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Research Article

Intuitionistic fuzzy Sombor indices: A novel approach for improving the performance of vaccination centers

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Abstract: Intuitionistic fuzzy graphs are generalizations of fuzzy graphs, in which each vertex is assigned an ordered pair whose first coordinate gives the membership value and the second coordinate gives the non-membership value. There are many theoretical parameters to study different types of graphs and fuzzy graphs, topological indices are one of them. Sombor indices are important in explaining the topology of a graph, and were found to possess useful applicative properties. The two versions of the Sombor indices (SO_3 and SO_4)are converted into an intuitionistic fuzzy framework, and then formulas for different kinds of graphs are calculated. Our study also involves setting up a network of vaccination centers during a pandemic and applying intuitionistic fuzzy-based topological indices to assess their performance. With the help of this application, we highlight the practical implication and benefits of employing intuitionistic fuzzy-based techniques in vaccination centers. Through a comparative analysis, we evaluate which index is more efficient.

Keywords: intuitionistic fuzzy graph, vaccination centers based on path, cycle, complete graph, Sombor indices, membership value, non-membership value.

AMS Subject classification: 05C09, 05C35, 05C38, 05C72, 05C76, 05C90, 05C92, 03B52, 03E72

1. Introduction

Fuzzy topological indices are a mathematical method to explain the topological structure of molecules in both chemistry and biology. They are based on fuzzy set theory,

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which allows for the assignment of elements to sets with various degrees of membership. Because they provide a quantitative evaluation of the structural traits of molecules that are linked to their physicochemical traits, fuzzy topological indices are important. These indices are employed in a variety of disciplines, such as drug design, bioinformatics, and material science, to predict molecular behaviour and develop new molecules with desired properties. One of the main advantages of fuzzy topological indices is their capacity to express the inherent uncertainty and haziness of molecular structures. They can clarify how atom-bond boundaries in molecules can fluctuate in strength and orientation and aren't always clearly defined [2, 9].

Fuzzy topological indices can also be used with other mathematical and computational techniques, such as machine learning and artificial intelligence, to improve their predictive power and build more accurate models of molecular behaviour.

A graph is a pair $G(V, E)$, where V represents the finite set of vertices and E stands for the finite set of edges. The number of vertices in a graph G is called the *order* of the graph, and the number of edges is called the *size* of the graph. The set of all vertices of a graph is called *vertex set*, and the set of all edges is called *edge set*. The *degree* of the vertex β_i, denoted by $d(\beta_i)$, is the number of edges incident on β_i. The edge connecting the vertices β_i and β_j will be denoted by $\beta_i\beta_j$.

In the current mathematical and chemical literature, several dozens of vertex-degree-based graph invariants are being studied, whose general forms is

$$TI(G) = \sum_{\beta_i\beta_j \in E(G)} F\big(d(\beta_i), d(\beta_j)\big) \tag{1.1}$$

where F is pertinently chosen function, with properties $F(x, y) = F(y, x)$, and $F(x, y) \geq 0$. Such graph invariants are usually referred to as topological indices, and serve as a kind of molecular descriptor in quantitative structure-activity relationship (QSAR) research in chemical informatics. They are founded on the graph theory premise and are used to explain the topology of molecules. A recently introduced group of such topological indices are the Sombor indices [38, 39].

The Sombor indices are conceived by noting that the vertex-degree-pair $d(\beta_i)$ and $d(\beta_j)$ in Eq. (1.1) can be interpreted as a point A in a coordinate system with coordinates $\big(d(\beta_i), d(\beta_j)\big)$. The original Sombor index [38] is then the sum over all edges of the distance of this point from the origin, i.e.,

$$SO(G) = \sum_{\beta_i\beta_j \in E(G)} \sqrt{d(\beta_i)^2 + d(\beta_j)^2}.$$

The mathematical properties [50] and chemical applications [43, 58] of $SO(G)$ and other Sombor indices have been studied in due detail. They have been shown to be excellent descriptors for a number of molecular properties, including boiling point, refractive index, molecular weight, and surface area. They can be used as input characteristics in QSAR models to predict the biological activity of a group of medications or other desirable qualities. Sombor indices have the advantage of being able to measure the complexity and symmetry of molecules. Furthermore, they are computationally efficient and frequently easy to understand. Sombor indices can therefore be used instead of or in addition to other descriptors that are often used in QSAR modeling; see for instance [1, 18, 55, 61].

Noting that in addition to the point A with coordinates $\big(d(\beta_i), d(\beta_j)\big)$, there is another point B with coordinates $\big(d(\beta_j), d(\beta_i)\big)$, and a third point O, the origin with coordinates $(0,0)$, one may consider the triangle ABO, and the topological indices related to it [39]. Thus, the Sombor index SO_3 corresponds to the perimeter of the cycle circumscribed to the triangle ABC, whereas SO_4 correspond to the are of this circle. It has been shown [39] that

$$SO_3 = \sum_{\beta_i\beta_j \in E(G)} \sqrt{2}\, \frac{d(\beta_i)^2 + d(\beta_j)^2}{d(\beta_i) + d(\beta_j)}\, \pi \quad \text{and} \quad SO_4 = \frac{1}{2} \sum_{\beta_i\beta_j \in E(G)} \left(\frac{d(\beta_i)^2 + d(\beta_j)^2}{d(\beta_i) + d(\beta_j)}\right)^2 \pi\,.$$

In this paper, for the first time, we introduce and examine the fuzzy Sombor indices, in particular the fuzzy variants of SO_3 and SO_4.

If G is the collection of objects represented by β. Then the fuzzy set $\mathcal{F}$ in G is the set of ordered pairs: $\mathcal{F} = \{(b, \zeta_{\mathcal{F}}(\beta)) | \beta \in G\}$, where $\zeta_{\mathcal{F}}(\beta)$ is called the membership degree of β, s.t., $0 \le \zeta_{\mathcal{F}}(\beta) \le 1$, given by [67].

Zadeh gave the first paper on fuzzy set theory in 1965; see [65]. In this study, Zadeh used the membership function, which assigns each member a membership degree that ranges from zero to one, to characterize this collection. His goal was to broaden the traditional concept of set theory. According to Zadeh and Goguen, human opinion, judgement, and appraisal minimize fuzziness. A fuzzy set is a method of handling issues where the presence of random variables rather than class membership is the primary cause of imperfection. It is a mathematical method that enables the scientific investigation of hazy conceptual problems.

In the 1960s and 1970s, this hypothesis only very slowly gained favor. However, its use in the regulation of technological processes by fuzzy rule-based systems in the late 1970s piqued researchers' interest in this area. People were inspired to conduct study in this area as a result of its successful applications in washing machines, video cameras, tube trains, etc. Up to 2000, there were more than 30,000 articles on fuzzy set theory. It expanded throughout the last ten years into both a formal theory and an application-focused theory.

Fuzzy set theory has applications and importance in many fields like in production and operations management [64]. It has a wide range of application in flood control system [31]. Fuzzy set theory can be used to make computers think like human beings [66].

As expansions of traditional fuzzy set theory and graphs, intuitionistic fuzzy sets (IFS) and intuitionistic fuzzy graphs (IGGs) were developed. For further information, see

[10, 12, 13, 19, 48]. In the 1980s, Atanassov introduced them to deal with circumstances where ambiguity and reluctance coexist and where a more flexible model of the degree of membership, non-membership, and hesitancy is required.

The significance of intuitionistic fuzzy sets and intuitionistic fuzzy graphs resides in their superiority over conventional fuzzy sets and graphs in handling and representing incomplete and uncertain information. They offer a way to communicate the idea of reluctance or indeterminacy, which is frequently present in real-world problem-solving situations.
Here are some key aspects of their importance:

Modeling uncertainty: By incorporating the idea of reluctance, intuitionistic fuzzy sets enable a more nuanced portrayal of doubt. IFS assigns membership and non-membership values to elements rather than only assigning membership values, allowing for a more accurate modeling of uncertain or imperfectly understood information.

Making decisions in the face of uncertainty: Intuitionistic fuzzy sets offer a paradigm for making choices in the face of ambiguity. Making decisions based on both membership and non-membership values allows decision-makers to assess the level of reluctance associated with various options and choose more wisely.

Handling imprecise and ambiguous information: When modeling imprecise or ambiguous information, intuitionistic fuzzy sets and graphs are helpful. They make it possible to describe information in a more flexible manner by capturing not only the degree of membership but also the degree of uncertainty or ambiguity related to the information.

The development of formal methods and techniques for the analysis and application of intuitionistic fuzzy sets and intuitionistic fuzzy graphs is made possible by the intuitionistic fuzzy objects' strong mathematical foundations. For usage with intuitionistic fuzzy sets and graphs, numerous operations, aggregation techniques, and algorithms have been created.
Applications include decision-making, pattern recognition, data mining, image processing, expert systems, and control systems. Intuitive fuzzy sets and graphs are used in these and other areas. It has been demonstrated that they enhance the precision and efficiency of decision-making processes and offer a potent toolkit for handling ambiguous and incomplete information in these domains.

In conclusion, by including the idea of hesitation or indeterminacy, intuitionistic fuzzy sets and intuitionistic fuzzy graphs enhance traditional fuzzy set theory and graphs. They offer a more sophisticated framework for dealing with uncertainty and insufficient data, allowing for more precise modeling and decision-making across a variety of applications.

Physical networks like road networks, electrical circuits, structures of organic compound, computer networks and social networking are represented by nodes and links. In such networks nodes represent objects and links represent the relationship between these objects. For example in road network nodes may represent cities and links represent the paths joining the cities. In a formal way such setups are designed by patterns called graphs consisting of vertices(nodes) and edges(links) [20–22].

1.1. Literature Review

While keeping the Wiener index for intuitionistic fuzzy graphs (IFG) parallel to the connectedness index, the paper [29] is to study it in some depth. For an IFG, some of the additional degree and distance-based topological indices have been established. Railway crimes in India are analyzed in [45] via F-index for fuzzy graphs. They compared the railway crime with three other topological indices. They concluded that first Zagreb index and F-index provide similar results while both indices provide realistic results then crisp graph indices. New connectivity index and Wiener index for bipolar fuzzy incidence graphs are discussed in [37]. The MWI and distance matrix algorithms are displayed. The use of these indexes is finally explained. Randić index and fuzzy harmonic fuzzy topological indices are discussed in [4]. They also developed several upper bounds for chosen fuzzy topological indices. Moreover, some graph operations like lexicographic product, cross product and Cartesian product are discussed and they provide some lowers bounds on the chosen indices and graph operations. Authors of [47], introduced some topological indices in the pure fuzzy environment. Indices are defined in fuzzy graphs are Randić, harmonic, Zagreb, Gutman, Schultz, hyper-Wiener, and modified Wiener indices.

Connectivity index of directed rough fuzzy graphs [7], Wiener index of a directed rough fuzzy graph [6], domination in rough fuzzy digraphs [3], a novel method for computing connection metrics in rough fuzzy network models [15], connectivity index of a fuzzy graph [26], Neighborhood-based graphs for topological visualization of rough sets and a heart application [30], Wiener index, hyper-Wiener index, fuzzy hyper-Wiener index defined in [44]. Networks involving illegal immigration are applied to the Wiener index of a fuzzy graph [27]. Wiener index of bipolar fuzzy incidence graphs are available in [36], Some distance-based indices of bipartite molecular graphs exhibit Nordhaus–Gaddum type inequalities in [34], using the graph's Wiener absolute index and bipolar fuzzy information, the order of the journeys is determined [57]. Authors define the operations of join, union, composition and Cartesian product on fuzzy subgraphs [54], they establish necessary and sufficient criteria for the same technique to also create an arbitrary fuzzy subgraph of G from fuzzy subgraphs. For some interesting work on the topic of Sombor indices, one can see [6, 28, 40, 49, 59, 60].

