overline{v_{n+1}}$ and $n-2$ black vertices are taken from $\{\overline{W} \setminus \{\overline{v_{n+1}}\}$ are taken in S. Then the degree one vertex $\overline{v_{n+1}}$ can forces its corresponding white neighbour v_{n+1} black in $W_{n+1}\overline{W_{n+1}}$ black. On the other hand there are exactly 2 white vertices in $\{\overline{W} \setminus \{\overline{v_{n+1}}\}$, so maximum two of these black vertices can force its corresponding white neighbour respectively black in $W \setminus v_{n+1}$. Hence any black vertex in the graph $W_{n+1}\overline{W_{n+1}}$ has either no white neighbour or 2 or more white neighbours. Hence the process halts, a contradiction.

Case 3. If these $n-1$ initial black vertices are taken from both W and $\overline{W}$.

Subcase 3.1. Suppose that first black vertex in S is taken from $\overline{W}$.
i) If the first chosen vertex is $\overline{v_{n+1}}$ (pendent vertex), then it will force the $n+1$ degree vertex v_{n+1} black (Central vertex of W). The remaining structure resembles the graph of $C_n\overline{C_n}$. According to Theorem 3 $Z(C_n\overline{C_n}) = n$, in other words we need extra n black vertices, a contradiction.
ii) If the first chosen vertex is from $\{\overline{W} \setminus \{\overline{v_{n+1}}\}$, then this vertex say $\overline{v_1}$ has a total of $n-2$ neighbours in $W_{n+1}\overline{W_{n+1}}$. So at least $n-3$ of them need to be initially colored black so that $\overline{v_1}$ can force the remaining one white neighbour. Now any black vertex in $W_{n+1}\overline{W_{n+1}}$ will have no or at least 2 white neighbour and there are 3 white vertices in $\overline{W}$. Taking any one of these vertex can force at most one white vertex black and then the forcing stops. On the other hand the black vertex in W that is v_1 has three white neighbours, a contradiction.

Subcase 3.2. Suppose that first black vertex is taken from W.
Clearly we cannot take the $n+1$ degree vertex v_{n+1} (central vertex). Any other vertex in W has four neighbour, let v_1 be such a vertex. Let v_1 be the first black vertex considered in S, now lets consider three of its neighbour initially black (say v_2, v_n, v_{n+1}). v_1 will now force its other white neighbour (say $\overline{v_1}$) black. To continue the process further we cannot consider the v_{n+1} (as it has n white neighbours), nor can we consider $\overline{v_1}$ as $\overline{v_1}$ has $n-3$ white neighbours. So only possibility is when we consider other two vertices of W. This is now reduced to case 1, a contradiction.
From the above cases, we can conclude that

$$Z(W_{n+1}\overline{W_{n+1}}) \geq n. \tag{4}$$

Theorem follows from equation (3) and (4). □

Theorem 5. *Let G be a complete bipartite graph $G = k_{p,q}$ of order $n = p + q$. Then the zero forcing number $Z(K_{p,q}\overline{K_{p,q}}) = n - 1$.*

Proof. Consider the complete bipartite graph $K_{p,q}$ and let $V = \{v_1, v_2, \ldots, v_q\}$ be the vertices in the first partite set and let $U = \{u_1, u_2, \ldots, u_p\}$ be the vertices in second partite set. Corresponding to $K_{p,q}$ its complement $\overline{K_{p,q}}$ consists of $\overline{V} = \{\overline{v_1}, \overline{v_2}, \ldots, \overline{v_q}\}$ corresponding to the first partite set V and $\overline{U} = \{\overline{u_1}, \overline{u_2}, \ldots, \overline{u_p}\}$ corresponding to the second partite set u.
Let S be the set of all vertices which are required to be colored initially black in order to force the entire graph $K_{p,q}\overline{K_{p,q}}$ black. Let all the q vertices of V be in S, remaining p vertices of U and $\overline{U}$ forms an induced graph $K_p\overline{K_p}$. By Theorem 2, we know that $Z(K_p\overline{K_p}) = p - 1$. Now by taking these $p - 1$ vertices in S, $K_p\overline{K_p}$ can be forced black the forcing process then continues to force the remaining q vertices of $\overline{V}$ black. Therefore at most $|S|$ number of vertices are required to force the entire graph black. That is, $|S| = q + p - 1 = n - 1$ implying that $Z(K_{p,q}\overline{K_{p,q}}) \leq n - 1$. In order to choose the initial black vertices of $K_{p,q}\overline{K_{p,q}}$ we have 4 options, that is one among q vertices of V or one among q vertices of $\overline{V}$ or one among p vertices of U or one among p vertices of $\overline{U}$.

Case 1. If one among the q vertices of V is considered black, clearly, each of these vertices has degree $p+1$. Therefore p of the neighbours are to be considered initially black in order for the forcing process to continue. Now any black vertices will have either no white neighbour or $q - 1$ white neighbour. At least $q - 2$ vertices must be considered black in order for the forcing process to continue. That is a minimum of $1 + p + q - 2 = p + q - 1 = n - 1$ black vertices are required to force the graph.

