Communications in Combinatorics and Optimization
Vol. 10, No. 2 (2025), pp. 335-354
https://doi.org/10.22049/cco.2023.28736.1692

CCO
Commun. Comb. Optim.

Research Article

On odd-graceful coloring of graphs

I Nengah Suparta[1,*], Yuqing Lin[2], Roslan Hasni[3], I Nyoman Budayana[1,†]

[1]Departement of Mathematics, Ganesha University of Education, Singaraja-Bali, Indonesia
*nengah.suparta@undiksha.ac.id
†nyoman.budayana@undiksha.ac.id

[2]College of Engineering, Science and Environment, The University of Newcastle, Australia
yuqing.lin@newcastle.edu.au

[3]Special Interest Group on Modeling and Data Analytics (SIGMDA)
Universiti Malaysia Terengganu, Malaysia
hroslan@umt.edu.my

Received: 9 June 2023; Accepted: 13 November 2023
Published Online: 18 November 2023

Abstract: For a graph $G(V, E)$ which is undirected, simple, and finite, we denote by $|V|$ and $|E|$ the cardinality of the vertex set V and the edge set E of G, respectively. A graceful labeling f for the graph G is an injective function $f : V \to \{0, 1, 2, \dots, |E|\}$ such that $\{|f(u) - f(v)| : uv \in E\} = \{1, 2, \dots, |E|\}$. A graph that has a graceful-labeling is called graceful graph. A vertex (resp. edge) coloring is an assignment of color (positive integer) to every vertex (resp. edge) of G such that any two adjacent vertices (resp. edges) have different colors. A graceful coloring of G is a vertex coloring $c : V \to \{1, 2, \dots, k\}$, for some positive integer k, which induces edge coloring $|c(u) - c(v)|$, $uv \in E$. If c also satisfies additional property that every induced edge color is odd, then the coloring c is called an odd-graceful coloring of G. If an odd-graceful coloring c exists for G, then the smallest number k which maintains c as an odd-graceful coloring, is called odd-graceful chromatic number for G. In the latter case we will denote the odd-graceful chromatic number of G as $\mathcal{X}_{og}(G) = k$. Otherwise, if G does not admit odd-graceful coloring, we will denote its odd-graceful chromatic number as $\mathcal{X}_{og}(G) = \infty$. In this paper, we derived some facts of odd-graceful coloring and determined odd-graceful chromatic numbers of some basic graphs.

Keywords: graceful graph, graceful coloring, odd-graceful coloring, odd-graceful chromatic number.

AMS Subject classification: 05C15, 05C78

1. Introduction

Let $G(V, E)$ be an undirected, finite simple graph. Denote by $|V|$ and $|E|$ the cardinality of the vertex set V and edge set E of G, respectively. A *graceful labeling* of G is

* *Corresponding Author*

an injective function f from vertex set V into the set $\{0, 1, 2, \ldots, |E|\}$ which induces a bijective function f' from the edge set E onto the set $\{1, 2, \ldots, |E|\}$ such that for every edge $uv \in E$ we have $f'(uv) = |f(u) - f(v)|$. A graph that has a graceful labeling is called *graceful*. A variation of graceful labeling is called *odd-graceful* labeling which was introduced by Gnanajothi [4] in 1991. Here, the codomain of f is the set $\{0, 1, 2, \ldots, 2|E| - 1\}$, instead of $\{0, 1, 2, \ldots, |E|\}$, which generates distinct odd-labels for edges of G, $\{|f(u) - f(v)| : uv \in E\} = \{1, 3, \ldots, 2|E| - 1\}$. A graph which admits an odd-graceful labeling is called *odd-graceful* graph. An application of odd-graceful labeling for information security can be seen in [9].

Furthermore, Pasotti [7] in 2012 generalized these two labeling concepts as the so-called *d-graceful labeling*. Let G be a graph of size $d \times m$, for some positive integers d and m. A labeling f of G is called *d-graceful* if f is an injection from V into the set $\{0, 1, \ldots, d(m + 1) - 1\}$ such that $\{|f(u) - f(v)| : uv \in E\} = \{1, 2, \ldots, d(m + 1) - 1\} - \{m + 1, 2(m + 1), \ldots, (d - 1)(m + 1)\}$. In terms of d-graceful labeling, we can see that graceful and odd-graceful labelings are 1-graceful and $|E|$-graceful labelings, respectively.

A *vertex* (resp. *edge*) *coloring* is an assignment of colors (positive integers) to every vertex (resp. edge) of G such that any two adjacent vertices (resp. edges) have different colors. Here, we will discuss a combination between the coloring concept and the concept of graceful labeling into new concept which is called *graceful coloring*. The word graceful refers to the way we induce edge colors which are different for all edges incident to a common vertex. In more precise formulation, we define graceful coloring as the following. A *graceful coloring* for a graph G is a vertex coloring $c : V \to \{1, 2, \ldots, k\}$, for some positive integer k, which induces edge coloring $|c(u) - c(v)|$, for every edge $uv \in E$. The smallest k for which c is a graceful coloring for G is called the *graceful chromatic number* of G, denoted by $\mathcal{X}_g(G)$. It is clear that any graph admits a graceful coloring.

If c also satisfies additional property that every induced edge colors are odd, then the coloring c is called an *odd-graceful coloring* of G. We will show later on that not every graph has odd-graceful coloring. If an odd-graceful coloring c exists for G, then the smallest number k which maintains c as an odd-graceful coloring for G, is called the *odd-graceful chromatic number* of G. In this later case, we will denote the odd-graceful chromatic number of G as $\mathcal{X}_{og}(G) = k$. Otherwise, if G does not admits any odd-graceful coloring, we will denote its odd-graceful chromatic number as $\mathcal{X}_{og}(G) = \infty$. Since an odd-graceful coloring for a graph G is a graceful coloring for G, it is obvious that $\mathcal{X}_g(G) \leq \mathcal{X}_{og}(G)$.