Some basics of fuzzy graphs such as, fuzzy line graphs [53], automorphisms of fuzzy graphs [24], arcs in a fuzzy graph [52], bipolar fuzzy graphs [8], Complete fuzzy graphs [17], directed rough fuzzy graph [5, 16], fuzzy tolerance relation, fuzzy rough digraphs, homomorphism [11], author discussed some characteristics of fuzzy graphs and define the terms eccentricity and center [23], geodesics in fuzzy graphs [25].

1.2. Methodology and Importance of Sombor Indices

Fuzzy Sombor indices are a development of the traditional Sombor indices, that have potentials of being utilized in chemoinformatics and quantitative structure-activity relationship (QSAR) research. Fuzzy Sombor indices are aimed at providing a more flexible and advanced technique of calculating molecular descriptors. In traditional Sombor indices, the separations between vertices in a molecule are expressed as binary values (0 if the vertices are not connected by a bond and 1 if they are). In contrast, the vertices of fuzzy Sombor indices are separated from one another by continuous values between 0 and 1, which show how linked or similar they are to one another.
In order to create fuzzy Sombor indices, the adjacency matrix of a molecule is transformed into a fuzzy adjacency matrix by replacing each binary value with a continuous value that symbolizes the degree of similarity between the vertices. This fuzzy adjacency matrix is then used to calculate the above defined SO_3 and SO_4 indices. They can also be used to describe the differences in molecular structure among a collection of chemicals, which aids in QSAR modeling predictions of biological activity or other desirable properties for a group of substances.

1.3. Prelimanaries

Kaufmann first described fuzzy graphs in 1973. Rosenfeld then presented his work on fuzzy graphs using fuzzy relations in 1975, [35]. He provided definitions for fuzzy subgraphs, trees, bridges, cut nodes, and forests, described each with the use of examples, and covered a variety of features associated with these ideas. Then, a large number of academics contributed significantly to the evaluation of fuzzy graphs and advanced numerous ideas. Researchers defined numerous operations on fuzzy graphs, such as union, join, and Cartesian product, among others, in [33].
Some definitions and notations related to fuzzy graph are presented in this section.

Definition 1. [62] A fuzzy graph $\mathcal{G} : \{V, \zeta, \psi\}$ is a pair of functions, where $\zeta : V \to [0, 1]$. ζ is a fuzzy subset of a non empty set V and ψ is a symmetric fuzzy relation on ζ s.t., $\psi : E \to [0, 1]$, $\forall\ E \subset V \times V$ and $\forall\ \beta_i, \beta_j \in V, \psi(\beta_i, \beta_j) \leq min\,(\zeta(\beta_i), \zeta(\beta_j))\,.$

Definition 2. [56] An intuitionistic fuzzy graph G is defined as $G = (V, E)$, where $V = \{\beta_1, \beta_2, \beta_3, \ldots, \beta_n\}$ such that $\psi_1 : V \to [0, 1]$ and $\zeta_1 : V \to [0, 1]$ represent the membership degree and non-membership degree of element $\beta_i \in V$ respectively, and $0 \leq \psi_1(\beta_i) + \zeta_1(\beta_i) \leq 1$.
$E \subset V \times V$ where $\psi_2 : V \times V \to [0, 1]$ and $\zeta_2 : E \times E \to [0, 1]$ are such that:

$$\psi_2(\beta_i, \beta_j) \leq \min\{\psi_1(\beta_i), \psi_1(\beta_j)\},$$
$$\zeta_2(\beta_i, \beta_j) \leq \max\{\zeta_1(\beta_i), \zeta_1(\beta_j)\}.$$

Also,$0 \leq \psi_2(\beta_i, \beta_j) + \psi_2(\beta_i, \beta_j) \leq 1$. for all $\beta_i, \beta_j \in V$, and $1 \leq i, j \leq \eta$.

Definition 3. [32] Let $G = (V, E)$, be an intuitionistic fuzzy graph. Then the degree of a vertex depends upon both, the membership degree and non-membership degree i.e., $d(\beta_i) = (d_{\psi_1}(\beta_i), d_{\zeta_1}(\beta_i))$, where $d_{\psi_1}(\beta_i) = \sum_{\beta_i \neq \beta_j} \psi_2(\beta_i, \beta_j)$ and $d_{\zeta_1}(\beta_i) = \sum_{\beta_i \neq \beta_j} \zeta_2(\beta_i, \beta_j)$.

Definition 4. The size of an intuitionistic fuzzy graph G also depends upon the membership size $\mathcal{Z}_\psi(G)$ and non-membership size $\mathcal{Z}_\zeta(G)$, i.e., $\mathcal{Z}(G) = (\mathcal{Z}_\psi(G), \mathcal{Z}_\zeta(G))$, and is defined as:

$$\mathcal{Z}_\psi(G) = \sum_{\beta_i \neq \beta_j} \psi_2(\beta_i, \beta_j),$$
$$\mathcal{Z}_\zeta(G) = \sum_{\beta_i \neq \beta_j} \zeta_2(\beta_i, \beta_j).$$

Definition 5. The order of an intuitionistic fuzzy graph G, denoted by $O(G)$, is defined with respect to membership degree as well as w.r.t. non-membership degree as $O(\beta_i) = (O_\psi(G), O_\zeta(G))$,

$$O_\psi(G) = \sum_{\beta_i \in V} \psi_1(\beta_i),$$
$$O_\zeta(G) = \sum_{\beta_i \in V} \zeta_1(\beta_i).$$

Definition 6. The intuitionistic fuzzy version of the Sombor index SO_3 is symbolized by $\mathcal{SO}_3(G)$ and for intuitionistic fuzzy graph G is defined as:

$$\mathcal{SO}_3(G) = \sum_{\beta_i\beta_j \in E(G)} \left[\frac{(\psi_1(\beta_i), \zeta_1(\beta_i))\, d^2(\beta_i) + (\psi_1(\beta_j), \zeta_1(\beta_j))\, d^2(\beta_j)}{(\psi_1(\beta_i), \zeta_1(\beta_i))\, d(\beta_i) + (\psi_1(\beta_j), \zeta_1(\beta_j))\, d(\beta_j)} \right] \sqrt{2}\pi. \tag{1.2}$$

Definition 7. The intuitionistic fuzzy version of the Sombor index SO_4" is symbolized by $\mathcal{SO}_4(G)$ and for intuitionistic fuzzy graph G is defined as:

$$\mathcal{SO}_4(G) = \frac{1}{2} \sum_{\beta_i\beta_j \in E(G)} \left[\frac{(\psi_1(\beta_i), \zeta_1(\beta_i))\, d^2(\beta_i) + (\psi_1(\beta_j), \zeta_1(\beta_j))\, d^2(\beta_j)}{(\psi_1(\beta_i), \zeta_1(\beta_i))\, d(\beta_i) + (\psi_1(\beta_j), \zeta_1(\beta_j))\, d(\beta_j)} \right]^2 \pi. \tag{1.3}$$

2. Main results on intuitionistic fuzzy versions of Sombor indices SO_3 and SO_4

In this section, we calculate the intuitionistic fuzzy versions of the Sombor indices SO_3 and SO_4 for IF path graph, IF cycle graph, IF complete graph, IF bipartite graph, IF wheel graph, and IF star graph.

2.1. Intuitionistic Fuzzy path graph and its results

A path graph $G = \{V, \psi, \zeta\}$ is a sequence of distinct vertices where $\beta_{i-1} \neq \beta_i$ satisfying one of the following conditions.

- $\psi_2(\beta_{i-1}, \beta_i) > 0$ and $\zeta_2(\beta_{i-1}, \beta_i) = 0$,
- $\psi_2(\beta_{i-1}, \beta_i) = 0$ and $\zeta_2(\beta_{i-1}, \beta_i) > 0$,
- $\psi_2(\beta_{i-1}, \beta_i) > 0$ and $\zeta_2(\beta_{i-1}, \beta_i) > 0$.

For some i such that $1 \leq i \leq \eta$.

Note: In this work, the terms $\{m_1, m_2\}_l = \min\{m_1, m_2\}$, and $\{m_1, m_2\}_h = \max\{m_1, m_2\}$ are used.

Lemma 1. *If $\mathcal{P}_\eta$ is a fuzzy path graph with $\beta_i \in V(\mathcal{P}_\eta)$, then the degree of a vertex is given by $d(\beta_i) = (d_\psi(\beta_i), d_\zeta(\beta_i))$. If the membership and non-membership degrees of a vertex β_i are m_i and n_i, then the degree of $\eta - 2$ vertices is defined as:*

$$d_\psi(\beta_i) \leq (\{m_{i-1}, m_i\}_l + \{m_i, m_{i+1}\}_l), \forall i, 2 \leq i \leq \eta - 1,$$

and,

$$d_\zeta(\beta_i) \leq (\{n_{i-1}, n_i\}_h + \{n_i, n_{i+1}\}_h), \forall i, 2 \leq i \leq \eta - 1.$$

For first vertex,

$$d_\psi(\beta_1) \leq \{m_1, m_2\}_l\,,$$
$$d_\zeta(\beta_1) \leq \{n_1, n_2\}_h\,.$$

For ηth vertex,

$$d_\psi(\beta_\eta) \leq \{m_{\eta-1}, m_\eta\}_l\,,$$
$$d_\zeta(\beta_\eta) \leq \{n_{\eta-1}, n_\eta\}_h$$

Theorem 1. *Let $\mathcal{P}_n$ be an intuitionistic fuzzy path graph with $V = \{\beta_1, \beta_2, \beta_3, \ldots, \beta_\eta\}$ the set of vertices and $E = \{\beta_1\beta_2, \beta_2\beta_3, \beta_3\beta_4, \ldots, \beta_{\eta-1}\beta_\eta\}$ the set of $\eta - 1$ edges, such that $m_i = \psi(\beta_i)$ denotes the membership degree and $n_i = \zeta(\beta_i)$ denotes the non-membership degree of vertex β_i. Then the third version of intuitionistic fuzzy Sombor index SO_3 is:*

$$\begin{aligned} \mathcal{SO}_3(\mathcal{P}_n) \leq & \sum_{i=2}^{\eta-2} \left[\frac{(m_i, n_i)\left(a_i^2, b_i^2\right) + (m_{i+1}, n_{i+1})\left(a_{i+1}^2, b_{i+1}^2\right)}{(m_i, n_i)(a_i, b_i) + (m_{i+1}, n_{i+1})(a_{i+1}, b_{i+1})} \right] \sqrt{2}\pi \\ & + \left[\frac{(m_1, n_1)\left(a_1^2, b_1^2\right) + (m_2, n_2)\left(a_2^2, b_2^2\right)}{(m_1, n_1)(a_1, b_1) + (m_2, n_2)(a_2, b_2)} \right] \sqrt{2}\pi \\ & + \left[\frac{(m_{\eta-1}, n_{\eta-1})\left(a_{\eta-1}^2, b_{\eta-1}^2\right) + (m_\eta, n_\eta)\left(a_\eta^2, b_\eta^2\right)}{(m_{\eta-1}, n_{\eta-1})(a_{\eta-1}, b_{\eta-1}) + (m_\eta, n_\eta)(a_\eta, b_\eta)} \right] \sqrt{2}\pi, \end{aligned}$$

where,

$$\begin{array}{ll}
\left[\{m_{i-1},m_i\}_l+\{m_i,m_{i+1}\}_l\right]=a_i & \left[\{n_{i-1},n_i\}_h+\{n_i,n_{i+1}\}_h\right]=b_i\\
\left[\{m_i,m_{i+1}\}_l+\{m_{i+1},m_{i+2}\}_l\right]=a_{i+1} & \left[\{n_i,n_{i+1}\}_h+\{n_{i+1},n_{i+2}\}_h\right]=b_{i+1}\\
\{m_1,m_2\}_l=a_1 & \{n_1,n_2\}_h=b_1,\\
\left[\{m_1,m_2\}_l+\{m_2,m_3\}_l\right]=a_2, & \left[\{n_1,n_2\}_h+\{n_2,n_3\}_h\right]=b_2,\\
\left[\{m_{\eta-2},m_{\eta-1}\}_l+\{m_{\eta-1},m_\eta\}_l\right]=a_{\eta-1}, & \left[\{n_{\eta-2},n_{\eta-1}\}_h+\{n_{\eta-1},n_\eta\}_h\right]=b_{\eta-1},\\
\{m_{\eta-1},m_\eta\}_l=a_\eta, & \{n_{\eta-1},n_\eta\}_h=b_\eta.
\end{array}$$