Case 2. If one among the p vertices of U is considered black, with a similar argument as in Case 1, we can show that at least $n - 1$ vertices have to be considered initially black in order for the forcing process to occur.

Case 3. If one among the q vertices of $\overline{V}$ is considered black, clearly, each of these vertices has degree q. Therefore at least $q - 1$ vertices have to be considered initially black in order for the forcing process to occur. After the forcing process, all the $q - 1$ vertices of $\overline{V}$ will have exactly one white neighbour. Therefore they can force the corresponding neighbours to be black. At this stage, each of the black vertices will have either no white neighbour or p white neighbours. Hence in order for the forcing process to continue, $p - 1$ of the white vertices of U has to be considered black. That is a minimum of $1 + q - 1 + p - 1 = p + q - 1 = n - 1$ black vertices are required to force the graph.

Case 4. If one among the p vertices of $\overline{U}$ is considered black, with a similar argument as in Case 3, we can show that at least $n - 1$ vertices have to be considered initially black in order for the forcing process to occur.
From all four cases, it is clear that $Z(K_{p,q}\overline{K_{p,q}}) \geq n - 1$. Therefore $Z(K_{p,q}\overline{K_{p,q}}) = n - 1$. □

2.2. More bounds and Inequality

In this section, we study the zero forcing number of a few complementary prisms of cut edge graph, disconnected graph etc.,

Theorem 6. *If G and $\overline{G}$ are connected and either G or $\overline{G}$ has a cut edge, then $Z(G\overline{G}) \leq n-1$.*

Proof. Without loss of generality let G be the graph with cut edge. Let uv be the cut edge such that u belongs to the first component, v belongs to the second component and the edge uv forms the bridge between these two components. The total number of vertices $u_1, u_2, \ldots, u_{p-1}, u$ in the first component of graph G is p, and the total number of vertices $v_1, v_2, \ldots, v_{q-1}, v$ in the second component of the graph G is q such that $p+q=n$ where n is the order of the graph G. In the complement part of G $\overline{u}, \overline{v}$ are the vertices corresponding to the vertices u, v respectively in G.
Consider the graph $G\overline{G}$. Let S be the zero forcing set. Initially we take all the p vertices $u_1, u_2, \ldots, u_{p-1}, u$ of $G\overline{G}$ in S, then $p-1$ vertices among them (i.e except u) force their corresponding vertices in $G\overline{G}$ black. In order to force $\overline{u}$ black we need to take v in S. Now $\overline{u}$ has exactly $q-1$ white neighbours. By taking $q-2$ of its neighbour (Say $v_2, v_3, \ldots, v_{q-1}$) black, $\overline{u}$ can force remaining white neighbour, making all its neighbour black. Also all the vertices adjacent to $\overline{v}$ are black and one of the vertices among $\overline{u_1}, \overline{u_2}, \ldots, \overline{u_{p-1}}$ will force $\overline{v}$ black. $v_1, v_2, \ldots, v_{q-1}$ can be now forced black by their corresponding neighbour in $G\overline{G}$. since all the vertices of $G\overline{G}$ are forced black, the set of vertices in $S = \{u_1, u_2, \ldots, u_{p-1}, u, v, v_2, v_3, \ldots, v_{q-1}\}$ are enough to force the entire graph black. Hence $Z(G\overline{G}) \leq |S| = n-1$

□

Theorem 7. *Let G and $\overline{G}$ be connected graphs. Then $Z(G\bar{G}) \leq n-1$ if and only if there exists two vertices $v_i, v_j \in V(G)$ and $i \neq j$ such that, either $N(v_i) \subseteq N(v_j)$ or $N[v_i] \subseteq N[v_j]$ in G.*

Proof. Let G be the graph such that $v_i, v_j \in V(G)$. Let V be a set containing vertices $\{v_1, v_2 \ldots, v_i, \ldots v_j, \ldots v_n\}$ and $\overline{V}$ be the set containing $\{\overline{v_1}, \overline{v_2}, \ldots, \overline{v_i}, \ldots \overline{v_j}, \ldots, \overline{v_n}\}$. Let S be the zero forcing set of $G\overline{G}$.

Case 1. $N(v_i) \subseteq N(v_j)$ and $i \neq j$.
Take all the $n-1$ vertices of V except v_i in S. Then all the vertices in V except $N[v_i]$ can force their corresponding vertices in $G\overline{G}$ black. Now, $\overline{v_j}$ can force $\overline{v_i}$ black, as $N(v_i) \subseteq N(v_j)$ implies the vertices which are not adjacent to $\overline{v_i}$ is not a subset of $N(\overline{v_j})$ and $\overline{v_i} \in N(\overline{v_j})$. Hence all the neighbouring vertices of $\overline{v_j}$ except the vertex $\overline{v_i}$ are black, according to the color change rule $\overline{v_j}$ forces $\overline{v_i}$ black. This vertex $\overline{v_i}$ in turn forces its corresponding neighbour v_i black. Since all the vertices in V are black they can force the remaining white vertices of $\overline{V}$ black (as they are the corresponding vertices). Hence entire graph can be forced black with the initially colored black vertex set $S = \{v_1, v_2, \ldots, v_j, \ldots, v_n\}$ and $|S| = n-1$. Therefore, $Z(G\overline{G}) \leq n-1$.