Graceful colorings have been studied by many researchers. Byers in [2] studied chromatic number of some classes of graphs, including cycles, wheel, and caterpillars. He also derived some upper bound of graceful chromatic numbers of trees related to their maximum degree. Alfarisi, et al in [1] derived graceful chromatic numbers of tadpole and sun graphs. At the last part of their paper, they proposed some open problems. Other researches on graceful coloring may also be seen in [3, 5, 6].

In [8] Su, Sun, and Yao introduced a variation of graceful coloring which they call as odd-graceful total coloring. This variant of graceful coloring has prospective application for the security authentication in generating passwords which resist some attacks: brute force, dictionary, guessing, reproduction, and denial of service attacks. In this paper, we study another variation of graceful coloring which may also be considered as a relaxation of odd-graceful total coloring. Instead of total coloring, here we introduce a vertex coloring, but keeping oddness and gracefulness properties of the coloring. As we mentioned above, we call this variation of graceful coloring as *odd-graceful coloring*. We study some characteristics of odd-graceful coloring and then derived some results on odd-graceful chromatic number of some basic graphs.

2. Main Results

We will start with an observation regarding dual coloring of an odd-graceful coloring. Let c be an odd-graceful coloring of a graph G. Define the dual coloring of c for G, $c' : V \to \{1, 2, \ldots, k\}$ as $c'(u) = k+1-c(u)$ for every $u \in V$. We can immediately see that the dual coloring is a proper vertex coloring for G. Now, we will prove that the dual coloring c' of an odd-graceful coloring will preserve the induced color of every edge, and therefore, c' is again an odd-graceful coloring for G. This is shown in the following line.

$$|c'(u) - c'(v)| = |[k+1-c(u)] - [k+1-c(v)]| = |c(v) - c(u)|.$$

This observation is formulated in the following lemma.

Lemma 1. *The dual coloring of an odd-graceful coloring of a graph G is an odd-graceful coloring of G.*

Assume that a graph G has an odd-graceful coloring c. If we restrict the coloring c to a subgraph H of G, then we may conclude that c is also an odd-graceful coloring for H. An immediate result is the following.

Lemma 2. *Let c be an odd-graceful coloring of G, and H be a subgraph of G. We have*

$$\mathcal{X}_{og}(H) \leq \mathcal{X}_{og}(G).$$

Consider a non-connected graph G with l components $G^1, G^2, \ldots, G^l$. It is clear that $\mathcal{X}_{og}(G) = \max\{\mathcal{X}_{og}(G^i) : i = 1, 2, \ldots, l\}$. So, whenever we are talking about odd-graceful chromatic number of some non-connected graph, in fact we are dealing with an odd-graceful chromatic number of some connected graph. Hence, in the sequel, we will be dealing with only connected simple graphs.
The degree of a graph G, denoted by $\deg(G)$, is equal to the maximum degree of its vertices. Now, assume that $\deg(G) = \Delta$ and let u be a vertex of G with $\deg(u) = \Delta$.

Let $N(v) = \{w \in V : d(v,w) \leq 1\}$, where $d(x,y)$ stands for the distance between vertex x and y. We call $N(v)$ as the *closed neighbourhood* of vertex v. If c is an odd-graceful coloring of G, then it is clear that $c(v) \geq 1$ for every $v \in V$. Furthermore, for maintaining the property of oddness, the parity of $c(v)$ is different from the parity of $c(w)$ for every $w \neq v, w \in N(v)$.
Let some pair of edges a and b both be incident to u, with $deg(u) = \Delta$. Since edges a and b must have different odd colors, we need Δ odd colors for all these such edges. This implies that there exists some edge uw, $w \in N(u)$, which has color $|c(u)-c(w)| \geq 2\Delta-1$. Thus, $c(u)$ or $c(w)$ is greater than or equal to $2\Delta-1+1 = 2\Delta$, since $c(v) \geq 1$ for all $v \in V$. Based on this observation, we have the following lemma.

Lemma 3. *If G is a graph with $\deg(G) = \Delta$, then $\mathcal{X}_{og}(G) \geq 2\Delta$.*

In the following theorem, we will formulate on how many colors we may assign to vertices adjacent to a certain vertex which has a given color.

Theorem 1. *Let G be a graph with $\deg(G) = \Delta$ and c be an odd-graceful coloring for G. Let a be an integer $1 \leq a \leq 2\Delta$, and $b = min\{a, 2\Delta + 1 - a\}$. If for some vertex $u \in V$ we have $c(u) = a$, then there are only $\Delta - \lfloor \frac{b}{2} \rfloor$ integers in $\{1, 2, \ldots, 2\Delta\}$ that can be assigned for neighbours of u.*

Proof. **Case 1.** $a \leq \Delta$. Here, $b = a$.
If a is odd, then the odd differences between a and integers $2, 4, \ldots, a-1$ are the same with those between a and integers $a+1, a+3, \ldots, 2a-2$. Integers $a-k$ and $a+k$, $2 \leq k \leq a-2$, can not be used simultaneously, since they give the same odd difference to a. Hence, only a half number, $(= (a-1)/2)$, of even integers $\{2, \ldots, a-1, a+1, \ldots, 2a-2\}$ that may be assigned for the color of the neighbours of u due to the odd-gracefulness property. So, there are exactly $\Delta - \frac{a-1}{2} = \Delta - \frac{b-1}{2}$ even integers in $\{1, 2, \ldots, 2\Delta\}$ which can be assigned for neighbours of u. (Since a is odd, odd integers can not be assigned to color any neighbour of u. The total of these odd integers in this set is equal to Δ.)
Now, let a be even. Based on a similar argument, only a half number $(= a/2)$ of odd integers $\{1, 3, \ldots, a-1, a+1, \ldots, 2a-1\}$ that may be assigned for the color of the neighbours of u. Thus, there are exactly $\Delta - \frac{a}{2} = \Delta - \frac{b}{2}$ integers in $\{1, 2, \ldots, 2\Delta\}$ which can be assigned for neighbours of u.
Case 2. $\Delta < a \leq 2\Delta$.
In this case $b = 2\Delta + 1 - a$. Consider the set of integers (colors)

$$X = \{a \pm k : 1 \leq k \leq 2\Delta - a, k \text{ odd}\}, \text{ if } a \text{ odd},$$

and

$$Y = \{a \pm k : 1 \leq k \leq 2\Delta - a - 1, k \text{ odd}\}, \text{ if } a \text{ even}.$$

