Communications in Combinatorics and Optimization
Vol. 10, No. 2 (2025), pp. 371-379
https://doi.org/10.22049/cco.2023.28880.1760

Research Article

Polycyclic codes over R

Gowdhaman Karthick

Presidency University, Bangalore, Karnatakka-560064, India
karthygowtham@gmail.com

Received: 28 July 2023; Accepted: 16 November 2023
Published Online: 28 November 2023

Abstract: In this paper, we discuss the structure of polycyclic codes over the ring $R = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q; u^2 = \alpha u, v^2 = v$ and $uv = vu = 0$, where α is an unit element in R. We introduce annihilator self-dual codes, annihilator self-orthogonal codes and annihilator LCD codes over R. Using a Gray map, we define a one to one correspondence between R and $\mathbb{F}_q$ and construct quasi polycyclic codes over the $\mathbb{F}_q$.

Keywords: semi-simple ring, polycyclic codes, hamming distances, gray maps, annihilator dual codes.

AMS Subject classification: 05C50, 05C09, 05C92

1. Introduction

An interesting subtype of linear codes are polycyclic codes of length n over a finite field $\mathbb{F}_q$ with q elements which are described by ideals of a polynomial rings $\mathbb{F}_q[x]/\langle f(x)\rangle$. In 2009, López-Permouth et al. [3] studied polycyclic codes and sequential codes, and showed that a linear code is polycyclic if and only if its Euclidean dual code is sequential which is not always polycyclic. In 2016, Alahmadi et al. [1] introduced the annihilator dual codes over $\mathbb{F}_q$ and showed that the annihilator dual codes of polycyclic codes over $\mathbb{F}_q$ are also polycyclic. In 2022, Wei Qi study the polycyclic codes over $\mathbb{F}_q + u\mathbb{F}_q$ with $u^2 = u$ and have constructed the annihilator self-dual codes, annihilator self-orthogonal codes and annihilator LCD codes. This motivated us to do the following works.
In this paper, we study Polycyclic codes and Sequential codes over the ring $R = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q; u^2 = \alpha u, v^2 = v$ and $uv = vu = 0$. We have introduced annihilator self-dual codes, annihilator self-orthogonal codes and annihilator LCD codes over R. Using a Gray map, we have defined a one to one correspondance between $\{1$, R and $\mathbb{F}_q^3\}$ and a few codes are constructed.

2. Preliminaries

Let $\mathbb{F}_q$ be a finite field of order q with characteristic p, then we define a ring $R = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$ with $u^2 = \alpha u, v^2 = v, uv = vu = 0$ where α is an unit element in R. The ring R is a semi-local and Frobenious ring. A linear code C is a R-module. $C^{\perp}$ is the Eucleadean dual of C. Let $e_1 = \frac{u}{\alpha},\ e_2 = v$ and $e_3 = (1 - \frac{u}{\alpha} - v)$. Then, we have $e_i^2 = e_i, e_ie_j = 0$ and $\sum_{i=1}^{3} e_i = 1$ where $i = 1, 2, 3$ and $i \neq j$. By using decomposition theorem of rings, we have $R = \bigoplus_{i=1}^{3} e_iR \cong \bigoplus_{i=1}^{3} e_i\mathbb{F}_q$. Therefore, any element in R can be uniquely expressed as $r = \sum_{i=1}^{3} e_ir_i$ where $r_i \in \mathbb{F}_q$.
Let C be a linear code of length n over R and $C_i = \{r_i \in \mathbb{F}_q^n \mid \ \sum_{i=1}^{3} e_ir_i \in C\}$ for some $r_j \in \mathbb{F}_q^n$ where $j \neq i$. Then C_i is a linear code of length n over $\mathbb{F}_q$ for $1 \leq i \leq 3$. Hence, C can be expressed as $C = \bigoplus_{i=1}^{3} e_iC_i$.