Proof. The degrees of vertices of an intuitionistic fuzzy path graph were already defined in Lemma 1. Applying these definitions, the intuitionistic fuzzy version of the Sombor index SO_3 for edge $\beta_1\beta_2$ is given by,

$$\mathcal{SO}_3(\beta_1\beta_2)\leq\frac{(m_1,n_1)\left(a_1^2,b_1^2\right)+(m_2,n_2)\left(a_2^2,b_2^2\right)}{(m_1,n_1)(a_1,b_1)+(m_2,n_2)(a_2,b_2)}\sqrt{2}\pi,$$

similarly for edge $\beta_2\beta_3$,

$$\mathcal{SO}_3(\beta_2\beta_3)\leq\frac{(m_2,n_2)\left(a_2^2,b_2^2\right)+(m_3,n_3)\left(a_3^2,b_3^2\right)}{(m_2,n_2)(a_2,b_2)+(m_3,n_3)(a_3,b_3)}\sqrt{2}\pi,$$

for edge $\beta_3\beta_4$,

$$\mathcal{SO}_3(\beta_3\beta_4)\leq\frac{(m_3,n_3)\left(a_3^2,b_3^2\right)+(m_4,n_4)\left(a_4^2,b_4^2\right)}{(m_3,n_3)(a_3,b_3)+(m_4,n_4)(a_4,b_4)}\sqrt{2}\pi,$$

and so on, for edge $\beta_{\eta-2}\beta_{\eta-1}$,

$$\mathcal{SO}_3(\beta_{\eta-2}\beta_{\eta-1})\leq\frac{(m_{\eta-2},n_{\eta-2})\left(a_{\eta-2}^2,b_{\eta-2}^2\right)+(m_{\eta-1},n_{\eta-1})\left(a_{\eta-1}^2,b_{\eta-1}^2\right)}{(m_{\eta-2},n_{\eta-2})(a_{\eta-2},b_{\eta-2})+(m_{\eta-1},n_{\eta-1})(a_{\eta-1},b_{\eta-1})}\sqrt{2}\pi,$$

and for edge $\beta_{\eta-1}\beta_\eta$,

$$\mathcal{SO}_3(\beta_{\eta-1}\beta_\eta)\leq\frac{(m_{\eta-1},n_{\eta-1})\left(a_{\eta-1}^2,b_{\eta-1}^2\right)+(m_\eta,n_\eta)\left(a_\eta^2,b_\eta^2\right)}{(m_{\eta-1},n_{\eta-1})(a_{\eta-1},b_{\eta-1})+(m_\eta,n_\eta)(a_\eta,b_\eta)}\sqrt{2}\pi,$$

Hence, by adding all these results, general form of third version of intuitionistic fuzzy Sombor index for IF path graph is:

$$\begin{aligned}\mathcal{SO}_3(\mathcal{P}_n)\leq&\sum_{i=2}^{\eta-2}\left[\frac{(m_i,n_i)\left(a_i^2,b_i^2\right)+(m_{i+1},n_{i+1})\left(a_{i+1}^2,b_{i+1}^2\right)}{(m_i,n_i)(a_i,b_i)+(m_{i+1},n_{i+1})(a_{i+1},b_{i+1})}\right]\sqrt{2}\pi\\
&+\left[\frac{(m_1,n_1)\left(a_1^2,b_1^2\right)+(m_2,n_2)\left(a_2^2,b_2^2\right)}{(m_1,n_1)(a_1,b_1)+(m_2,n_2)(a_2,b_2)}\right]\sqrt{2}\pi\\
&+\left[\frac{(m_{\eta-1},n_{\eta-1})\left(a_{\eta-1}^2,b_{\eta-1}^2\right)+(m_\eta,n_\eta)\left(a_\eta^2,b_\eta^2\right)}{(m_{\eta-1},n_{\eta-1})(a_{\eta-1},b_{\eta-1})+(m_\eta,n_\eta)(a_\eta,b_\eta)}\right]\sqrt{2}\pi,\end{aligned}$$

where,

$[\{m_{i-1}, m_i\}_l + \{m_i, m_{i+1}\}_l] = a_i,$ $[\{n_{i-1}, n_i\}_h + \{n_i, n_{i+1}\}_h] = b_i,$
$[\{m_i, m_{i+1}\}_l + \{m_{i+1}, m_{i+2}\}_l] = a_{i+1},$ $[\{n_i, n_{i+1}\}_h + \{n_{i+1}, n_{i+2}\}_h] = b_{i+1},$
$\{m_1, m_2\}_l = a_1,$ $\{n_1, n_2\}_h = b_1,$
$[\{m_1, m_2\}_l + \{m_2, m_3\}_l] = a_2,$ $[\{n_1, n_2\}_h + \{n_2, n_3\}_h] = b_2,$
$\left[\{m_{\eta-2}, m_{\eta-1}\}_l + \{m_{\eta-1}, m_\eta\}_l\right] = a_{\eta-1},$ $\left[\{n_{\eta-2}, n_{\eta-1}\}_h + \{n_{\eta-1}, n_\eta\}_h\right] = b_{\eta-1},$
$\{m_{\eta-1}, m_\eta\}_l = a_\eta,$ $\{n_{\eta-1}, n_\eta\}_h = b_\eta.$

□

Theorem 2. *Let $\mathcal{P}_\eta$ be an intuitionistic fuzzy path graph with $V = \{\beta_1, \beta_2, \beta_3, \dots, \beta_\eta\}$ the set of vertices and $E = \{\beta_1\beta_2, \beta_2\beta_3, \beta_3\beta_4, \dots, \beta_{\eta-1}\beta_\eta\}$ the set of $\eta - 1$ edges. Denote the membership degree by $m_i = \psi(\beta_i)$ and the non-membership degree of vertex β_i by $n_i = \zeta(\beta_i)$. Then, the fourth version of intuitionistic fuzzy Sombor index is:*

$$
\begin{aligned}
SO_4(P_\eta) \leq & \frac{1}{2}\sum_{i=2}^{n-2}\left[\frac{(m_i, n_i)\left(a_i^2, b_i^2\right) + (m_{i+1}, n_{i+1})\left(a_{i+1}^2, b_{i+1}^2\right)}{(m_i, n_i)(a_i, b_i) + (m_{i+1}, n_{i+1})(a_{i+1}, b_{i+1})}\right]^2 \pi + \\
& \frac{1}{2}\left[\frac{(m_1, n_1)\left(a_1^2, b_1^2\right) + (m_2, n_2)\left(a_2^2, b_2^2\right)}{(m_1, n_1)(a_1, b_1) + (m_2, n_2)(a_2, b_2)}\right]^2 \pi + \\
& \frac{1}{2}\left[\frac{(m_{\eta-1}, n_{\eta-1})\left(a_{\eta-1}^2, b_{\eta-1}^2\right) + (m_\eta, n_\eta)\left(a_\eta^2, b_\eta^2\right)}{(m_{\eta-1}, n_{\eta-1})(a_{\eta-1}, b_{\eta-1}) + (m_\eta, n_\eta)(a_\eta, b_\eta)}\right]^2 \pi,
\end{aligned}
$$

where,

$\left[\{m_{i-1}, m_i\}_l + \{m_i, m_{i+1}\}_l\right] = a_i,$ $\left[\{n_{i-1}, n_i\}_h + \{n_i, n_{i+1}\}_h\right] = b_i,$
$\left[\{m_i, m_{i+1}\}_l + \{m_{i+1}, m_{i+2}\}_l\right] = a_{i+1},$ $\left[\{n_i, n_{i+1}\}_h + \{n_{i+1}, n_{i+2}\}_h\right] = b_{i+1},$
$\{m_1, m_2\}_l = a_1,$ $\{n_1, n_2\}_h = b_1,$
$\left[\{m_1, m_2\}_l + \{m_2, m_3\}_l\right] = a_2,$ $\left[\{n_1, n_2\}_h + \{n_2, n_3\}_h\right] = b_2,$
$\left[\{m_{\eta-2}, m_{\eta-1}\}_l + \{m_{\eta-1}, m_\eta\}_l\right] = a_{\eta-1},$ $\left[\{n_{\eta-2}, n_{\eta-1}\}_h + \{n_{\eta-1}, n_\eta\}_h\right] = b_{\eta-1},$
$\{m_{\eta-1}, m_\eta\}_l = a_\eta,$ $\{n_{\eta-1}, n_\eta\}_h = b_\eta.$

Proof. This can be proven similarly by taking steps used in the proof of Theorem 1. □

2.2. Intuitionistic fuzzy cycle graph and its results

In an intuitionistic fuzzy graph $G = \{V, \psi, \zeta\}$, a path is a cycle if $\beta_1\beta_\eta \in E(G)$, where $\eta \geq 3$. A cycle is an *intuitionistic fuzzy cycle* if it contains more than one weakest edge with least membership degree and greatest non-membership degree in the path.

Lemma 2. *If C_η is a fuzzy cycle graph with $\beta_i \in V(C_\eta)$. Then degree of all the vertices is given by:*

$$d_\psi(\beta_i) \leq (\{m_{i-1}, m_i\}_l + \{m_i, m_{i+1}\}_l), \forall i, \leq i \leq \eta,$$

and

$$d_{\zeta}(\beta_i) \leq (\{n_{i-1}, n_i\}_h + \{n_i, n_{i+1}\}_h), \forall i, 1 \leq i \leq \eta,$$

where $m_{\eta+1} = m_1$, $n_{\eta+1} = n_1$.

Theorem 3. *Let* $C_\eta = \{V, E\}$ *be an intuitionistic fuzzy cycle graph with* $V = \{\beta_1, \beta_2, \beta_3, \dots, \beta_\eta\}$ *such that* $m_i = \psi(\beta_i)$ *denotes the membership degree and* $n_i = \zeta(\beta_i)$ *denotes the non-membership degree of vertex* β_i. *Also* $\beta_1 = \beta_\eta$. *Then*

$$\mathcal{SO}_3(C_\eta) \leq \sum_{i=1}^{\eta} \left[\frac{(m_i, n_i)\left(c_i^2, d_i^2\right) + (m_{i+1}, n_{i+1})\left(c_{i+1}^2, d_{i+1}^2\right)}{(m_i, n_i)\left(c_i, d_i\right) + (m_{i+1}, n_{i+1})\left(c_{i+1}, d_{i+1}\right)} \right] \sqrt{2}\pi,$$

where,

$$\begin{array}{ll} \left[\{m_{i-1}, m_i\}_l + \{m_i, m_{i+1}\}_l\right] = c_i, & \left[\{n_{i-1}, n_i\}_h + \{n_i, n_{i+1}\}_h\right] = d_i, \\ \left[\{m_i, m_{i+1}\}_l + \{m_{i+1}, m_{i+2}\}_l\right] = c_{i+1}, & \left[\{n_i, n_{i+1}\}_h + \{n_{i+1}, n_{i+2}\}_h\right] = d_{i+1}, \\ m_{\eta+1} = m_1, & m_{\eta+2} = m_2, \\ n_{\eta+1} = n_1, & n_{\eta+2} = n_2. \end{array}$$