If a is odd, the vertex colors $a+k$ and $a-k$, for every odd $1 \leq k \leq 2\Delta - a$, will induce the same edge colors $|a-(a-k)| = |a-(a+k)| = k$. The same thing will happen for a is even. Based on the same argument, we may again conclude that only a half $(= \lfloor \frac{b}{2} \rfloor)$ elements of the integer sets X and Y which can be assigned for colors of neighbours of u. Thus, the theorem is proven. □

Corollary 1. *Let G be a graph with $\deg(G) = \Delta$ and c be an odd-graceful coloring of G with $\mathcal{X}_{og}(G) = 2\Delta$. If $\deg(u) = \Delta$ for some $u \in V$, then $c(u) = 1$ or $c(u) = 2\Delta$.*

Proof. By Theorem 1, it is clear that $\deg(u) \leq \Delta - \lfloor \frac{b}{2} \rfloor$, with $b = min\{c(u), 2\Delta + 1 - c(u)\}$. If $c(u)$ is not equal to 1 or 2Δ, then, again based on Theorem 1, we get $deg(u) \leq \Delta - \lfloor \frac{b}{2} \rfloor < \Delta$. This implies that $\deg(u) < \Delta$, a contradiction. □

Now, we will observe odd-graceful chromatic number of some basic graphs. We will start from path graph.

Theorem 2. *Let P_n be the path on n vertices with $n \geq 2$. Then,*

$$\mathcal{X}_{og}(P_n) = \begin{cases} 2, & \text{if } n = 2, \\ 4, & \text{if } n = 3, 4, \\ 5, & \text{if } n \geq 5. \end{cases}$$

Proof. Assume the coloring we set for the path P_n is $c : V(P_n) \to \{1, 2, \ldots, k\}$, for some positive integer k. Let $x_1, x_2, \ldots, x_n$, be the vertices of the path P_n where $x_i x_{i+1}$, $1 \leq i \leq n-1$, are the edges of the path. If $n = 2$, we set $c(x_i) = i, i = 1, 2$. It is obvious that $\mathcal{X}_{og}(P_2) = 2$.
For $n = 3$, we define the coloring c as $c(x_1) = 2, c(x_2) = 1, c(x_3) = 4$, and for $n = 4$, we define c as $c(x_1) = 2, c(x_2) = 1, c(x_3) = 4$, and $c(x_4) = 3$. By using Lemma 3, we get $\mathcal{X}_{og}(P_3) = 4 = \mathcal{X}_{og}(P_4)$.
For $n \geq 5$, we define the coloring as follows.

$$c(x_i) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod 4, \\ 2, & \text{if } i \equiv 2 \pmod 4, \\ 5, & \text{if } i \equiv 3 \pmod 4, \\ 4, & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

We can immediately check that this above coloring defines an odd-graceful coloring for $P_n, n \geq 5$. From here, we conclude that $\mathcal{X}_{og}(P_n) \leq 5$, with $n \geq 5$.
First, we focus on the path of five vertices, P_5. We will show that $\mathcal{X}_{og}(P_5) = 5$.
Let $V(P_5) = \{x_1, x_2, x_3, x_4, x_5\}$ and $E(P_5) = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5\}$. Note that the vertices x_2, x_3, and x_4 have maximum degree $\Delta = 2$. Suppose that $\mathcal{X}_{og}(P_5) = 2\Delta = 4$. Based on Corollary 1, it must be either $c(x_2) = 1 = c(x_4)$ and $c(x_3) = 4$, or $c(x_2) = 4 = c(x_4)$ and $c(x_3) = 1$. But this implies $c(x_2x_3) = 3 = c(x_3x_4)$ which contradicts the gracefulness property. This insists us to add one more color which is

greater than 4. From this, we then conclude that $\mathcal{X}_{og}(P_5) \geq 5$. Thus, for $n = 5$ we infer that $\mathcal{X}_{og}(P_5) = 5$.
Furthermore, by Lemma 2, we have that $\mathcal{X}_{og}(P_n) \geq \mathcal{X}_{og}(P_5) = 5$. Therefore, we can conclude that $\mathcal{X}_{og}(P_n) = 5$, for $n \geq 5$. □

The next basic graph we will observe is cycle on $n \geq 3$ vertices, C_n.

Theorem 3. *Let C_n, $n \geq 3$, be the cycle on n vertices. Then,*

$$\mathcal{X}_{og}(C_n) = \begin{cases} 5, & \text{if } n \equiv 0 \pmod 4, \\ 6, & \text{if } n \equiv 2 \pmod 4, \\ \infty, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let the vertices of C_n be $x_1, x_2, \ldots, x_n$ where x_i is adjacent with x_{i+1}, $1 \leq i \leq n-1$, and x_n and x_1 are adjacent. Now we divide the proof into three cases.
Case 1. $n \equiv 0 \pmod 4$.
In this case we define a coloring c for cycle C_n, $n \geq 4$, as follows.

$$c(x_i) = \begin{cases} 2, & \text{if } i \equiv 1 \pmod 4, \\ 1, & \text{if } i \equiv 2 \pmod 4, \\ 4, & \text{if } i \equiv 3 \pmod 4, \\ 5, & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

Under this definition we can see immediately that every two adjacent edges in C_n have different induced colors: 1 and 3. Therefore, in this case we may conclude that $\mathcal{X}_{og}(C_n) = 5$.