Definition 1. Let C be a linear code over R and let $a = (a_0, a_1, \ldots, a_{n-1}) \in R^n$ with the condition that a_0 as a unit element of R

- then C is *a-polycyclic code* if it satisfies the right polycyclic shift operator given by

$$\sigma_a(c_0, c_1, \ldots, c_{n-1}) = (0, c_1, c_2, \ldots, c_{n-2}) + c_{n-1}(a_0, a_1, \ldots, a_{n-1})$$

- then C is *a-sequential code* if it satisfies the right sequential shift operator given by

$$\tau_a(c_0, c_1, \ldots, c_{n-1}) = (c_1, c_2, \ldots, c_{n-1}, c_0a_0 + c_1a_1 + \cdots + c_{n-1}a_{n-1}).$$

Hereafter, we denote $R[x]/\langle x^n - a(x)\rangle$ as R^a. Then the map $\phi : R^n \longrightarrow R^a$ defined by

$$(c_0, c_1, c_2, \ldots, c_{n-1}) \mapsto c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1},$$

is a module isomorphism and we have the following result.

Theorem 1. *Let C be a polycyclic code over the ring R, then the corresponding image sets ϕ is an $R[x]$-module over R^a.*

Definition 2 ([4]). Let C be a polycyclic code of length n.

1. Let $\alpha(x), \beta(x) \in R^a$, then the annihilator product of $\alpha(x)$ and $\beta(x)$ is defined as

$$\langle \alpha(x), \beta(x)\rangle_a = r(0)$$

   where $\alpha(x)\beta(x) \equiv r(x)(\mod x^n - a(x))$ and $deg(r(x)) \leq n - 1$.

2. The annihilator dual code C' of an a-polycyclic code C is defined to be

$$C' = \{\beta(x) \in R^a \mid \langle \alpha(x), \beta(x)\rangle_a = r(0) = 0 \text{ for all } \alpha(x) \in C\}.$$

3. The a-polycyclic code C is said to be an *annihilator self-orthogonal code (resp., annihilator self-dual code, annihilator LCD code)* provided that $C \subseteq C'$ (resp., $C = C', C \cap C' = \{0\}$).

4. The annihilator of C is

$$Ann(C) = \{\beta(x) \in R_a \mid \alpha(x)\beta(x) = 0 \in R^a \text{ for all } \alpha(x) \in C\}.$$

Theorem 2. *[[4]] Let C be an a-polycyclic code of length n over $\mathbb{F}_q$. Let $g(x)$ be the generator polynomial and $h(x)$ the check polynomial of C, then $C' = \langle h(x) \rangle$.*

Lemma 1 ([1]). *Let $a = (a_0, a_1, \cdots, a_{n-1}) \in \mathbb{F}_q^n$ with $a_0 \neq 0$, C be an a-polycyclic code of length n over $\mathbb{F}_q$, then $\alpha(x)\beta(x)$ is non-degenerate, and thus $C' = Ann(C)$.*

Lemma 2 ([1]). *Let C_1 and C_2 be a-polycyclic codes over $\mathbb{F}_q$, g_1, g_2 their generator polynomials, respectively, then $C_1 \subseteq C_2$ if and only if $g_2 | g_1$.*

Lemma 3 ([1]). *Let C be an a-polycyclic code over $\mathbb{F}_q$, then C is an annihilator self-orthogonal code if and only if $h(x)|g(x)$ where $g(x)$ and $h(x)$ are the generator polynomial and check polynomial of C, respectively.*

Lemma 4 ([1]). *Let C be an a-polycyclic code over $\mathbb{F}_q$, then C is an annihilator LCD code if and only if* $\gcd(g(x), h(x)) = 1$ *where $g(x)$ and $h(x)$ are the generator polynomial and check polynomial of C, respectively.*