Proof. Suppose C_η is an intuitionistic fuzzy cycle with η vertices and $\eta - 1$ edges, degree of vertices is defined in Lemma 2.
Now we calculate third version of intuitionistic fuzzy Sombor index for different edge. For edge $\beta_1\beta_2$,

$$\mathcal{SO}_3(\beta_1\beta_2) \leq \left[\frac{(m_1, n_1)\left(c_1^2, d_1^2\right) + (m_2, n_2)\left(c_2^2, d_2^2\right)}{(m_1, n_1)\left(c_1, d_1\right) + (m_2, n_2)\left(c_2, d_2\right)} \right] \sqrt{2}\pi,$$

similarly for edge $\beta_2\beta_3$,

$$\mathcal{SO}_3(\beta_2\beta_3) \leq \frac{(m_2, n_2)\left(c_2^2, d_2^2\right) + (m_3, n_3)\left(c_3^2, d_3^2\right)}{(m_2, n_2)\left(c_2, d_2\right) + (m_3, n_3)\left(c_3, d_3\right)} \sqrt{2}\pi,$$

and so on. Now the third version of intuitionistic fuzzy Sombor index for edge $\beta_{\eta-1}\beta_\eta$ is

$$\mathcal{SO}_3(\beta_{\eta-1}\beta_\eta) \leq \frac{(m_{\eta-1}, n_{\eta-1})\left(c_{\eta-1}^2, d_{\eta-1}^2\right) + (m_\eta, n_\eta)\left(c_\eta^2, d_\eta^2\right)}{(m_{\eta-1}, n_{\eta-1})\left(c_{\eta-1}, d_{\eta-1}\right) + (m_\eta, n_\eta)\left(c_\eta, d_\eta\right)} \sqrt{2}\pi,$$

Hence, adding up all above inequalities, we have:

$$\mathcal{SO}_3(C_\eta) \leq \sum_{i=1}^{\eta} \left[\frac{(m_i, n_i)\left(c_i^2, d_i^2\right) + (m_{i+1}, n_{i+1})\left(c_{i+1}^2, d_{i+1}^2\right)}{(m_i, n_i)\left(c_i, d_i\right) + (m_{i+1}, n_{i+1})\left(c_{i+1}, d_{i+1}\right)} \right] \sqrt{2}\pi$$

where

$$\left[\{m_1, m_\eta\}_l + \{m_1, m_2\}_l\right] = c_1, \quad \left[\{n_1, n_\eta\}_l + \{n_1, n_2\}_l\right] = d_1,$$
$$\left[\{m_1, m_2\}_l + \{m_2, m_3\}_l\right] = c_2, \quad \left[\{n_1, n_2\}_h + \{n_2, n_3\}_h\right] = d_2,$$
$$\left[\{m_2, m_3\}_l + \{m_3, m_4\}_l\right] = c_3, \quad \left[\{n_2, n_3\}_h + \{n_3, n_4\}_h\right] = d_3,$$
$$\left[\{m_{\eta-2}, m_{\eta-1}\}_l + \{m_{\eta-1}, m_\eta\}_l\right] = c_{\eta-1}, \quad \left[\{n_{\eta-2}, n_{\eta-1}\}_h + \{n_{\eta-1}, n_\eta\}_h\right] = d_{\eta-1},$$
$$\left[\{m_{\eta-1}, m_\eta\}_l + \{m_\eta, m_1\}_l\right] = c_\eta, \quad \left[\{n_{\eta-1}, n_\eta\}_h + \{n_\eta, n_1\}_h\right] = d_\eta,$$
$$m_{\eta+1} = m_1,\ m_{\eta+2} = m_2, \quad n_{\eta+1} = n_1,\ n_{\eta+2} = n_2.$$

□

Theorem 4. *If $C_\eta = \{V, E\}$ be an intuitionistic fuzzy cycle graph with $V = \{\beta_1, \beta_2, \beta_3, \dots, \beta_\eta\}$ such that $m_i = \psi(\beta_i)$ and $n_i = \zeta(\beta_i)$ denote the membership degree and non-membership degree of vertex β_i, respectively, with $\beta_1 = \beta_\eta$, then*

$$\mathcal{SO}_4(C_\eta) \leq \frac{1}{2}\sum_{i=1}^{n}\left[\frac{(m_i, n_i)\left(c_i^2, d_i^2\right) + (m_{i+1}, n_{i+1})\left(c_{i+1}^2, d_{i+1}^2\right)}{(m_i, n_i)\left(c_i, d_i\right) + (m_{i+1}, n_{i+1})\left(c_{i+1}, d_{i+1}\right)}\right]^2 \pi,$$

where,

$$\left[\{m_{i-1}, m_i\}_l + \{m_i, m_{i+1}\}_l\right] = c_i, \quad \left[\{n_{i-1}, n_i\}_h + \{n_i, n_{i+1}\}_h\right] = d_i,$$
$$\left[\{m_i, m_{i+1}\}_l + \{m_{i+1}, m_{i+2}\}_l\right] = c_{i+1}, \quad \left[\{n_i, n_{i+1}\}_h + \{n_{i+1}, n_{i+2}\}_h\right] = d_{i+1},$$
$$m_{\eta+1} = m_1, \quad m_{\eta+2} = m_2,$$
$$n_{\eta+1} = n_1, \quad n_{\eta+2} = n_2.$$

Proof. This can be proved similarly by taking steps used in the proof of Theorem 3. □

2.3. Complete intuitionistic fuzzy graph and its results

A pair $G = (V, E)$, is said to be complete graph, if

$$\psi_2(\beta_i, \beta_j) = min\left[\psi_1(\beta_i), \psi_1(\beta_j)\right],$$
$$\zeta_2(\beta_i, \beta_j) = max\left[\zeta_1(\beta_i), \zeta_1(\beta_j)\right],$$

for all $(\beta_i, \beta_j) \in E(G)$

Lemma 3. *If K_η is a complete intuitionistic fuzzy graph and $\beta_i \in V(K_\eta)$. Then degree of β_i vertex in K_η is defined as:*

$$d_\psi(\beta_i) = \sum_{j=1}^{\eta-1}\{m_i, m_j\}_l, \forall i \neq j,$$

and,

$$d_\zeta(\beta_i) = \sum_{j=1}^{\eta-1}\{n_i, n_j\}_h, \forall i \neq j.$$

Theorem 5. *Let $K_\eta = \{V, E\}$ be a complete intuitionistic fuzzy graph with $\{\beta_1, \beta_2, \beta_3, \beta_4, \dots, \beta_\eta\}$, set of vertices in which degree of each vertex is defined as in Lemma 3. Then for all $i \neq j \neq k$,*

$$\mathcal{SO}_3(K_\eta) = \sum_{i=1}^{\eta}\sum_{k=1}^{\eta}\left[\frac{(m_i, n_i)\,\chi^2_{i,\eta} + (m_k, n_k)\,\chi^2_{k,\eta}}{(m_i, n_i)\,\chi^1_{i,\eta} + (m_k, n_k)\,\chi^1_{k,\eta}}\right]\sqrt{2}\pi,$$

where $\chi^r_{i,p} = \left(\left[\sum_{j=1}^{p}\{m_i, m_j\}_l\right]^r, \left[\sum_{j=1}^{p}\{n_i, n_j\}_h\right]^r\right)$ for $r \in \{1, 2\}$.

Proof. We prove the above theorem by induction. For $\eta = 3$, the complete graph is the triangle. Therefore the third version of intuitionistic Sombor fuzzy index for K_3 is:

$$\begin{aligned}\mathcal{SO}_3(K_3) = &\left[\frac{(m_1, n_1)\,\chi^2_{1,3} + (m_2, n_2)\,\chi^2_{2,3}}{(m_1, n_1)\,\chi^1_{1,3} + (m_2, n_2)\,\chi^1_{2,3}}\right]\sqrt{2}\pi + \left[\frac{(m_2, n_2)\,\chi^2_{2,3} + (m_3, n_3)\,\chi^2_{3,3}}{(m_2, n_2)\,\chi^1_{2,3} + (m_3, n_3)\,\chi^1_{3,3}}\right]\sqrt{2}\pi + \\ &\left[\frac{(m_1, n_1)\,\chi^2_{1,3} + (m_3, n_3)\,\chi^2_{3,3}}{(m_1, n_1)\,\chi^1_{1,3} + (m_3, n_3)\,\chi^1_{3,3}}\right]\sqrt{2}\pi.\end{aligned}$$

For $\eta = 4$, the complete graph is symbolized by K_4. The third version of the intuitionistic Sombor fuzzy index for this graph is given by,

$$\begin{aligned}\mathcal{SO}_3(K_4) = &\left[\frac{(m_1, n_1)\,\chi^2_{1,4} + (m_2, n_2)\,\chi^2_{2,4}}{(m_1, n_1)\,\chi^1_{1,4} + (m_2, n_2)\,\chi^1_{2,4}}\right]\sqrt{2}\pi + \left[\frac{(m_2, n_2)\,\chi^2_{2,4} + (m_3, n_3)\,\chi^2_{3,4}}{(m_2, n_2)\,\chi^1_{2,4} + (m_3, n_3)\,\chi^1_{3,4}}\right]\sqrt{2}\pi + \\ &\left[\frac{(m_3, n_3)\,\chi^2_{3,4} + (m_4, n_4)\,\chi^2_{4,4}}{(m_3, n_3)\,\chi^1_{3,4} + (m_4, n_4)\,\chi^1_{4,4}}\right]\sqrt{2}\pi + \left[\frac{(m_1, n_1)\,\chi^2_{1,4} + (m_4, n_4)\,\chi^2_{4,4}}{(m_1, n_1)\,\chi^1_{1,4} + (m_4, n_4)\,\chi^1_{4,4}}\right]\sqrt{2}\pi + \\ &\left[\frac{(m_1, n_1)\,\chi^2_{1,4} + (m_3, n_3)\,\chi^2_{3,4}}{(m_1, n_1)\,\chi^1_{1,4} + (m_3, n_3)\,\chi^1_{3,4}}\right]\sqrt{2}\pi + \left[\frac{(m_2, n_2)\,\chi^2_{2,4} + (m_4, n_4)\,\chi^2_{4,4}}{(m_2, n_2)\,\chi^1_{2,4} + (m_4, n_4)\,\chi^1_{4,4}}\right]\sqrt{2}\pi,\end{aligned}$$

or it can be written as,

$$\mathcal{SO}_3(K_4) = \sum_{i=1}^{4}\sum_{k=1}^{4}\left[\frac{(m_i, n_i)\,\chi^2_{i,4} + (m_k, n_k)\,\chi^2_{k,4}}{(m_i, n_i)\,\chi^1_{i,4} + (m_k, n_k)\,\chi^1_{k,4}}\right]\sqrt{2}\pi.$$

for all $i \neq j \neq k$, such that $1 \leq i, j \leq 4$.
Hence, by generalizing the result for complete intuitionistic fuzzy graph of η vertices we can write,

$$\mathcal{SO}_3(K_\eta) = \sum_{i=1}^{\eta}\sum_{k=1}^{\eta}\left[\frac{(m_i, n_i)\,\chi^2_{i,\eta} + (m_k, n_k)\,\chi^2_{k,\eta}}{(m_i, n_i)\,\chi^1_{i,\eta} + (m_k, n_k)\,\chi^1_{k,\eta}}\right]\sqrt{2}\pi.$$

for all $i \neq j \neq k$, where $1 \leq i, j \leq \eta$. □

Theorem 6. *Let $K_\eta = \{V, E\}$ be a complete intuitionistic fuzzy graph with $\{\beta_1, \beta_2, \beta_3, \beta_4, \ldots, \beta_\eta\}$, set of vertices. Then for all $i \neq j \neq k$, fourth version of intuitionistic fuzzy Sombor index is given by:*

$$\mathcal{SO}_4(K_\eta) = \frac{1}{2} \sum_{i=1}^{\eta} \sum_{k=1}^{\eta} \left[\frac{\chi_i^2 + \chi_k^2}{\chi_i^1 + \chi_k^1} \right] \sqrt{2}\pi$$

where $\chi_i^r = (m_i, n_i) \left(\left[\sum_{j=1}^{\eta} \{m_i, m_j\}_l \right]^r, \left[\sum_{j=1}^{\eta} \{n_i, n_j\}_h \right]^r \right)$ *for* $r \in \{1, 2\}$.