Case 2. $n \equiv 2 \pmod 4$.
Here we will consider C_n as a subdivision graph from C_{n-2}, through twice edge subdivisions. Let $n = 4k + 2$, for some positive integer k. Consider the colored C_{4k} as in Case 1. We will do subdivision on an edge which has end vertex colors 4 and 5. Assume that the end vertices of this edge are x and y where $c(x) = 4$ and $c(y) = 5$. This edge xy has induced color 1. This means that if vertex $w \neq y$ is adjacent to x, and vertex $z \neq x$ is adjacent to y, then $c(wx) \neq 1$ and $c(yz) \neq 1$.
Let us add two different vertices a and b on this edge xy, such that each new pair of vertices $\{x, a\}, \{a, b\}$, and $\{b, y\}$ are adjacent vertices in the new graph C_n. Extend the coloring c into c^* for the subdivision graph C_n as follows.

$$c^*(v) = \begin{cases} c(v), & if\ v \in V(C_{4k}), \\ 3, & \text{if } v = a, \\ 6, & \text{if } v = b. \end{cases}$$

Observe that the color of the edge xa is $c(xa) = |c(x) - c(a)| = 4 - 3 = 1$, and the one of the edge by is $c(by) = |c(b) - c(y)| = |5 - 6| = 1$. On the other hand, the color

of the edge ab is $c(ab) = |c(a) - c(b)| = 3 - 6 = 3$. This implies that the coloring c^* is an odd-graceful coloring for C_n.
Thus, for $n \equiv 2 \pmod 4$, we can conclude that $\mathcal{X}_{og}(C_n) = 6$.

Case 3. $n \geq 3$ is odd.
For inducing edge color to become odd, any pair of adjacent vertices should get colors with different parity: odd and even parities. Let $\{x_1, x_2, \ldots, x_n\}$ be the vertices of the graph C_n. Without loss of generality, assume that $c(x_1)$ is odd. For maintaining the odd-graceful coloring property, then for every even integer $i, 1 \leq i \leq n-1$, $c(x_i)$ is even, and otherwise $c(x_i)$ is odd. Therefore, $c(x_n)$ is odd. But, this will imply that the induced label for the edge x_1x_n to be even. This violates the requirement of c for being an odd-graceful coloring. Hence, we may conclude that $\mathcal{X}_{og}(C_n) = \infty$ whenever n is odd. So, in all cases suggest that the theorem is proved. □

It is well known that graph contains a cycle of odd length if and only if it is not bipartite. Thus, we have the following corollary.

Corollary 2. *If G is not bipartite graph, then $\mathcal{X}_{og}(G) = \infty$.*

Proof. Theorem 3 and Lemma 2 immediately imply Corollary 2. □

A *caterpillar* is a connected simple graph which becomes a path after removing all leafs from the graph. This path is called the *spine* of the caterpillar. Each vertex in the spine is an internal vertex of the original caterpillar. If each end vertex of the spine of a caterpillar is reconnected with one pendant edge (which was removed before), then we obtain a new path of length two more than the length of the spine. This new path will be named as the *extended spine* of the caterpillar. The length of a caterpillar is equal to the length of its extended spine. Note that caterpillar of length 2 is a star.
If all vertices of the spine of a caterpillar have the same degree Δ in the related caterpillar, then the caterpillar is called *uniform caterpillar*. It is well known that a caterpillar is bipartite graph. This implies that every caterpillar has an odd-graceful coloring. Moreover, any two vertices in the same partition must get colors with the same parity, and they are at even distance one another. Any path on more than two vertices is a caterpillar.

Theorem 4. *Let G be a caterpillar with* $\deg(G) = \Delta \geq 2$*, then* $2\Delta \leq \mathcal{X}_{og}(G) \leq 2\Delta + 1$.

Proof. Based on Theorem 3 we have $\mathcal{X}_{og}(G) \geq 2\Delta$. To complete the proof, we will now prove that $\mathcal{X}_{og}(G) \leq 2\Delta + 1$. First, let $x_1, x_2, \ldots, x_n$ be the vertices of the extended spine of G. Color these vertices as follow:

$$c(x_i) = \begin{cases} 2, & \text{if } i \equiv 1 \pmod 4, \\ 1, & \text{if } i \equiv 2 \pmod 4, \\ 2\Delta, & \text{if } i \equiv 3 \pmod 4, \\ 2\Delta + 1, & \text{if } i \equiv 0 \pmod 4. \end{cases} \tag{1}$$

Now, the coloring c for leafs of the caterpillar using procedure below.

i. Every leaf which is adjacent to $x_i, i \equiv 1 (\text{mod } 4)$ is assigned with color from the odd color set $\{5, 7, \ldots, 2\Delta - 1\}$. Together with colors 1 and $2\Delta + 1$(which have already been assigned for x_{i-1} and x_{i+1}, $2 \leq i \leq n - 1$), we have Δ odd colors for all vertices which are adjacent to this x_i. So, the provided colors are enough for coloring the mentioned leafs, since the degree of G is Δ.

ii. Each leaf which is adjacent to $x_i, i \equiv 2 \pmod 4$ at the extended spine is assigned with color from the even color set $\{4, 6, \ldots, 2\Delta - 2\}$. Similarly to i.) the provided colors are enough for coloring the mentioned leafs.

iii. Any leaf adjacent to vertex $x_i, i \equiv 3 \pmod 4$ is assigned with odd color from the color set $\{3, 5, 7, \ldots, 2\Delta - 3\}$. Again, we can argue that the provided colors are enough for coloring the mentioned leafs.

iv. Any leaf adjacent to vertex $x_i, i \equiv 0 \pmod 4$ at the extended spine is assigned with even color from the color set $\{4, 6, \ldots, 2\Delta - 2\}$. Here, we can also see that the provided colors are enough for coloring the mentioned leafs.

We can see immediately that c defines an odd graceful coloring for the caterpillar G, and gives $\mathcal{X}_{og}(G) \leq 2\Delta + 1$. □

Below we will show that the bounds of the odd-graceful chromatic number for caterpillars as formulated in Theorem 4, are sharp. This is shown by observing odd-graceful chromatic numbers of some classes of caterpillars.