3. Codes over the ring R

A unique representation of an element in R is defined as $r = r_1e_1 + r_2e_2 + r_3e_3$. Each coordinate a_j in $a = (a_0, a_1, \ldots, a_{n-1})$ can be written as $a_j = a_j^1 e_1 + a_j^2 e_2 + a_j^3 e_3,\ 1 \leq j \leq n-1$ in a unique way and c_j in $c = (c_0, c_1, \ldots, c_{n-1}) \in C$ as $c_j = c_j^1 e_1 + c_j^2 e_2 + c_j^3 e_3,\ 1 \leq j \leq n-1$. On applying the polycyclic operator,

$$\begin{aligned}
\sigma_a(c) &= (0, c_1, c_2, \ldots, c_{n-2}) + c_{n-1}(a_0, a_1, \ldots, a_{n-1}) \\
&= (0, c_1^1e_1 + c_1^2e_2 + c_1^3e_3, c_2^1e_1 + c_2^2e_2 + c_2^3e_3, \ldots, c_{n-2}^1e_1 + c_{n-2}^2e_2 + c_{n-2}^3e_3) \\
&\quad +(c_{n-1}^1e_1 + c_{n-1}^2e_2 + c_{n-1}^3e_3)(a_0^1e_1 + a_0^2e_2 + a_0^3e_3, a_1^1e_1 + a_1^2e_2 + a_1^3e_3, \cdots, \\
&\quad a_{n-1}^1e_1 + a_{n-1}^2e_2 + a_{n-1}^3e_3) \\
&= (0, c_1^1e_1, e_1c_2^1, \ldots, e_1c_{n-2}^1) + e_1c_{n-1}^1(a_0^1e_1, a_1^1e_1, \ldots, a_{n-1}^1e_1) \\
&\quad +(0, c_1^2e_2, e_2c_2^2, \ldots, e_2c_{n-2}^2) + e_2c_{n-1}^2(a_0^2e_2, a_1^2e_2, \ldots, a_{n-1}^2e_2) \\
&\quad +(0, c_1^3e_3, e_3c_2^3, \ldots, e_3c_{n-2}^3) + e_3c_{n-1}^3(a_0^3e_3, a_1^3e_3, \ldots, a_{n-1}^3e_3) \\
&= e_1((0, c_1^1, c_2^1, \ldots, c_{n-2}^1) + c_{n-1}^1(a_0^1, a_1^1, \ldots, a_{n-1}^1)) \\
&\quad +e_2((0, c_1^2, c_2^2, \ldots, c_{n-2}^2) + c_{n-1}^2(a_0^2, a_1^2, \ldots, a_{n-1}^2)) \\
&\quad +e_3((0, c_1^3, c_2^3, \ldots, c_{n-2}^3) + c_{n-1}^3(a_0^3, a_1^3, \ldots, a_{n-1}^3)) \\
&= e_1(\sigma_{a_1}(c^1)) + e_2(\sigma_{a_2}(c^2)) + e_3(\sigma_{a_3}(c^3)).
\end{aligned}$$

Thus, $\sigma_{a_1}(c^1) \in C_1, \sigma_{a_2}(c^2) \in C_2$ and $\sigma_{a_3}(c^3) \in C_3$ and vice versa. Thus, we have the following Theorem.

Theorem 3. *Let C be a linear code over R of length n, then C is an a-polycyclic code of length n if and only if every C_i is an a_i-polycyclic codes over $\mathbb{F}_q$ $(1 \leq i \leq 3)$.*

Theorem 4. *Let C be a linear code of length n over R, then C is a-sequential over R if and only if every C_i is a_i-sequential over $\mathbb{F}_q$.*

Proof. Proof is similar to that of Theorem 3. □

Lemma 5. *Let C be an a-polycyclic code of length n over $\mathbb{F}_q$, then C is a principal ideal $\langle g(x)\rangle$ of $\mathbb{F}_q[x]/\langle x^n - a(x)\rangle$ generated by some monic polynomial and a divisor of $x^n - a(x)$. In this case, $g(x)$ is said to be a generator polynomial of C.*

Theorem 5. *Let $C = \bigoplus_{i=1}^{3} e_iC_i$ be a a-polycyclic code of length n over R, then $C = \langle g(x) = e_1g_1(x) + e_2g_2(x) + e_3g_3(x)\rangle$ of $R[x]/\langle x^n - a(x)\rangle$ where $g_i(x) = \langle C_i\rangle, g_i(x)|x^n - a_i(x), 1 \leq i \leq 3$ over $\mathbb{F}_q$.*