Proof. This can be proved similarly by taking steps used in the proof of Theorem 5. □

2.4. Complete bipartite intuitionistic fuzzy graph

An IFG $G = \{V, \psi, \zeta\}$ is bipartite with $\beta_i, \alpha_j \in V(K_{\eta,p})$, where vertex set of graph is partitioned into two sets $\mathcal{H}$ and $\mathcal{I}$ such that:

- $\psi_2(\beta_i, \alpha_j) = 0$ and $\zeta_2(\beta_i, \alpha_j) = 0$, if $(\beta_i, \alpha_j) \in \mathcal{H}$ **or** $(\beta_i, \alpha_j) \in \mathcal{I}$, **or**
- $\psi_2(\beta_i, \alpha_j) > 0$, $\zeta_2(\beta_i, \alpha_j) < 0$, if $\beta_i \in \mathcal{H}$ and $\alpha_j \in \mathcal{H}$ for some i and j, **or**
- $\psi_2(\beta_i, \alpha_j) = 0$, $\zeta_2(\beta_i, \alpha_j) < 0$, if $\beta_i \in \mathcal{H}$ and $\alpha_j \in \mathcal{H}$ for some i and j, **or**
- $\psi_2(\beta_i, \beta_j) > 0$, $\zeta_2(\beta_i, \alpha_j) = 0$, if $\beta_i \in \mathcal{H}$ and $\alpha_j \in \mathcal{I}$ for some i and j,

and a bipartite intuitionistic fuzzy graph is complete if,

$$\begin{aligned} \psi_2(\beta_i, \alpha_j) &= \min(\psi_1(\beta_i), \psi_1(\alpha_j)), \\ \zeta_2(\beta_i, \alpha_j) &= \max(\zeta_1(\beta_i), \zeta_1(\alpha_j)). \end{aligned}$$

Where, $\beta_i \in \mathcal{H}$ and $\alpha_j \in \mathcal{I}$.

Lemma 4. *Let $K_{\eta,p}$ denote a complete bipartite intuitionistic fuzzy graph with $\beta_i \in \mathcal{H}$ s.t., $1 \leq i \leq \eta$, and $\alpha_j \in \mathcal{I}$ where, $1 \leq j \leq p$. Suppose m_i and t_j denote the membership degrees of vertices β_i and α_j respectively, whereas n_i and f_j denote the non-membership degrees of same vertices, then the membership degree of the vertex $\beta_i \in V(\mathcal{H})$ is:*

$$d_\psi(\beta_i) = \sum_{j=1}^{p} \{m_i, t_j\}_l,$$

whereas the non-membership degree is given by:

$$d_\zeta(\beta_i) = \sum_{j=1}^{p} \{n_i, f_j\}_h.$$

Similarly, for the vertex $\beta_j \in \mathcal{I}$, the membership and non-membership degrees are defined as:

$$d_{\psi}(\alpha_j) = \sum_{i=1}^{\eta} \{t_j, m_i\}_l \,,$$
$$d_{\zeta}(\alpha_j) = \sum_{i=1}^{\eta} \{f_j, n_i\}_h \,.$$

Theorem 7. *Let $K_{\eta,p} = \{V, E\}$ be a complete bipartite intuitionistic fuzzy graph with set of vertices partitioned in two sets $\mathcal{H} = \{\beta_1, \beta_2, \beta_3, \dots, \beta_\eta\}$ and $\mathcal{I} = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_p\}$ with m_i, t_j being the membership degrees of vertices β_i and α_j, whereas n_i and f_j are the non-membership degrees of the same vertices. Then its third version of intuitionistic Sombor fuzzy index is:*

$$\mathcal{SO}_3(K_{\eta,p}) = \sum_{i=1}^{\eta}\sum_{j=1}^{p} \left[\frac{(m_i, n_i)\,\chi_i^2 + (t_j, f_j)\,\chi_i^2}{(m_i, n_i)\,\chi_i^1 + (t_j, f_j)\,\chi_i^1}\right]\sqrt{2}\pi.$$

where $\chi_{i,p}^r = \left(\left[\sum_{j=1}^{p}\{m_i, t_j\}_l\right]^r, \left[\sum_{j=1}^{p}\{n_i, f_j\}_h\right]^r\right)$ *for* $r \in \{1, 2\}$.

Proof. Suppose $\eta = 3$ and $p = 2$. Then the vertex sets would be $\mathcal{H} = \{\beta_1, \beta_2, \beta_3\}$ $\mathcal{I} = \{\alpha_1, \alpha_2\}$ .Then,

$$\begin{aligned}
\mathcal{SO}_3(K_{2,3}) = &\left[\frac{(m_1,n_1)\,\chi_{1,p}^2 + (t_1,f_1)\left(\left[\sum_{i=1}^3\{t_1,m_i\}_l\right]^2, \left[\sum_{i=1}^3\{f_1,n_i\}_h\right]^2\right)}{(m_1,n_1)\,\chi_{1,p}^1 + (t_1,f_1)\left(\left[\sum_{i=1}^3\{t_1,m_i\}_l\right], \left[\sum_{i=1}^3\{f_1,n_i\}_h\right]\right)}\right]\sqrt{2}\pi + \\
&\left[\frac{(m_1,n_1)\,\chi_{1,p}^2 + (t_2,f_2)\left(\left[\sum_{i=1}^3\{t_2,m_i\}_l\right]^2, \left[\sum_{i=1}^3\{f_2,n_i\}_h\right]^2\right)}{(m_1,n_1)\,\chi_{1,p}^1 + (t_2,f_2)\left(\left[\sum_{i=1}^3\{t_2,m_i\}_l\right], \left[\sum_{i=1}^3\{f_2,n_i\}_h\right]\right)}\right]\sqrt{2}\pi + \\
&\left[\frac{(m_1,n_1)\,\chi_{1,p}^2 + (t_3,f_3)\left(\left[\sum_{i=1}^3\{t_3,m_i\}_l\right]^2, \left[\sum_{i=1}^3\{f_3,n_i\}_h\right]^2\right)}{(m_1,n_1)\,\chi_{1,p}^1 + (t_3,f_3)\left(\left[\sum_{i=1}^3\{t_3,m_i\}_l\right], \left[\sum_{i=1}^3\{f_3,n_i\}_h\right]\right)}\right]\sqrt{2}\pi + \\
&\left[\frac{(m_2,n_2)\,\chi_{2,p}^2 + (t_1,f_1)\left(\left[\sum_{i=1}^3\{t_1,m_i\}_l\right]^2, \left[\sum_{i=1}^3\{f_1,n_i\}_h\right]^2\right)}{(m_2,n_2)\,\chi_{2,p}^1 + (t_1,f_1)\left(\left[\sum_{i=1}^3\{t_1,m_i\}_l\right], \left[\sum_{i=1}^3\{f_1,n_i\}_h\right]\right)}\right]\sqrt{2}\pi + \\
&\left[\frac{(m_2,n_2)\,\chi_{2,p}^2 + (t_2,f_2)\left(\left[\sum_{i=1}^3\{t_2,m_i\}_l\right]^2, \left[\sum_{i=1}^3\{f_2,n_i\}_h\right]^2\right)}{(m_2,n_2)\,\chi_{2,p}^1 + (t_2,f_2)\left(\left[\sum_{i=1}^3\{t_2,m_i\}_l\right], \left[\sum_{i=1}^3\{f_2,n_i\}_h\right]\right)}\right]\sqrt{2}\pi + \\
&\left[\frac{(m_2,n_2)\,\chi_{2,p}^2 + (t_3,f_3)\left(\left[\sum_{i=1}^3\{t_3,m_i\}_l\right]^2, \left[\sum_{i=1}^3\{f_3,n_i\}_h\right]^2\right)}{(m_2,n_2)\,\chi_{2,p}^1 + (t_3,f_3)\left(\left[\sum_{i=1}^3\{t_3,m_i\}_l\right], \left[\sum_{i=1}^3\{f_3,n_i\}_h\right]\right)}\right]\sqrt{2}\pi.
\end{aligned}$$

Similarly, in general for $K_{\eta,p}$ third version of Sombor fuzzy index is given by,

$$\begin{aligned}\mathcal{SO}_3(K_{\eta,p}) = &\left[\frac{(m_1,n_1)\,\chi^2_{1,p} + (t_1,f_1)\left(\left[\sum_{i=1}^{\eta}\{t_1,m_i\}_l\right]^2, \left[\sum_{i=1}^{\eta}\{f_1,n_i\}_h\right]^2\right)}{(m_1,n_1)\,\chi^1_{1,p} + (t_1,f_1)\left(\left[\sum_{i=1}^{\eta}\{t_1,m_i\}_l\right], \left[\sum_{i=1}^{\eta}\{f_1,n_i\}_h\right]\right)}\right]\sqrt{2}\pi + \\ &\left[\frac{(m_1,n_1)\,\chi^2_{1,p} + (t_2,f_2)\left(\left[\sum_{i=1}^{\eta}\{t_2,m_i\}_l\right]^2, \left[\sum_{i=1}^{\eta}\{f_2,n_i\}_h\right]^2\right)}{(m_1,n_1)\,\chi^1_{1,p} + (t_2,f_2)\left(\left[\sum_{i=1}^{\eta}\{t_2,m_i\}_l\right], \left[\sum_{i=1}^{\eta}\{f_2,n_i\}_h\right]\right)}\right]\sqrt{2}\pi + \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\ &\left[\frac{(m_\eta,n_\eta)\,\chi^2_{\eta,p} + (t_p,f_p)\left(\left[\sum_{i=1}^{\eta}\{t_p,m_i\}_l\right]^2, \left[\sum_{i=1}^{\eta}\{f_p,n_i\}_h\right]^2\right)}{(m_\eta,n_\eta)\,\chi^1_{\eta,p} + (t_p,f_p)\left(\left[\sum_{i=1}^{\eta}\{t_p,m_i\}_l\right], \left[\sum_{i=1}^{\eta}\{f_p,n_i\}_h\right]\right)}\right]\sqrt{2}\pi.\end{aligned}$$

Above result can be written as:

$$\mathcal{SO}_3(K_{\eta,p}) = \sum_{i=1}^{\eta}\sum_{j=1}^{p}\left[\frac{(m_i,n_i)\,\chi^2_{i,p} + (t_j,f_j)\,\chi^2_{i,\eta}}{(m_i,n_i)\,\chi^1_{i,p} + (t_j,f_j)\,\chi^1_{i,\eta}}\right]\sqrt{2}\pi.$$

□

Theorem 8. *Let $K_{\eta,p} = \{V,E\}$ be a complete bipartite intuitionistic fuzzy graph with set of vertices $\mathcal{H} = \{\beta_1,\beta_2,\beta_3,\ldots,\beta_\eta\}$ and $\mathcal{I} = \{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_l\}$,such that $\mathcal{H}\cup\mathcal{I} = V$ and $\mathcal{H}\cap\mathcal{I} = 0$. Also, if m_i, t_j represent the membership degrees of vertices β_i and α_j, whereas n_i and f_j are non-membership degrees of the same vertices, then the fourth version of intuitionistic Sombor fuzzy index is:*

$$SO_4(K_{n,l}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{l}\left[\frac{(m_i,n_i)\,\chi^2_{i,p} + (t_j,f_j)\,\chi^2_{i,\eta}}{(m_i,n_i)\,\chi^1_{i,p} + (t_j,f_j)\,\chi^1_{i,\eta}}\right]^2\pi.$$

Proof. This can be proven similarly by taking steps used in the proof of Theorem 7. □

2.5. Intuitionistic fuzzy wheel graph and its results

Definition 8. A cycle graph C_n can be used to construct a wheel graph W_n by inserting a vertex at the center of cycle graph, and by joining all the vertices of cycle graph with this new vertex. This new vertex is called center or Hub vertex. A wheel graph would be an intuitionistic fuzzy wheel graph, if for membership value m_γ and non-membership values n_γ of center vertex γ and for membership value m_i and non-membership value of vertices β_i, following conditions hold.