Case 2. $N[v_i] \subseteq N[v_j]$ and $i \neq j$.
Then it is evident that $N(\overline{v_j}) \subseteq N(\overline{v_i})$. Using the same argument of Case 1 by taking v_i as $\overline{v_j}$ and v_j as $\overline{v_i}$ we can show that $Z(G\overline{G}) \leq n - 1$.

Conversely, assume that if $Z(G\overline{G}) \leq n - 1$, then there exist no $v_i, v_j \in V(G)$ or $\overline{v_i}, \overline{v_j} \in V(\overline{G})$ such that either $N(v_i) \subseteq N(v_j) \Rightarrow N[\overline{v_j}] \subseteq N[\overline{v_i}]$ or $N[v_i] \subseteq N[v_j] \Rightarrow N(\overline{v_j}) \subseteq N(\overline{v_i})$. Without loss of generality let us start the process of zero forcing by taking a vertex in $G\overline{G}$ say v_i black. In order to carry out the forcing process we need to take all of its neighbours in V black, then v_i can force its corresponding neighbour $\overline{v_i}$ black. Further in order to continue the forcing process we need to either consider the white neighbours of the black vertices in V or white neighbours of the black vertices of $\overline{V}$ in $G\overline{G}$.

Case 3. consider the white neighbours of any black vertex which is distinct form v_i in V.
According to our assumption there will be at least two white neighbour for every black neighbours of v_i in V. Hence at each stage any vertex which is black in V will have at least two white neighbour or no white neighbours in $G\overline{G}$ until all the n vertices are taken in S, a contradiction.

Case 4. consider the white neighbours of $\overline{v_i}$ in $\overline{V}$.
According to our assumption, for any two vertices in V, say v_i and v_j there exist at least one vertex in $N(v_i)$ which doesn't belong to $N(v_j)$ in G. It can be observed that any two vertices of $\overline{V}$ also satisfies the above condition since vertex set of $\overline{V}$ in $G\overline{G}$ forms an induced graph $\overline{G}$. Now if we consider neighbours of $\overline{v_i}$ to be initially black, (that is in S). The total number of black vertices in S will be $n - 1$. Since the graph cannot be disconnected and $\overline{G}$ also satisfies the assumption that for any two vertices in $\overline{G}$ there exist at least one neighbour which is different from the neighbour of the other. Hence in order to continue the forcing process we need to consider one more vertex in the initial forcing set S, in other words we need at least n vertices to force the entire graph, a contradiction. □

From the above Theorem it can be observed that, for any two vertices in graph G. If$N(v_i) \subsetneq N(v_j)$ or $N[v_i] \subsetneq N[v_j]$ in G, then $Z(G\overline{G}) = n$.
It is known that either G and $\overline{G}$ will be connected for any graph classes.

Annexure

Code to find the Complementary prism of a given graph

This Python program outputs the complementary prism graph of a given graph.

```
import networkx as nx
import matplotlib.pyplot as plt


def complementary_prism(G):
    n=G.number_of_nodes()
    h= range(n,n+n)
    G.add_nodes_from(h)
     for i in range(n):
```

```
        G.add_edge(i,i+n)
        for j in range(n):
            if i != j:
                if G.has_edge(i,j)==False:
                    G.add_edge(i+n,j+n)

complementary_prism(G)
# G is a graph for which complementary prism graph is to be obtained.
```

Few example of graph and its complementarty prism
Cycle

```
H = nx.cycle_graph(5)

complementary_prism(H)
nx.draw_networkx(
        H,
        pos=nx.circular_layout(H))
plt.title("Complementary prism graph")
plt.show()
```

Wheel

```
H = nx.wheel_graph(5)
complementary_prism(H)
plt.subplot(1,3,3)
nx.draw_networkx(
        H,
        pos=nx.circular_layout(H))
plt.title("Complementary prism graph")
plt.show()
```

3. Conclusion

In section 2, we found some bounds of the complementary prism $Z(G\overline{G})$. In section 2.1, we discussed the zero forcing number of the Complementary prism graph of a cycle, complete graph, wheel graph and bipartite graph. In section 2.2, we found more bounds based on the graph's number of vertices $|V(G)|$. The following problems are open

Conjecture 1. If G is a connected graph, then $Z(G\overline{G}) \geq \max\{\delta(G), \delta(\overline{G})\} + 1$.

Problem 1. Determine the zero forcing number of the complementary prism of other graph classes such as path, generalised Petersen graphs etc.

The python code to obtain the complementary prism graph of a graph is given. Finding the code to obtain the zero forcing number of complementary prism graph is an open problem.

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