Theorem 5. *Let G be a caterpillar of degree Δ. If there are two vertices in G with maximum degree having distance 2 one another, then $\mathcal{X}_{og}(G) = 2\Delta + 1$.*

Proof. In order that the caterpillar G may contain two vertices as in the theorem, the length of G must be at least 4. Based on Corollary 1, if we use only colors $1, 2, \ldots, 2\Delta$, for coloring G odd-gracefully, then there are only two colors, that are 1 and 2Δ, which can be assigned to vertex of maximum degree Δ. But, two vertices which are at distance 2 can not get the same color. Therefore, we need at least one bigger color to complete the coloring for G into odd-graceful coloring. This says that $\mathcal{X}_{og}(G) \geq 2\Delta + 1$. So, based on Theorem 4, we can conclude that $\mathcal{X}_{og}(G) = 2\Delta + 1$ if G contains two vertices with maximum degree Δ which are at distance 2. □

Except for caterpillars of length less than 4, any uniform caterpillar satisfies the condition of Theorem 5. Therefore, the following theorem can be considered as a corollary of Theorem 5.

Theorem 6. *Let G be a uniform caterpillar of degree Δ, then $\mathcal{X}_{og}(G) = 2\Delta$ if the length of G is at most 3, and $\mathcal{X}_{og}(G) = 2\Delta + 1$ otherwise.*

Proof. Let $P_n, n \geq 3$, be the extended spine of the caterpillar G. If $n = 3$ (G is a star), assign color 1 for the vertex with maximum degree Δ, and its Δ leafs with colors $\{2, 4, \ldots, 2\Delta\}$. Here, we conclude that $\mathcal{X}_{og}(G) = 2\Delta$.
If $n = 4$, let w, x, y and z in order be the vertices of the extended spine of G. Here $\deg(x) = \deg(y) = \Delta$ and $\deg(w) = \deg(z) = 1$. Color x with 1 and y with 2Δ. Then, color leafs which are adjacent to x, including the vertex w, using colors from $\{2, 4, \ldots, 2(\Delta - 1)\}$, and color all leafs adjacent to y, including z, with $\{3, 5, \ldots, 2\Delta - 1\}$.
It is clear that the above two colorings are odd-graceful colorings with $\mathcal{X}_{og}(G) = 2\Delta$. As it is mentioned just right after Theorem 5, for $n \geq 5$, graph G satisfies the premise of Theorem 5, and hence the theorem is proved. □

In Figure 1 we depict an example of a uniform caterpillar G of length 7 and of degree 5 with $\mathcal{X}_{og}(G) = 11 = 2\Delta + 1$.

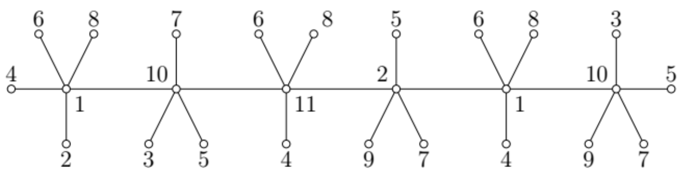

Figure 1. An example of uniform caterpillar G of degree $\Delta = 5$ with an odd-graceful coloring having $\mathcal{X}_{og}(G) = 11$.

In the following theorem we show the exact value of odd-graceful chromatic number of another class of caterpillar.

Theorem 7. *Let G be a non-trivial caterpillar with $deg(G) = \Delta \geq 3$. If any two vertices with maximum degrees Δ in the same partition set are at distance congruent to $0(mod\ 4)$, then $\mathcal{X}_{og}(G) = 2\Delta$.*

Before proceeding to a proof of the theorem, we show an example of the theorem in Figure 2. We see that the blue and the red vertices have maximum degree 6. The blue vertices are at distance 4 one another. So are the red vertices. Here, we may consider that the bipartition set A contains the blue vertices, and the other bipartion set, B, contains the red vertices. The coloring for the caterpillar graph in the figure is an odd-graceful coloring with $\mathcal{X}_{og}(G) = 12$.

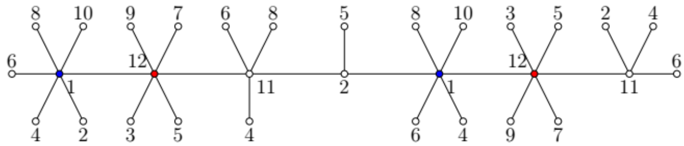

Figure 2. An example of caterpillar G of degree $\Delta = 6$ which satisfies Theorem 7 having $\mathcal{X}_{og}(G) = 12$.

Proof. Let G be a caterpillar with $\deg(G) = \Delta$, where A and B be the partition sets of its vertex set. As it is noticed just right before the Theorem 4, any two vertices in the same partition set have even distance one another. Moreover, it is clear that vertices with maximum degree are always internal vertex of the extended spine of the caterpillar G.

Let P_n be the extended spine of G and let $V = \{x_1, x_2, \ldots, x_{n-1}, x_n\}$ in this order be the vertices of P_n. Let $s, 2 \leq s \leq n-1$, be the smallest index such that $\deg(x_s) = \Delta$. Assume, without loss of generality, that $x_s \in A$, and $t, s+1 \leq t \leq n-1$, be the smallest index such that $x_t \in B$ with $\deg(x_t) = \Delta$. We have two cases.

Case 1. $t - s \equiv 1 \pmod 4$

Define a vertex coloring c for G as follows.

$$c(x_i) = \begin{cases} 1 & \text{if } i - s \equiv 0 \pmod 4, \\ 2\Delta & \text{if } i - s \equiv 1 \pmod 4, \\ 2\Delta - 1 & \text{if } i - s \equiv 2 \pmod 4, \\ 2 & \text{if } i - s \equiv 3 \pmod 4. \end{cases}$$

From the above coloring we see immediately that every vertex in the extended spine which is at distance congruent to $0 \pmod 4$ from vertex x_s (resp. x_t) will get color 1 (resp. 2Δ). So, all these vertices are eligible to have maximum degree.