- $\psi_2(\beta_i,\beta_j) \le min\,(\psi_1(\beta_i),\psi_1(\beta_j))$,
- $\zeta_2(\beta_i,\beta_j) \le max\,(\zeta_1(\beta_i),\zeta_1(\beta_j))$, where $1 \le i,j \le \eta-1$, and $i \ne j$
- $\psi_2(\beta_i,\gamma) \le min\,(\psi_1(\beta_i),\psi_1(\gamma))$,

- $\zeta_2(\beta_i,\gamma) \leq max(\zeta_1(\beta_i),\zeta_1(\gamma))$.

Lemma 5. *In an intuitionistic fuzzy wheel graph W_η with $\beta_i \in V(W_\eta)$, if m_i denotes the membership value and n_i the non-membership value of the vertex β_i, then the membership and non-membership degrees of $\eta-1$ vertices are defined as:*

$$\begin{aligned} d_\psi(\beta_i) &\leq \{m_{i-1}, m_i\}_l + \{m_i, m_{i+1}\}_l + \{m_i, m_\gamma\}_l\,, \\ d_\zeta(\beta_i) &\leq \{n_{i-1}, n_i\}_h + \{n_i, n_{i+1}\}_h + \{n_i, n_\gamma\}_h\,. \end{aligned}$$

Where, $1 \leq i \leq \eta-1, m_\eta = m_1, n_\eta = n_1$ and m_γ, n_γ denote the membership and non-membership values of the central vertex γ which is connected with all other vertices of the cycle $C_{\eta-1}$. The membership and non-membership degrees of this vertex are denoted by $d_\psi(\gamma)$ and $d_\zeta(\gamma)$ and are defined as:

$$\begin{aligned} d_\psi(\gamma) &\leq \sum_{i=1}^{\eta-1}\{m_\gamma, m_i\}_l\,, \\ d_\zeta(\gamma) &\leq \sum_{i=1}^{\eta-1}\{n_\gamma, n_i\}_h\,, \end{aligned}$$

Theorem 9. *Let $W_\eta = \{V, E\}$ be an intuitionistic fuzzy wheel graph with set of vertices $\{\beta_1, \beta_2, \beta_3, \beta_4, \ldots, \beta_{\eta-1}, \gamma\}$, in which the degrees of all the vertices are defined as in Lemma 5. Then*

$$\begin{aligned} \mathcal{SO}_3(W_\eta) = &\sum_{i=1}^{\eta-1}\left[\frac{(m_i,n_i)\left(u_i^2,v_i^2\right) + (m_{i+1},n_{i+1})\left(u_{i+1}^2,v_{i+1}^2\right)}{(m_i,n_i)(u_i,v_i) + (m_{i+1},n_{i+1})(u_{i+1},v_{i+1})}\right]\sqrt{2}\pi + \\ &\left[\frac{(m_{\eta-1},n_{\eta-1})\left(u_{\eta-1}^2,v_{\eta-1}^2\right) + (m_1,n_1)\left(u_1^2,v_1^2\right)}{(m_{\eta-1},n_{\eta-1})(u_{\eta-1},v_{\eta-1}) + (m_1,n_1)(u_1,v_1)}\right]\sqrt{2}\pi + \\ &\sum_{j=1}^{\eta-1}\left[\frac{(m_\gamma,n_\gamma)\left(u_\gamma^2,v_\gamma^2\right) + (m_j,n_j)\left(u_j^2,v_j^2\right)}{(m_\gamma,n_\gamma)(u_\gamma,v_\gamma) + (m_j,n_j)(u_j,v_j)}\right]\sqrt{2}\pi, \end{aligned}$$

where, $\left[\{m_{i-1}, m_i\}_l + \{m_i, m_{i+1}\}_l + \{m_i, m_\gamma\}_l\right] = u_i$, $[\{n_{i-1}, n_i\}_h + n_i, n_{i+1}{}_l + \{n_i, n_\gamma\}_l]$ $= v_i$, $\left[\{m_{\eta-2}, m_{\eta-1}\}_l + \{m_{\eta-1}, m_1\}_l + \{m_{\eta-1}, m_\gamma\}_l\right] = u_{\eta-1}$, $[n_{\eta-2}, n_{\eta-1}{}_h + n_{\eta-1}, n_1{}_h +$ $n_{\eta-1}, n_{\gamma}{}_h] = v_{\eta-1}$, $[m_i, m_{i+1}{}_l + m_{i+1}, m_{i+2}{}_l + m_{i+1}, m_{\gamma}{}_l] = u_{i+1}$, $[n_i, n_{i+1}{}_h + n_{i+1}, n_{i+2}{}_h$ $+ n_{i+1}, n_{\gamma}{}_h] = v_{i+1}$, $[m_1, m_{\eta-1}{}_l + m_1, m_2{}_l + m_1, m_{\gamma}{}_l] = u_1$, $[n_1, n_{\eta-1}{}_h + n_1, n_2{}_h + n_1, n_{\gamma}{}_h]$ $= v_1$, $[\sum_{i=1}^{\eta-1} m_\gamma, m_i{}_l] = u_\gamma$, $[\sum_{i=1}^{\eta-1} n_\gamma, n_i{}_h] = v_\gamma$, $[m_j - 1, m_j{}_l + m_j, m_j + 1_l + m_j, m_{\gamma}{}_l] = u_j$, $[n_j - 1, n_j{}_h + n_j, n_j + 1_h + n_j, n_{\gamma}{}_h] = v_j$, $m_\eta = m_1$, $n_\eta = n_1$.

Proof. Let $\beta_i \in V(W_\eta)$ be the set of $\eta-1$ vertices on the boundary of a wheel graph with m_i and n_i membership and non-membership values and γ be the vertex at the center of wheel graph with membership value m_γ and non-membership value n_γ. Then the following three cases need to be distinguished.

Case 1. First we calculate the third version of intuitionistic fuzzy Sombor index for $\eta - 2$ edges. By applying formula for edge $\beta_1\beta_2$, we have

$$\mathcal{SO}_3(\beta_1\beta_2) \leq \left[\frac{(m_1, n_1)\left(u_1^2, v_1^2\right) + (m_2, n_2)\left(u_2^2, v_2^2\right)}{(m_1, n_1)(u_1, v_1) + (m_2, n_2)(u_2, v_2)}\right]\sqrt{2}\pi,$$

similarly, for edge $\beta_2\beta_3$,

$$\mathcal{SO}_3(\beta_2\beta_3) \leq \left[\frac{(m_2, n_2)\left(u_2^2, v_2^2\right) + (m_3, n_3)\left(u_3^2, v_3^2\right)}{(m_2, n_2)(u_2, [v]_2) + (m_3, n_3)(u_3, v_3)}\right]\sqrt{2}\pi,$$

continuing in the same way, the third version of Sombor fuzzy index for edge $\beta_{\eta-2}\beta_{\eta-1}$ is given by:

$$\mathcal{SO}_3(\beta_{\eta-2}\beta_{\eta-1}) \leq \left[\frac{(m_{\eta-2}, n_{\eta-2})\left(u_{\eta-2}^2, v_{\eta-2}^2\right) + (m_{\eta-1}, n_{\eta-1})\left(u_{\eta-1}^2, v_{\eta-1}^2\right)}{(m_{\eta-2}, n_{\eta-2})(u_{\eta-2}, v_{\eta-2}) + (m_{\eta-1}, n_{\eta-1})(u_{\eta-1}, v_{\eta-1})}\right]\sqrt{2}\pi,$$

By adding all the above results, the third version of the intuitionistic Sombor fuzzy index for $\eta - 2$ edges would be less than or equal to

$$\sum_{i=1}^{\eta-2}\left[\frac{(m_i, n_i)\left(u_i^2, v_i^2\right) + (m_{i+1}, n_{i+1})\left(u_{i+1}^2, v_{i+1}^2\right)}{(m_i, n_i)(u_i, v_i) + (m_{i+1}, n_{i+1})(u_{i+1}, v_{i+1})}\right]\sqrt{2}\pi.$$

In this case, we considered $m_\eta = m_1$.

Case 2. Now $\mathcal{SO}_3(\beta_{\eta-1}\beta_1)$ would be:

$$\mathcal{SO}_3(\beta_{\eta-1}\beta_1) \leq \left[\frac{(m_{\eta-1}, n_{\eta-1})\left(u_{\eta-1}^2, v_{\eta-1}^2\right) + (m_1, n_1)\left(u_1^2, v_1^2\right)}{(m_{\eta-1}, n_{\eta-1})(u_{\eta-1}, v_{\eta-1}) + (m_1, n_1)(u_1, v_1)}\right]\sqrt{2}\pi.$$

Case 3. Now we calculate the third version of the intuitionistic fuzzy Sombor index for the central vertex γ with membership value m_γ and non-membership value n_γ. For the edge $\gamma\beta_1$,

$$\mathcal{SO}_3(\gamma\beta_1) \leq \left[\frac{(m_\gamma, n_\gamma)\left(u_\gamma^2, v_\gamma^2\right) + (m_1, n_1)\left(u_1^2, v_1^2\right)}{(m_\gamma, n_\gamma)(v_\gamma, v_\gamma) + (m_1, n_1)(u_1, v_1)}\right],$$

for edge $\gamma\beta_2$,

$$\mathcal{SO}_3(\gamma\beta_2) \leq \left[\frac{(m_\gamma, n_\gamma)\left(u_\gamma^2, v_\gamma^2\right) + (m_2, n_2)\left(u_2^2, v_2^2\right)}{(m_\gamma, n_\gamma)(v_\gamma, v_\gamma) + (m_2, n_2)(u_2, v_2)}\right].$$

and so on, for the edge $\gamma\beta_{\eta-1}$,

$$\mathcal{SO}_3(\gamma\beta_{\eta-1}) \leq \left[\frac{(m_\gamma, n_\gamma)\left(u_\gamma^2, v_\gamma^2\right) + (m_{\eta-1}, n_{\eta-1})\left(u_{\eta-1}^2, v_{\eta-1}^2\right)}{(m_\gamma, n_\gamma)(u_\gamma, v_\gamma) + (m_{\eta-1}, n_{\eta-1})(u_{\eta-1}, v_{\eta-1})}\right]\sqrt{2}\pi,$$

Hence, for wheel graph,

$$\begin{aligned}\mathcal{SO}_3(W_\eta) \sum_{i=1}^{\eta-2} & \left[\frac{(m_i, n_i)\left(u_i^2, v_i^2\right) + (m_{i+1}, n_{i+1})\left(u_{i+1}^2, v_{i+1}^2\right)}{(m_i, n_i)(u_i, v_i) + (m_{i+1}, n_{i+1})(u_{i+1}, v_{i+1})}\right]\sqrt{2}\pi + \\ & \left[\frac{(m_{\eta-1}, n_{\eta-1})\left(u_{\eta-1}^2, v_{\eta-1}^2\right) + (m_1, n_1)\left(u_1^2, v_1^2\right)}{(m_{\eta-1}, n_{\eta-1})(u_{\eta-1}, v_{\eta-1}) + (m_1, n_1)(u_1, v_1)}\right]\sqrt{2}\pi + \\ \sum_{j=1}^{\eta-1} & \left[\frac{(m_\gamma, n_\gamma)\left(u_\gamma^2, v_\gamma^2\right) + (m_j, n_j)\left(u_j^2, v_j^2\right)}{(m_\gamma, n_\gamma)(u_\gamma, v_\gamma) + (m_j, n_j)(u_j, v_j)}\right]\sqrt{2}\pi,\end{aligned}$$

where, $[m_1, m_{2_l} + m_1, m_{\eta-1_l} + m_1, m_{\gamma_l}] = u_1$, $[n_1, n_{2_h} + n_1, n_{\eta-1_h} + n_1, n_{\gamma_h}] = v_1$, $[m_1, m_{2_l} + m_2, m_{3_l} + m_2, m_{\gamma_l}] = u_2$, $[m_1, m_{2_l} + m_2, m_{3_l} + m_2, m_{\gamma_l}] = v_2$, $[m_2, m_{3_l} + m_3, m_{4_l} + m_3, m_{\gamma_l}] = u_3$, $[n_2, n_{3_h} + n_3, n_{4_h} + n_3, n_{\gamma_h}] = v_3$, $[m_{\eta-3}, m_{\eta-2_l} + m_{\eta-2}, m_{\eta-1_l} + m_{\eta-2}, m_{\gamma_l}] = u_{\eta-2}$, $[n_{\eta-3}, n_{\eta-2_h} + n_{\eta-2}, n_{\eta-1_h} + n_{\eta-2}, n_{\gamma_h}] = v_{\eta-2}$, $m_\eta = m_1$, $n_\eta = n_1$. □