Proof. Let $C = \bigoplus_{i=1}^{3} e_iC_i$ be an a-polycyclic code over R. Let $c(x) \in C = \bigoplus_{i=1}^{3} e_iC_i$, then there exists $p_i(x) \in \mathbb{F}_q[x]/\langle x^n - a_i(x)\rangle$ such that

$$\sum_{i=1}^{3} e_ip_i(x)g_i(x) = c(x)$$

$$\left(\sum_{i=1}^{3} e_ip_i(x)\right)\left(\sum_{i=1}^{3} e_ig_i(x)\right) = c(x)$$

Then $c(x) \in \langle g(x)\rangle, \langle g(x)\rangle \subseteq \bigoplus_{i=1}^{3} e_iC_i$.
Let $C = \bigoplus_{i=1}^{3} e_iC_i$ be a a-polycyclic code over R, then by Theorem 3, C_i is a_i-polycyclic code of length n over $\mathbb{F}_q$. So by Lemma 5, we have $g_i(x) = \langle C_i\rangle$ and $g_i(x)|x^n - a_i(x)$. Then there exists $h_i(x) \in R[x]/\langle x^n - a_i(x)\rangle$ such that $g_i(x)h_i(x) = x^n - a_i(x)$. Therefore $e_ig_i(x)h_i(x) = e_i(x^n - a_i(x))$ and hence

$$\begin{aligned}\sum_{i=1}^{3} e_ig_i(x)h_i(x) &= x^n - a(x)\\ \left(\sum_{i=1}^{3} e_ig_i(x)\right)\left(\sum_{i=1}^{3} e_ih_i(x)\right) &= x^n - a(x).\end{aligned}$$

Thus, we have $C = \langle \sum_{i=1}^{3} e_ig_i(x)h_i(x)\rangle$. □

Theorem 6 ([2]). *If $f(0) \neq 0$, then the bilinear form $\langle ., .\rangle$ is non degenerate.*

Theorem 7. *Let $\alpha(x), \beta(x) \in R^a$. Then $\langle \alpha(x), \beta(x)\rangle$ is a non-degenerate symmetric R-bilinear form.*

Proof. For any $\alpha, \beta, \gamma \in R^n,\ k \in R$, $\langle k(\alpha+\beta), \gamma\rangle = r(0)$,

$$\begin{aligned} \text{where } [k(\alpha+\beta)\gamma](x) &\equiv r(x)(\mod x^n - a(x)) \\ k[\alpha(x)\gamma(x)] + k[\beta(x)\gamma(x)] &\equiv r(x)(\mod x^n - a(x)) \end{aligned}$$

on the other hand,

$$\begin{aligned} \langle k\alpha(x), \gamma(x)\rangle = r_1(0) \text{ where } k[\alpha(x)\gamma(x)] &\equiv r_1(0) \mod x^n - a(x), \\ \langle k\beta(x), \gamma(x)\rangle = r_1(x) \text{ where } k[\beta(x)\gamma(x)] &\equiv r_2(x) \mod x^n - a(x), \end{aligned}$$

using the property compatibility with addition, we have $r(x) = r_1(x) + r_2(x)$. Thus, $\langle k(\alpha+\beta), \gamma\rangle = k\langle\alpha,\gamma\rangle + k\langle\beta,\gamma\rangle$ is bilinear. Since the ring R is commutative, we have $\langle\beta,\gamma\rangle = \langle\gamma,\beta\rangle$. To show $\langle .,.\rangle$ is non-degenerate, it is enough to show that the Radicals of R is $\{0\}$. Suppose not, that is, there exists $\beta \neq 0 \in R(R^n)$ such that $\langle\alpha,\beta\rangle = 0$ for all $\alpha \in R$. Since $\alpha, \beta \in R^n$, it can be uniquely represented by $\alpha = e_1\alpha_1 + e_2\alpha_2 + e_3\alpha_3,\ \alpha = e_1\beta_1 + e_2\beta_2 + e_3\beta_3$. Therefore, by using the bilinear property, one can write $\langle\alpha,\beta\rangle = 0$ as

$$\langle\alpha,\beta\rangle = \sum_{i=1}^{3} e_i\langle\alpha_i,\beta_i\rangle = 0,$$

which contradicts 6. Thus, $\langle .,.\rangle_a$ is a non-degenrate symmetric R-bilinear form. □