Case 2. $t - s \equiv 3 \pmod 4$

Define a vertex coloring c for G as follows.

$$c(x_i) = \begin{cases} 1 & \text{if } i - s \equiv 0 \pmod 4, \\ 2 & \text{if } i - s \equiv 1 \pmod 4, \\ 2\Delta - 1 & \text{if } i - s \equiv 2 \pmod 4, \\ 2\Delta & \text{if } i - s \equiv 3 \pmod 4. \end{cases}$$

Using this above coloring we can see a similar thing with Case 1., that every vertex in the extended spine which is at distance congruent to $0 \pmod 4$ with vertex x_s (resp. x_t) will again get color 1 (resp. 2Δ). Hence, we can also conclude that all these vertices are eligible to have maximum degree.

The other internal vertices in the spine will get color 2 or $2\Delta - 1$ which are valid for having degree less than the maximum degree Δ. Thus, the theorem is proved. $\square$

In [3], English, et al define a specific tree which is denoted by $T_{\Delta,h}$ for some positive integer $h \geq 2$ where $T_{\Delta,1}$ is the star of $\Delta+1$ vertices. This $T_{\Delta,h}$ is obtained by adding $\Delta-1$ pendant edges to any leaf of $T_{\Delta,h-1}$. The tree $T_{\Delta,h}$ has $1+\Delta[1+\sum_{i=0}^{h-1}(\Delta-1)^i]$ vertices. For instance, the tree $T_{2,3}$ is the path with seven vertices.
The following theorem may be considered as a corollary of the first part of Theorem 6.

Theorem 8. *The star $T_{\Delta,1}$ with $\Delta+1$ vertices has $\mathcal{X}_{og}(T_{\Delta,1}) = 2\Delta$.*

Proof. Let x_0 be center vertex of the star and $x_1, x_2, \ldots, x_\Delta$ be its related leafs. Define vertex coloring $c : V(T_{\Delta,1}) \to \{1, 2, \ldots, 2\Delta\}$ as the following

$$c(x_i) = \begin{cases} 1, & \text{if } i = 0, \\ 2i, & \text{otherwise.} \end{cases}$$

We can see immediately that c defines an odd-graceful coloring for $T_{\Delta,1}$. By applying Theorem 3, then we may conclude that $\mathcal{X}_{og}(T_{\Delta,1}) = 2\Delta$. □

Below we observe $\mathcal{X}_{og}(T_{\Delta,h})$ for $h = 2$ and leave the case $h \geq 3$ as an open problem which will be formulated in the last section of the paper.

Theorem 9. *For a positive integer $\Delta \geq 2$, we have*

$$\mathcal{X}_{og}(T_{\Delta,2}) = 3\Delta - 1.$$

Proof. Let $V(T_{\Delta,2})$ be $\{x_0, x_i, y_j^i : i = 1, 2, \ldots, \Delta; j = 1, 2 \ldots, \Delta - 1\}$, and $E(T_{\Delta,2})$ be $\{x_0x_i, x_iy_j^i : i = 1, 2, \ldots, \Delta; j = 1, 2 \ldots, \Delta - 1\}$. We can see that $\deg(x_i) = \Delta$, for every $i = 0, 1, 2, \ldots, \Delta$, and $\deg(y_j^i) = 1$ for every $i = 1, 2, \ldots, \Delta$ and $j = 1, 2 \ldots, \Delta - 1$.
Below we define a coloring $c : V(T_{\Delta,2}) \to \{1, 2, \ldots, 3\Delta - 1\}$. First, we set c for all internal vertices of $T_{\Delta,2}$.

$$\begin{aligned} c(x_0) &= 1, \\ c(x_i) &= \begin{cases} 2i, & \text{if } i \leq \lfloor \frac{\Delta}{2} \rfloor, \\ 2(\Delta + (i - \lfloor \frac{\Delta}{2} \rfloor) - 1), & \text{if } \lfloor \frac{\Delta}{2} \rfloor + 1 \leq i \leq \Delta. \end{cases} \end{aligned}$$

Since $c(x_0)$ is odd and $c(x_i)$ is even for every $1 \leq i \leq \Delta$, it is clear that the induced edge label $|c(x_0) - c(x_i)|$ is odd for every $1 \leq i \leq \Delta$ and $|c(x_0) - c(x_i)| \neq |c(x_0) - c(x_k)|$ for every $i \neq k, 1 \leq i, k \leq \Delta$. Now we will define c for the leafs $y_j^i, 1 \leq i \leq \Delta; 1 \leq j \leq \Delta - 1$, as follows.
For $i = 1, 2, \ldots, \lfloor \frac{\Delta}{2} \rfloor$,

$$c(y_j^i) = \begin{cases} 2j + 1, & j = 1, 2, \ldots, i - 1, \\ 2(i + j) + 1, & j = i, i + 1, \ldots, \Delta - 1, \end{cases}$$

and for $i = \lfloor \frac{\Delta}{2} \rfloor + 1, \ldots, \Delta$,

$$c(y_j^i) = 2j + 1, \text{for every } j = 1, 2, \ldots, \Delta - 1.$$

By inspection we can see that c defines an odd-graceful coloring for graph $T_{\Delta,2}$. Furthermore, the largest color here is $3\Delta - 1$ which is assigned to vertex x_Δ if Δ is odd, and to vertex $y_{\Delta-1}^{\lfloor \frac{\Delta}{2} \rfloor}$ if Δ is even. Therefore, we can conclude that $\mathcal{X}_{og}(T_{\Delta,2}) \leq 3\Delta - 1$. Now, we will show that $3\Delta - 1$ is the smallest maximum possible color such that c is odd graceful coloring for $T_{\Delta,2}$. This is explained as below.
If there is a vertex v, $\deg(v) = \Delta$ with color $\Delta \leq c(v) \leq 2\Delta - 1$, then based on Theorem 1, the color $\geq 3\Delta - 1$ occurs for a vertex adjacent to v. To show this, we will divide the case on the parity of Δ: even or odd. Remember that, according to Theorem 1, if vertices x, y, with $\deg(x) = \Delta = \deg(y)$ and with $\Delta \leq c(x) < c(y) \leq 2\Delta - 1$, then the smallest maximum color needed by vertices adjacent to y is greater than the one adjacent to x. So, we will only see for $c(v) = \Delta$.
First let Δ is even. Here b in Theorem 1 is equal to Δ. Therefore, to complete all colors for vertices adjacent to v, we need $\Delta/2$ odd colors which are greater than 2Δ. Thus, the greatest color will be $\geq 2\Delta - 1 + 2(\Delta/2) = 3\Delta - 1$.
Now let Δ is odd. Based on Theorem 1, again b is equal to Δ, and therefore we need $(\Delta - 1)/2$ even colors which are greater than 2Δ to complete all colors for vertices adjacent to v. Thus, the greatest color will be $\geq 2\Delta + 2((\Delta - 1)/2) = 3\Delta - 1$.
In any case, we have shown that the color $\geq 3\Delta - 1$ occurs in odd-graceful coloring for the graph $T_{\Delta,2}$. In other words, here we obtain that $\mathcal{X}_{og}(T_{\Delta,2}) \geq 3\Delta - 1$. Hence, the theorem is proved. □