Theorem 10. *Let $W_\eta = \{V, E\}$ be an intuitionistic fuzzy wheel graph with $\{\beta_1, \beta_2, \beta_3, \beta_4, \ldots, \beta_{\eta-1}, \gamma\}$, set of vertices in which degree of all vertices are as defined in Lemma 5. Membership and non-membership values of vertices β_i are denoted by m_i and n_i respectively, where $1 \leq i \leq \eta - 1$. Also m_γ and n_γ denote the membership and non-membership values of central vertex γ, then the fourth version of intuitionistic fuzzy Sombor index for wheel graph is defined as:*

$$\begin{aligned}\mathcal{SO}_4(W_\eta) \leq \frac{1}{2}\sum_{i=1}^{n-2} & \left[\frac{(m_i, n_i)\left(u_i^2, v_i^2\right) + (m_{i+1}, n_{i+1})\left(u_{i+1}^2, v_{i+1}^2\right)}{(m_i, n_i)(u_i, v_i) + (m_{i+1}, n_{i+1})(u_{i+1}, v_{i+1})}\right]^2 \pi + \\ & \frac{1}{2}\left[\frac{(m_i, n_i)\left(u_i^2, v_i^2\right) + (m_{i+1}, n_{i+1})\left(u_{i+1}^2, v_{i+1}^2\right)}{(m_i, n_i)(u_i, v_i) + (m_{i+1}, n_{i+1})(u_{i+1}, v_{i+1})}\right]^2 \pi + \\ & \frac{1}{2}\sum_{j=1}^{\eta-1}\left[\frac{(m_\gamma, n_\gamma)\left(u_\gamma^2, v_\gamma^2\right) + (m_j, n_j)\left(u_j^2, v_j^2\right)}{(m_\gamma, n_\gamma)(u_\gamma, v_\gamma) + (m_j, n_j)(u_j, v_j)}\right]^2 \pi,\end{aligned}$$

where, $m_\eta = m_1$, $n_\eta = n_1$. $[m_{i-1}, m_{i_l} + m_i, m_{i+1_l} + m_i, m_{\gamma_l}] = u_i$, $[n_{i-1}, n_{i_h} + n_i, n_{i+1_l} + n_i, n_{\gamma_l}] = v_i$, $[m_{\eta-2}, m_{\eta-1_l} + m_{\eta-1}, m_{1_l} + m_{\eta-1}, m_{\gamma_l}] = u_{\eta-1}$, $[n_{\eta-2}, n_{\eta-1_h} + n_{\eta-1}, n_{1_h} + n_{\eta-1}, n_{\gamma_h}] = v_{\eta-1}$, $[m_i, m_{i+1_l} + m_{i+1}, m_{i+2_l} + m_{i+1}, m_{\gamma_l}] = u_{i+1}$, $[n_i, n_{i+1_h} + n_{i+1}, n_{i+2_h} + n_{i+1}, n_{\gamma_h}] = v_{i+1}$, $[m_1, m_{\eta-1_l} + m_1, m_{2_l} + m_1, m_{\gamma_l}] = u_1$, $[n_1, n_{\eta-1_h} + n_1, n_{2_h} + n_1, n_{\gamma_h}] = v_1$, $[\sum_{i=1}^{\eta-1} m_\gamma, m_{i_l}] = u_\gamma$, $[\sum_{i=1}^{\eta-1} n_\gamma, n_{i_h}] = v_\gamma$, $[m_j - 1, m_{j_l} + m_j, m_j + 1_l + m_j, m_{\gamma_l}] = u_j$, $[n_j - 1, n_{j_h} + n_j, n_j + 1_h + n_j, n_{\gamma_h}] = v_j$.

Proof. This can be proved similarly by taking steps used in the proof of Theorem 9. □

2.6. Intuitionistic fuzzy star graph and its results

A star graph of order η is a graph with $\eta - 1$ vertices of degree 1 and degree of one vertex is $\eta - 1$. An intuitionistic fuzzy graph $G = \{V, \psi, \zeta\}$, is a star graph if it consists of two vertex sets $\mathcal{H}(V, \psi, \zeta)$, and $\mathcal{I}(V, \psi, \zeta)$, with $|\mathcal{H}| = 1$ and $|\mathcal{I}| = \eta - 1$ where $\beta_i \in \mathcal{I}$, for $1 \le i \le \eta - 1$ and $\alpha \in \mathcal{H}$, such that:

- $\psi_2(\beta_i, \beta_j) = 0$ and $\zeta_2(\beta_i, \beta_j) = 0$,
- $\psi_2(\beta_i, \alpha) \le min\,(\psi_1(\beta_i), \psi_1(\alpha))$ and $\zeta_2(\beta_i, \alpha) \le max\,(\zeta_1(\beta_i), \zeta_1(\alpha))$

Lemma 6. *If S_η is an intuitionistic fuzzy star graph with $V(S_\eta) = \{\beta_1, \beta_2, \ldots, \beta_{\eta-1}, \alpha\}$ and vertex α is connected with remaining $\eta - 1$ vertices, m_i and n_i represent the membership and non-membership values of vertices β_i, where $1 \le i \le \eta - 1$. Also membership and non-membership values of vertex α are denoted by m_α and n_α respectively. Then degrees of these vertices are defined in terms of membership and non-membership degrees, denoted by $d_\psi(\beta_i)$, $d_\psi(\alpha)$, $d_\zeta(\beta_i)$ and $d_\zeta(\alpha)$ respectively for vertices β_i and α.*

$$d_\psi(\alpha_i) \le \sum_{i=1}^{\eta-1} \{m_\alpha, m_i\}_l\,,$$
$$d_\zeta(\alpha_i) \le \sum_{i=1}^{\eta-1} \{n_\alpha, n_i\}_h\,,$$

$$d_\psi(\beta_i) \le \{m_i, m_\alpha\}_l\,,$$
$$d_\zeta(\beta_i) \le \{n_i, n_\alpha\}_h\,,$$

for all $1 \le i \le \eta - 1$.

Theorem 11. *Let $S_\eta = \{V, E\}$ be an intuitionistic fuzzy star graph having vertex set $V = \{\beta_1, \beta_2, \ldots, \beta_{\eta-1}, \alpha\}$, with m_i, n_i as membership and non-membership values of vertex β_i, and m_α, n_α are the membership and non-membership value of vertex α, then*

$$\mathcal{SO}_3(S_\eta) \le \sum_{j=1}^{\eta-1} \left[\frac{(m_j, n_j)\left(\{m_j, m_\alpha\}_l^2, \{n_j, n_\alpha\}_h^2\right) + (m_\alpha, n_\alpha)\,\chi_{i,\eta}^2}{(m_j, n_j)\left(\{m_j, m_\alpha\}_l, \{n_j, n_\alpha\}_h\right) + (m_\alpha, n_\alpha)\,\chi_{i,\eta}^1} \right] \sqrt{2}\pi$$

where $\chi^r_{i,\eta} = \left(\left[\sum_{i=1}^{\eta-1} \{m_i, m_\alpha\}_l\right]^2, \left[\sum_{i=1}^{\eta-1} \{n_i, n_\alpha\}_h\right]^2 \right)$ *for* $r \in \{1, 2\}$.

Proof. Let $V = \{\beta_1, \beta_2, \ldots, \beta_{\eta-1}, \alpha\}$ be the set of vertices of an intuitionistic fuzzy star graph S_η, where the vertex α is attached to all $\beta_i's$ such that $1 \leq i \leq n-1$. The membership and non-membership values are as defined in Lemma 6.
Now, the third version of intuitionistic Sombor fuzzy index for the edge $\beta_1\alpha$ is:

$$\mathcal{SO}_3(\beta_1\alpha) \leq \left[\frac{(m_1, n_1)\left[\{m_1, m_\alpha\}_l^2, \{n_1, n_\alpha\}_h^2\right] + (m_\alpha, n_\alpha)\,\chi^2_{i,\eta}}{(m_1, n_1)\left[\{m_1, m_\alpha\}_l, \{n_1, n_\alpha\}_h\right] + (m_\alpha, n_\alpha)\,\chi^1_{i,\eta}} \right] \sqrt{2}\pi.$$

For edge $\beta_2\alpha$,

$$\mathcal{SO}_3(\beta_2\alpha) \leq \left[\frac{(m_2, n_2)\left[\{m_2, m_\alpha\}_l^2, \{n_2, n_\alpha\}_h^2\right] + (m_\alpha, n_\alpha)\,\chi^2_{i,\eta}}{(m_2, n_2)\left[\{m_2, m_\alpha\}_l, \{n_2, n_\alpha\}_h\right] + (m_\alpha, n_\alpha)\,\chi^1_{i,\eta}} \right] \sqrt{2}\pi.$$

Continuing in the same way, the third version of intuitionistic Sombor fuzzy index for the edge $\beta_{\eta-1}\alpha$ is

$$\mathcal{SO}_3(\beta_{\eta-1}\alpha) \leq \left[\frac{(m_{\eta-1}, n_{\eta-1})\left[\{m_{\eta-1}, m_\alpha\}_l^2, \{n_{\eta-1}, n_\alpha\}_h^2\right] + (m_\alpha, n_\alpha)\,\chi^2_{i,\eta}}{(m_{\eta-1}, n_{\eta-1})\left[\{m_{\eta-1}, m_\alpha\}_l, \{n_{\eta-1}, n_\alpha\}_h\right] + (m_\alpha, n_\alpha)\,\chi^1_{i,\eta}} \right] \sqrt{2}\pi.$$

Hence, by adding all above inequalities, we have

$$\mathcal{SO}_3(S_\eta) \leq \sum_{j=1}^{\eta-1} \left[\frac{(m_j, n_j)\left[\{m_j, m_\alpha\}_l^2, \{n_j, n_\alpha\}_h^2\right] + (m_\alpha, n_\alpha)\,\chi^2_{i,\eta}}{(m_j, n_j)\left[\{m_j, m_\alpha\}_l, \{n_j, n_\alpha\}_h\right] + (m_\alpha, n_\alpha)\,\chi^1_{i,\eta}} \right] \sqrt{2}\pi.$$

Hence, proved. □

Theorem 12. *Let $S_\eta = \{V, E\}$ be an intuitionistic fuzzy star graph with vertex set $V = \{\beta_1, \beta_2, \ldots, \beta_{\eta-1}, \alpha\}$,and m_i, n_i are the membership and non-membership values of vertex β_i, and m_α, n_α are the membership and non-membership value of vertex α, then thee fourth version of intuitionistic fuzzy Sombor index for star graph is given by:*

$$\mathcal{SO}_4(S_\eta) \leq \frac{1}{2} \sum_{j=1}^{n-1} \left[\frac{(m_j, n_j)\left[\{m_j, m_\alpha\}_l^2, \{n_j, n_\alpha\}_h^2\right] + (m_\alpha, n_\alpha)\,\chi^2_{i,\eta}}{(m_j, n_j)\left[\{m_j, m_\alpha\}_l, \{n_j, n_\alpha\}_h\right] + (m_\alpha, n_\alpha)\,\chi^1_{i,\eta}} \right]^2 \pi.$$

Proof. This can be proved similarly by taking steps used in the proof of Theorem 11. □

3. Application

In this section, we will present an application of the third and fourth versions of intuitionistic fuzzy Sombor indices. Suppose during a pandemic, Government has decided to set up some vaccination centers in different parts of the country in order to vaccinate the people in a quick and efficient way. For this purpose, experts checked all possible ways of building these centers, in such a way that the number of beneficiaries is maximum and entertained in the best possible way and people do not have to deal with any form of staff or medication shortages. They can use graph theory to help with this. The third and fourth versions of intuitionistic fuzzy Sombor indices would be quite useful for this purpose.