Theorem 8. *Let C be an a-polycyclic code over S and let $\epsilon_1 = (1,0,\cdots,0), \epsilon_2 = (0,1,\cdots,0), \cdots, \epsilon_n = (0,0,\cdots,1)$ and $A = (\langle\epsilon_i,\epsilon_j\rangle_a) 1 \leq i,j \leq n$. Let $CA = \{cA \mid c \in C\}$. Then $C' = (CA)^\perp$. Consequently, $(C')' = C$.*

Proof. Note that $\langle u,v\rangle_a = uAv^t = \langle u, Av\rangle_a$. Thus $C' = (CA)^\perp$. Using the equality, $C' = (CA)^\perp$. Since A is invertible, it follows that $(C')' = (C'A)^\perp = (C')^\perp A^{-1} = ((CA)^\perp)^\perp A^{-1} = C$. □

Theorem 9. *Let C be a polycyclic code of length n. Then $C' = e_1C_1' \bigoplus e_2C_2' \bigoplus e_3C_3'$.*

Proof. Since every element in $d \in R$ can be represented as $d = e_1d_1 + e_2d_2 + e_3d_3$, it can be written as a matrix A uniquely as $A = e_1A_{e_1} + e_2A_{e_2} + e_3A_{e_3}$ where every A_{e_i} is a matrix over $\mathbb{F}_q$. Consider

$$\begin{aligned} (C') &= (e_1C_1 \bigoplus e_2C_2 \bigoplus e_3C_3)^\perp (e_1A_{e_1} + e_2A_{e_2} + e_3A_{e_3})^{-1} \\ &= (e_1C_1A_{e_1} \bigoplus e_2C_2A_{e_2} \bigoplus e_3C_3A_{e_3}) \\ &= e_1C_1' \bigoplus e_2C_2' \bigoplus e_3C_3' \end{aligned}$$

Thus, $C' = e_1C_1' \bigoplus e_2C_2' \bigoplus e_3C_3'$. □

Theorem 10. *Let C be a linear code over R. Then C is a-polycyclic if and only if C' is a-polycyclic.*

Proof. Since C is a polycyclic code over R, by Theorem 3, every C_i is a polycyclic codes over $\mathbb{F}_q$. Then, by [[2], Proposition 3], we have C_i' as polycyclic code over $\mathbb{F}_q$ and again by Theorem 3 it is obvious that C' is a polycyclic codes. □

Theorem 11. *Let C be a linear code of length n over R. Then C is an a-polycyclic code over R if and only if $C^{\perp}$ is an a-sequential code over R.*

Proof. By Theorem 3 if C is an a-polycyclic codes then every C_i is a a_i-polycyclic code over $\mathbb{F}_q$. By Theorem[3.2] in [3], every C_i is a a_i-polycyclic code over $\mathbb{F}_q$ if and only if every $C_i^{\perp}$ is a a_i- sequential code over $\mathbb{F}_q$. Thus by from Theorem4 $C^{\perp}$ is an a-sequential code. □

Theorem 12. *Let C be an a-polycyclic code over R generated by $g(x)$. Suppose $h(x)$ is a check polynomial of C. Then C' is an a-polycyclic code generated by $h(x)$.*

Proof. It follows from the proof of Theorems 10 and 5. □

4. Gray map

In this section, we define a Gray map from R to $\mathbb{F}_q^3$. We have shown that Gray map enjoy certain properties. Let $x = x_1e_1 + x_2e_2 + x_3e_3 \in R$, then we define $\phi : R \longrightarrow \mathbb{F}_q^3$ by

$$\phi(x_1e_1 + x_2e_2 + x_3e_3) = (x_1, x_2, x_3).$$

It can be easily extended to any length n. Define $\Phi : R^n \mapsto \mathbb{F}_q^{3n}$ by

$$\text{by } \Phi(c_0, c_1, \ldots, c_{n-1}) = (\phi(c_0), \phi(c_1), \ldots, \phi(c_{n-1})).$$