An illustration for Theorem 9 is shown in Figure 3.

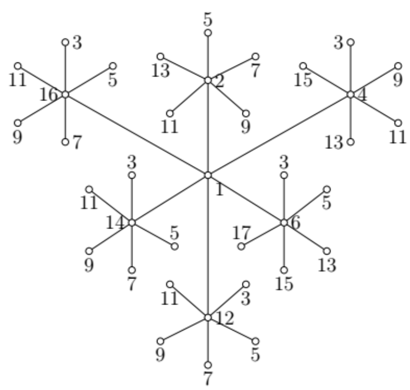

Figure 3. **An odd-graceful labeling of the graph $T_{6,2}$ having $\mathcal{X}_{og}(T_{6,2}) = 3\Delta - 1 = 17$.**

In the rest part of the paper, we will focus on the odd-graceful chromatic numbers of ladder and prism graphs. We first formulate our result on ladder $P_n \times P_2, n \geq 1$.

Theorem 10. *If n is even we have,*

$$\mathcal{X}_{og}(P_n \times P_2) = \begin{cases} 2, & \text{if } n = 1, \\ 5, & \text{if } n = 2, \\ 6, & \text{if } n = 3, \\ 7, & \text{if } n = 4, \\ 8, & \text{if } n = 5, \\ 9, & \text{if } n \geq 6. \end{cases}$$

Proof. For every integer $n \geq 1$, let c be an odd-graceful coloring for graph $P_n \times P_2$. For $n = 1$ is trivial. For $n = 2$, the graph $P_2 \times P_2$ is a cycle on four vertices. So, by referring to Theorem 3, we confirm the theorem.
For $n = 3$, there are two vertices of maximum degree $\Delta = 3$. Based on Theorem 3, we conclude that $\mathcal{X}_{og}(P_3 \times P_2) \geq 6$. By using diagram in Figure 4, we obtain that $\mathcal{X}_{og}(P_3 \times P_2) = 6$.
Now, let us observe for $n = 4$. The graph $P_4 \times P_2$ contains a cycle the vertices of which all have degree 3 in $P_4 \times P_2$. Thus, if we apply only even colors $2, 4$, and 6, the even color 2 or 4 must be the color of some vertex of degree 3. Based on Theorem 3, if some vertex v of degree 3 has color 2 or 4, then there is a vertex adjacent to v with odd color ≥ 7. This means that $\mathcal{X}_{og}(P_4 \times P_2) \geq 7$. But, then, using the odd-graceful coloring for $P_4 \times P_2$ shown in Figure 5, we may confirm that the theorem is true for $n = 4$.

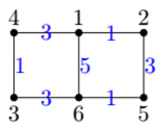

Figure 4. **An odd-graceful coloring for** $P_3 \times P_2$**.**

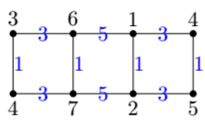

Figure 5. **An odd-graceful coloring for** $P_4 \times P_2$**.**

The argument for concluding $\mathcal{X}_{og}(P_5 \times P_2) = 8$ is similar with graph $P_4 \times P_2$. But here, we observe the occurrence of odd color 3 or 5 for vertex of maximum degree 3. See Figure 6 for an odd-graceful coloring of $P_5 \times P_2$.

Now, let us observe the graph $P_6 \times P_2$. Note that if even color 4 is the color of some vertex of maximum degree 3 in $P_6 \times P_2$, then odd color ≥ 9 must appear. Suppose that we use only even colors $2, 6$, and 8, for relevant vertices of degree 3.

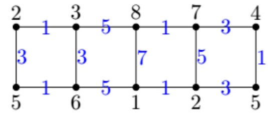

Figure 6. **An odd-graceful coloring for $P_5 \times P_2$.**

Let $V = \{x_i, y_i : i = 1, 2, \ldots, 6\}$ and $E = \{x_i x_{i+1}, y_i y_{i+1} : i = 1, 2, \ldots, 5\} \cup \{x_i y_i : i = 1, 2, \ldots, 6\}$. Recall that to maintain gracefulness property, any two vertices at distance 2 to each other can not get the same color.
Without loss of generality, assume $c(x_2) = 2$, $c(x_4) = 8$, $c(x_6) = 6$, $c(y_3) = 6$, $c(y_5) = 2$. See Figure 7 for a visualization. Consider x_3. We see that $c(x_3)$ can not be 5 or 7. Let $c(x_3) = 1$. Now, $c(y_4)$ can not be $1, 5$, or 7. Assign $c(y_4) = 3$. This implies that $c(x_5)$ can not be $1, 3, 5$, and 7. Hence, $c(x_5)$ must be odd color ≥ 9. This means that $\mathcal{X}_{og}(P_6 \times P_2) \geq 9$.