3.1. Network of vaccination centers

Experts are interested in finding out which network is more efficient. So they analyzed different types of graphical networks by considering vertices as vaccination centers to get an efficient network so that the maximum number of people get vaccinated within a short period without facing serious issues.
First, consider a network of six vaccination centers in the form of a cycle graph shown in Fig 1. Here vertices represent the vaccination centers, links or edges between them denote the distances between them.

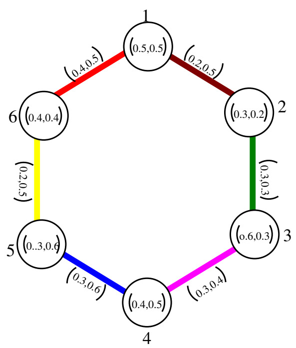

Figure 1. **Network of vaccination centers in the form of cycle G_1**

For all the vaccination centers in the system, the membership and non-membership values of each vertex provide information about the performance of each vaccination center. If the membership value is greater than the non-membership value, then that center performs more efficiently. The overall performance of the network can be calculated with the help of the third and fourth versions of intuitionistic fuzzy Sombor indices. Here, the product of two ordered pairs is obtained by applying the dot product formula. First, the square of the degrees of all vertices is calculated as:

$$d^2(1) = \left(\{0.2 + 0.4\}^2, \{0.5 + 0.5\}^2\right) = (0.36, 1),$$
$$d^2(2) = \left(\{0.2 + 0.3\}^2, \{0.3 + 0.5\}^2\right) = (0.25, 0.64),$$
$$d^2(3) = \left(\{0.3 + 0.3\}^2, \{0.3 + 0.7\}^2\right) = (0.36, 0.49),$$
$$d^2(4) = \left(\{0.3 + 0.3\}^2, \{0.6 + 0.4\}^2\right) = (0.36, 1),$$
$$d^2(5) = \left(\{0.3 + 0.2\}^2, \{0.6 + 0.5\}^2\right) = (0.25, 1.21),$$
$$d^2(6) = \left(\{0.2 + 0.4\}^2, \{0.5 + 0.5\}^2\right) = (0.36, 1).$$

Now applying the formula of the third version of the intuitionistic fuzzy Sombor index, we have:

$$\begin{aligned}\mathcal{SO}_3(G_1) =& \Bigg(\left[\frac{(0.5,0.5)(0.36,1.0)+(0.3,0.2)(0.25,0.64)}{(0.5,0.5)(0.6,1.0)+(0.3,0.2)(0.5,0.8)}\right] + \left[\frac{(0.3,0.2)(0.25,0.64)+(0.6,0.3)(0.36,0.49)}{(0.3,0.2)(0.5,0.8)+(0.6,0.3)(0.6,0.7)}\right] \\ &+ \left[\frac{(0.6,0.3)(0.36,0.49)+(0.4,0.5)(0.36,1)}{(0.6,0.3)(0.6,0.7)+(0.4,0.5)(0.6,1)}\right] + \left[\frac{(0.4,0.5)(0.36,1)+(0.3,0.6)(0.25,1.21)}{(0.4,0.5)(0.6,1)+(0.3,0.6)(0.5,1.1)}\right] + \\ &\left[\frac{(0.3,0.6)(0.25,1.21)+(0.4,0.4)(0.36,1)}{(0.3,0.6)(0.5,1.1)+(0.4,0.4)(0.6,1)}\right] + \left[\frac{(0.4,0.4)(0.36,1)+(0.5,0.5)(0.36,1.0)}{(0.4,0.4)(0.6,1)+(0.5,0.5)(0.6,1.0)}\right]\Bigg)\sqrt{2}\pi.\end{aligned}$$

Hence, $\mathcal{SO}_3(G_1) = 21.24$. Now, the value of the fourth version of the intuitionistic fuzzy Sombor index is: $\mathcal{SO}_4(G_1) = 6.123$. Now experts add one more vertex, assuming a new vaccination center is being added to the system, as shown in Figure 2.

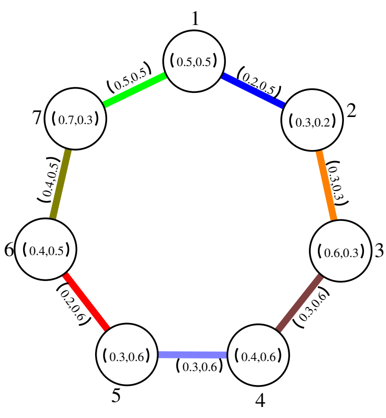

Figure 2. **Network of vaccination center with the addition of a new center G_2**

Now

$$d^2(1) = \left(\{0.2 + 0.5\}^2, \{0.5 + 0.5\}^2\right) = (0.49, 1),$$
$$d^2(2) = \left(\{0.2 + 0.3\}^2, \{0.3 + 0.5\}^2\right) = (0.25, 0.64),$$
$$d^2(3) = \left(\{0.3 + 0.3\}^2, \{0.3 + 0.6\}^2\right) = (0.36, 0.81),$$

$$d^2(4) = \left(\{0.3+0.3\}^2, \{0.6+0.6\}^2\right) = (0.36, 1.44),$$
$$d^2(5) = \left(\{0.3+0.2\}^2, \{0.6+0.6\}^2\right) = (0.25, 1.44),$$
$$d^2(6) = \left(\{0.2+0.4\}^2, \{0.6+0.5\}^2\right) = (0.36, 1.21),$$
$$d^2(7) = \left(\{0.5+0.4\}^2, \{0.5+0.5\}^2\right) = (0.81, 1).$$

By putting all the values in equations (1.2) and (1.3), the results so obtained provide information about the performance of the network of seven vaccination centers, i.e. $\mathcal{SO}_3(G_2) = 29.197$ and $\mathcal{SO}_4(G_2) = 9.813$.
In the next arrangement, one vaccination center is considered the hub vertex of a wheel graph, i.e., it has a connection with the remaining five centers and serves as the headquarter to provide necessary medicine and to fulfill the deficiency of staff where ever required, as shown in Figure 3.

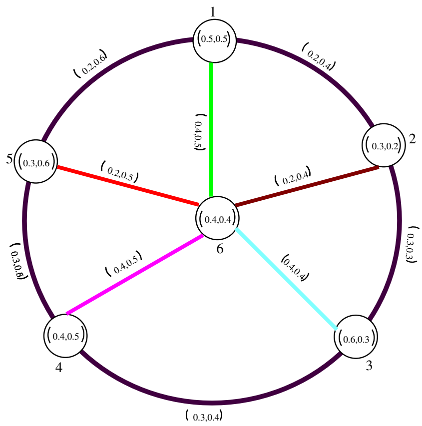

Figure 3. **Network of vaccination center as a wheel graph G_3**

then,

$$d^2(1) = \left(\{0.2+0.2+0.4\}^2, \{0.4+0.5+0.6\}^2\right) = (0.64, 2.25),$$
$$d^2(2) = \left(\{0.2+0.3+0.2\}^2, \{0.4+0.3+0.4\}^2\right) = (0.49, 1.21),$$
$$d^2(3) = \left(\{0.3+0.4+0.3\}^2, \{0.3+0.6+0.4\}^2\right) = (1, 1.69),$$
$$d^2(4) = \left(\{0.3+0.4+0.3\}^2, \{0.5+0.4+0.6\}^2\right) = (1, 2.25),$$
$$d^2(5) = \left(\{0.3+0.2+0.2\}^2, \{0.6+0.5+0.6\}^2\right) = (0.49, 2.89),$$
$$d^2(6) = \left(\{0.4+0.2+0.4+0.4+0.2\}^2, \{0.5+0.4+0.4+0.6+0.7\}^2\right) = (2.56, 5.29).$$

Then, for wheel graph, $\mathcal{SO}_3(G_3) = 68.227$ and $\mathcal{SO}_4(G_3) = 37.89$.
Next, the system of six vaccination centers is built in the form of a complete graph, so that all vaccination centers are linked with each other to facilitate communication in case of an emergency, as shown in Figure 4. In this graph, degrees of vertices are

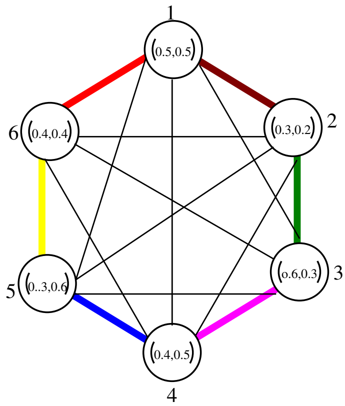

Figure 4. **Network of vaccination centers as complete graph** G_4

calculated by definition of complete fuzzy graph.

$$d^2(1) = \left(\{0.3+0.4+0.4+0.5+0.3\}^2, \{0.5+0.5+0.5+0.5+0.6\}^2\right) = (3.61, 6.76),$$
$$d^2(2) = \left(\{0.2+0.3+0.2+0.2+0.2\}^2, \{0.3+0.5+0.5+0.7+0.4\}^2\right) = (2.25, 5.29),$$
$$d^2(3) = \left(\{0.3+0.4+0.3+0.3+0.4\}^2, \{0.3+0.6+0.4+0.5+0.6\}^2\right) = (3.61, 5.29),$$
$$d^2(4) = \left(\{0.3+0.4+0.3+0.2+0.4\}^2, \{0.6+0.6+0.6+0.5+0.5\}^2\right) = (3.61, 7.84),$$
$$d^2(5) = \left(\{0.3+0.2+0.2+0.3+0.3\}^2, \{0.6+0.7+0.6+0.4+0.6\}^2\right) = (1.69, 8.41),$$
$$d^2(6) = \left(\{0.4+0.2+0.4+0.4+0.2\}^2, \{0.5+0.4+0.4+0.6+0.7\}^2\right) = (2.56, 6.76).$$

The performance of the network is calculated by the third and fourth versions of the intuitionistic fuzzy Sombor index, $\mathcal{SO}_3(G_4) = 153.29$ and $\mathcal{SO}_4(G_4) = 125.376$.
By comparing all of the computed values, it is possible to draw the conclusion that, even with the construction of two or three more vaccination centers, a complete graph significantly increases the number of beneficiaries and greatly improves the efficiency of the network of vaccination centers. The cost of opening new immunization centers is high since they need new infrastructure, employees, and other resources. When a graph of connections is fully formed, every center is connected to every other center, making it simple to go to any emergency. People receive vaccinations swiftly and effectively as a result.

4. Conclusion

In this paper, we introduce two topological indices for intuitionistic fuzzy graphs and then calculated these indices for different kinds of graphical structures. We also discussed the application of setting up a network of vaccination centers during a pandemic in an intuitionistic fuzzy framework and concluded that the performance of vaccination centers increases much more in the case of the complete graph. It is also concluded that the third version of the intuitionistic fuzzy Sombor index gives much better results than the fourth version. Also, the strength of the network is increased for both indices in the case of the complete graph.

5. Future Directions and Open Problems

In this section, we will guide and give direction to researchers in fuzzy graph theory, particularly those working in topological descriptors of different fuzzy environments. For different fuzzy environments, one can see the articles of [14, 41, 42, 46, 51, 63].

- This research work can be conducted for the different fuzzy environments like, picture fuzzy (three degree, like membership, abstinence and non-membership), and others as well.
- This research can also be extended for the other variants of fuzzy Sombor indices.

Conflict of Interest: The authors declare that there is no competing interest related to this paper.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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