The Gray weight w_G of $c \in R^n$ is defined by $w_G(c) = \sum_{i=0}^{n-1} w_G(c_i) = \sum_{i=0}^{n-1} w_H(\phi(c_i))$, where w_H is the Hamming weight in $\mathbb{F}_q$, and the distance between two codewords $c, d \in C$ is $d_G(c, d) = w_G(c - d)$. The minimum Gray distance of C is

$$d_G(C) = \min\{w_G(c) \mid 0 \neq c \in C\}.$$

For any two elements $c, d \in R^n$, $d_G(c, d) = w_G(c - d) = w_H(\Phi(c - d)) = w_H(\Phi(c) - \Phi(d)) = d_H(\Phi(c), \Phi(d))$. Hence, Φ is a linear distance preserving map from (R^n, d_G) to (F_q^{3n}, d_H).

Theorem 13. *Let $C = \bigoplus_{i=1}^{3} e_i C_i$ be a linear code with parameter $[n, k, d_G]$, then $\Phi(C)$ is a linear code over $\mathbb{F}_q^{3n}$ with the parameter $[3n, k, d_H]$.*

Definition 3. Let C be a linear code and let $a = a^1 e_1 + a^2 e_2 + a^3 e_3 \in R$, then C is called a-quasicyclic code of index 3 over $\mathbb{F}_q$ if it satisfies the shift operator given by

$$\begin{aligned}\tau^3(x_0, x_1, \ldots x_{n-1}, y_0, y_1, \ldots y_{n-1}, z_0, z_1, \ldots z_{n-1}) = \; & ((0, x_1, x_2, \ldots, x_{n-2}) + x_{n-1}(a_0^1, a_1^1, \ldots, a_{n-1}^1),\\ & (0, y_1, y_2, \ldots, y_{n-2}) + y_{n-1}(a_0^2, a_1^2, \ldots, a_{n-1}^2),\\ & (0, z_1, z_2, \ldots, z_{n-2}) + z_{n-1}(a_0^3, a_1^3, \ldots, a_{n-1}^3)).\end{aligned}$$

Theorem 14. *Let C be a linear code over R of length $3n$. Then C is an a-polycyclic code if and only if $\Phi(C)$ is a-quasi cyclic code over $\mathbb{F}_q$, $(\tau^3(\Phi(c)) = \Phi(\sigma_a(c)))$.*

Proof. Let C be an a-polycyclic code of length n, then it satisfies the cyclic shift operator for every $c \in C$,

$$\begin{aligned}\sigma_a(c) &= (0, c_1, c_2, \ldots, c_{n-2}) + c_{n-1}(a_0, a_1, \ldots, a_{n-1})\\ &= (0, c_1^1 e_1 + c_1^2 e_2 + c_1^3 e_3, c_2^1 e_1 + c_2^2 e_2 + c_2^3 e_3, \ldots, c_{n-2}^1 e_1 + c_{n-2}^2 e_2 + c_{n-2}^3 e_3)\\ &\quad + (c_{n-1}^1 e_1 + c_{n-1}^2 e_2 + c_{n-1}^3 e_3)(a_0^1 e_1 + a_0^2 e_2 + a_0^3 e_3, a_1^1 e_1 + a_1^2 e_2 + a_1^3 e_3, \cdots,\\ &\quad a_{n-1}^1 e_1 + a_{n-1}^2 e_2 + a_{n-1}^3 e_3)\\ &= (0, c_1^1 e_1, e_1 c_2^1, \ldots, e_1 c_{n-2}^1) + e_1 c_{n-1}^1 (a_0^1 e_1, a_1^1 e_1, \ldots, a_{n-1}^1 e_1)\\ &\quad + (0, c_1^2 e_2, e_2 c_2^2, \ldots, e_2 c_{n-2}^2) + e_2 c_{n-1}^2 (a_0^2 e_2, a_1^2 e_2, \ldots, a_{n-1}^2 e_2)\\ &\quad + (0, c_1^3 e_3, e_3 c_2^3, \ldots, e_3 c_{n-2}^3) + e_3 c_{n-1}^3 (a_0^3 e_3, a_1^3 e_3, \ldots, a_{n-1}^3 e_3)\\ &= e_1((0, c_1^1, c_2^1, \ldots, c_{n-2}^1) + c_{n-1}^1 (a_0^1, a_1^1, \ldots, a_{n-1}^1))\\ &\quad + e_2((0, c_1^2, c_2^2, \ldots, c_{n-2}^2) + c_{n-1}^2 (a_0^2, a_1^2, \ldots, a_{n-1}^2))\\ &\quad + e_3((0, c_1^3, c_2^3, \ldots, c_{n-2}^3) + c_{n-1}^3 (a_0^3, a_1^3, \ldots, a_{n-1}^3))\\ \Phi(\sigma_a(c)) &= ((0, c_1^1, c_2^1, \ldots, c_{n-2}^1) + c_{n-1}^1 (a_0^1, a_1^1, \ldots, a_{n-1}^1),\\ &\quad (0, c_1^2, c_2^2, \ldots, c_{n-2}^2) + c_{n-1}^2 (a_0^2, a_1^2, \ldots, a_{n-1}^2),\\ &\quad (0, c_1^3, c_2^3, \ldots, c_{n-2}^3) + c_{n-1}^3 (a_0^3, a_1^3, \ldots, a_{n-1}^3)).\end{aligned}$$

Let $c' \in \Phi(C)$, then there exists an $c \in C$ such that $\Phi(c) = c'$. Consider

$$\begin{aligned}\Phi(c) &= (c_0^1, c_1^1, \ldots, c_{n-1}^1, c_0^2, c_1^2, \ldots, c_{n-1}^2, c_0^3, c_1^3, \ldots, c_{n-1}^3)\\ \tau^3(\Phi(c)) &= ((0, c_1^1, c_2^1, \ldots, c_{n-2}^1) + c_{n-1}^1 (a_0^1, a_1^1, \ldots, a_{n-1}^1),\\ &\quad (0, c_1^2, c_2^2, \ldots, c_{n-2}^2) + c_{n-1}^2 (a_0^2, a_1^2, \ldots, a_{n-1}^2),\\ &\quad (0, c_1^3, c_2^3, \ldots, c_{n-2}^3) + c_{n-1}^3 (a_0^3, a_1^3, \ldots, a_{n-1}^3))\\ \text{Hence, } \tau^3(\Phi(c)) &= \Phi(\sigma_a(c)).\end{aligned}$$

□

Definition 4. Let C be an a-quasi polycyclic code of length n over $\mathbb{F}_q$.

1. Let $\alpha_{a_i}(x), \beta_{a_i}(x) \in \mathbb{F}_q^{a_i}$, then the annihilator product is defined as

$$\sum_{i=1}^{3}\langle \alpha_{a_i}(x), \beta_{a_i}(x)\rangle_{a_i} = \sum_{i=1}^{3} r_{a_i}(0)$$

where $\alpha_{a_i}(x), \beta_{a_i}(x) \equiv r_{a_i}(x)(\mod x^n - a_i(x))$ and $deg(r_{a_i}(x)) \leq n-1$

2. The annihilator dual code C' of an a-quasi polycyclic code C is defined to be $C' = \{(\beta_{a_1}(x), \beta_{a_2}(x), \beta_{a_3}(x)) \in (\mathbb{F}_q^{a1}, \mathbb{F}_q^{a2}, \mathbb{F}_q^{a3}) \mid \sum_{i=1}^{3}\langle \alpha_{a_i}(x), \beta_{a_i}(x)\rangle_{a_i} = \sum_{i=1}^{3} r_{a_i}(0) = 0 \text{ for all } \alpha_{a_i}(x) \in C_i\}$

Theorem 15. *Let C be a polycyclic code. If C' is annihilator dual of C, then $\Phi(C')$ is annihilator dual for an a-quasi cyclic code $\Phi(C)$.*

Proof. Let $\beta(x) \in C'$, then for every $\alpha(x) \in C$, $\langle \alpha(x), \beta(x)\rangle_a = r(0) = 0$. Since $\alpha(x), \beta(x)$ is an element of R^a, $\alpha(x) = \sum_{i=1}^{3} e_i\alpha_{a_i}(x)$, $\beta(x) = \sum_{i=1}^{3} e_i\beta_{a_i}(x)$.

$$\langle\sum_{i=1}^{3} e_i\alpha_{a_i}(x), \sum_{i=1}^{3} e_i\beta_{a_i}(x)\rangle_a = \sum_{i=1}^{3} e_i\langle \alpha_{a_i}(x), \beta_{a_i}(x)\rangle_a = \sum_{i=1}^{3} e_i r_{a_i}(0) = 0$$

where $\alpha_{a_i}(x), \beta_{a_i}(x) \equiv r_{a_i}(x)(\mod x^n - a_i(x))$ which shows that $r_{a_i}(0) = 0$ for all i. To show $\Phi(\beta(x)) = (\beta_{a_1}(x), \beta_{a_2}(x), \beta_{a_3}(x)) \in \Phi(C')$, let $\alpha_{a_i}(x) \in C_i$ then

$$\sum_{i=1}^{3}\langle \alpha_{a_i}(x), \beta_{a_i}(x)\rangle_{a_i} = \sum_{i=1}^{3} r_{a_i}(0) = 0.$$

Thus, $\Phi(\beta(x)) = (\beta_{a_1}(x), \beta_{a_2}(x), \beta_{a_3}(x)) \in \Phi(C')$. □

Theorem 16. *Let C be an a-polycyclic code over R, then*

- *C is annihilator self-orthogonal if and only if both C_{a_1}, C_{a_2} and C_{a_3} are annihilator self-orthogonal over $\mathbb{F}_q$.*
- *C is annihilator self-dual if and only if both C_{a_1}, C_{a_2} and C_{a_3} are annihilator self-dual over $\mathbb{F}_q$.*
- *C is annihilator LCD if and only if C_{a_1}, C_{a_2} and C_{a_3} are annihilator LCD over $\mathbb{F}_q$.*

Proof. The proof of this similar to that of Theorem 15. □

Example 1. Let $a(x) = 4x^3 + 1$, then $R^a = \frac{\mathbb{F}_5[x]}{\langle x^6 - a(x)\rangle}$. Let $C = \langle g_{a_i}(x)\rangle = \langle x^2 + 4x + 4\rangle$, then $C' = \langle h_{a_i}(x)\rangle = \langle (x^2 + 3x + 4)^2\rangle$. Since $(g_{a_i}(x), h_{a_i}(x)) = 1$, there exists a LCD annihilator code of parameter $[18, 12, 2]_5$.

Example 2. Let $a(x) = -(x^4 + x^6 - 1)$, then $R^a = \frac{\mathbb{F}_3[x]}{\langle x^8 - a(x)\rangle}$. Let $C = \langle g_{a_i}(x)\rangle = \langle x^4 + 2x^2 + 2\rangle$, then $C' = \langle h_{a_i}(x)\rangle = \langle (x^2+1)^2\rangle$. Since $(g_{a_i}(x), h_{a_i}(x)) = 1$, there exists a LCD annihilator code of parameter $[24, 15, 3]_3$.

Example 3. Let $a(x) = -(x^4 - 1)$, then $R^a = \frac{\mathbb{F}_3[x]}{\langle x^6 - a(x)\rangle}$. Let $C = \langle g_{a_i}(x)\rangle = \langle x^3 + 2x^2 + x + 1\rangle$, then $(g_{a_i}(x), h_{a_i}(x)) = 1$ and hence there exists a LCD annihilator code of parameter $[18, 9, 3]_3$.

Conflict of Interest: The author declares no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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