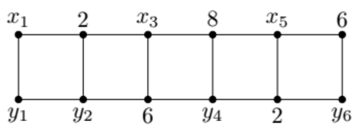

Figure 7. **Odd-graceful coloring of $P_6 \times P_2$ with $\mathcal{X}_{og}(P_6 \times P_2) \geq 9$.**

From here, based on Theorem 2, we can conclude that $\mathcal{X}_{og}(P_6 \times P_2) \geq 9$ for every $n \geq 6$.
It is clear that $P_n \times P_2$, $n \geq 6$ contains $P_6 \times P_2$. Therefore, $\mathcal{X}_{og}(P_n \times P_2) \geq 9$. Then, for concluding that $\mathcal{X}_{og}(P_n \times P_2) = 9$, we apply the following odd-graceful coloring for $P_n \times P_2$, $n \geq 6$ which shows that $\mathcal{X}_{og}(P_n \times P_2) \leq 9$.
We start by naming vertices and edges of $P_n \times P_2$ as $V = \{x_i, y_i : i = 1, 2, \ldots, n\}$ and $E = \{x_i x_{i+1}, y_i y_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{x_i y_i : i = 1, 2, \ldots, n\}$, respectively.
Define coloring c for $P_n \times P_2$, $n \geq 6$, as follows.

$$c(x_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 4, \\ 6 & \text{if } i \equiv 2 \pmod 4, \\ 3 & \text{if } i \equiv 3 \pmod 4, \\ 8 & \text{if } i \equiv 0 \pmod 4, \end{cases} \text{ and } c(y_i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod 4, \\ 7 & \text{if } i \equiv 2 \pmod 4, \\ 4 & \text{if } i \equiv 3 \pmod 4, \\ 9 & \text{if } i \equiv 0 \pmod 4. \end{cases} \tag{2}$$

See Figure 8 for a visualization. From the diagram in Figure 8 we can immediately conclude that $\mathcal{X}_{og}(P_4 \times P_2) = 9$ for every $n \geq 6$. Thus, we proved the theorem. $\square$

Below we will observe the odd-graceful chromatic number of prism graphs $C_n \times P_2$, $n \geq 3$. First, we only see cases $n \equiv 0 \pmod 4$ and n odd. The case $n \equiv 2 (\text{mod } 4)$, will be discussed separately. Here, we have the following theorem.

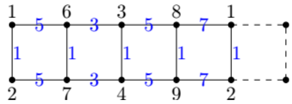

Figure 8. **Odd-graceful coloring for $P_n \times P_2, n \geq 6$ with $\mathcal{X}_{og}(P_n \times P_2) \leq 9$.**

Theorem 11. *For positive integer $n \geq 3$, we have,*

$$\mathcal{X}_{og}(C_n \times P_2) = \begin{cases} 9, & \text{if } n \equiv 0 \pmod 4, \\ \infty, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. For n odd, it is obvious that $\mathcal{X}_{og}(C_n \times P_2) = \infty$. This is because the graph is not bipartite (see Corollary 2).
Now, let $n \equiv 0 \pmod 4$. We know that all vertices of $C_n \times P_2$ each has degree 3. First, we observe for $n = 4$. Note that the vertices of graph $C_4 \times P_2$ may be partitioned into A and B, each with four vertices. For complying with the odd-graceful coloring property, all vertices in each partition get colors of the same parity. Assume, without loss of generality, that vertices in A get even colors. Since, each of these vertices is at distance 2 to the others in A, every vertex must be assigned with different colors. This insists the occurrence of a vertex with color 4 or greater than 8. Thus, by Theorem 1, $\mathcal{X}_{og}(C_4 \times P_2) \geq 9$. But the odd-graceful coloring for $C_4 \times P_2$ in Figure 9 confirms that $\mathcal{X}_{og}(C_4 \times P_2) = 9$.

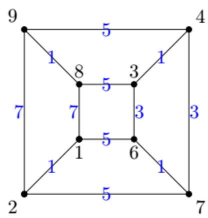

Figure 9. **An odd-graceful coloring for $C_4 \times P_2$.**

Now, we continue our observation on graph $C_n \times P_2$ with $n \equiv 0 \pmod 4), n \geq 8$. Based on Theorem 10, for the ladder $P_n \times P_2$, we have $\mathcal{X}_{og}(P_n \times P_2) \geq 9$ for every $n \geq 8$. This implies that $\mathcal{X}_{og}(C_n \times P_2) \geq 9$. But, using the odd-graceful coloring in Eq. 2 (or in Figure 8), we may conclude that $\mathcal{X}_{og}(C_n \times P_2) = 9$ for every integer $n \geq 8$. Combining this last result and the result on the odd-graceful chromatic number of graph $C_4 \times P_2$, we proved already the theorem. □

Finally, we will observe the odd-graceful chromatic number of $C_n \times P_2$, $n \equiv 2 (\text{mod } 4), n \geq 6$.
First, we show that $\mathcal{X}_{og}(C_6 \times P_2) = 10$, by showing that using colors $1, 2, 3, 4, 6, 7, 8, 9$ is not possible. Note that color 5 is not included, otherwise, since every vertex has

degree 3, even color ≥ 10 must occur.
From odd color combinations in any odd-graceful coloring c of $C_6 \times P_2$, we will have four possible combinations shown in Figure 10, where $c(u)$ is an even color for some vertex u in $C_6 \times P_2$.

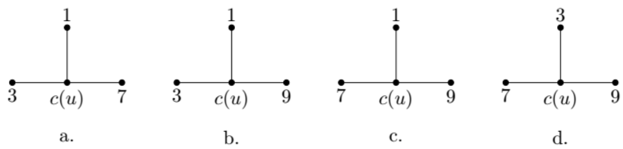

Figure 10. Four possible combinations of odd colors on odd-graceful coloring for $C_6 \times P_2$, with $c(u)$ even for some vertex u in $C_6 \times P_2$.

In the next discussion, we will only describe combination a. for the impossibility of using only colors $1, 2, 3, 4, 6, 7, 8$ and 9. The other three combinations are left to the reader for the sake of space efficiency and for the similarity of process. The impossibility of using only colors $1, 2, 3, 4, 6, 7, 8$, and 9, to produce an odd-graceful coloring for $C_6 \times P_2$, will be done by showing diagram in Figure 12 with the following explanations:

1) For some colors a and b, $a|b$ means that we assign color a for the related vertex between two possible colors, a and b, we may use.
2) The number written in red color stands for the only one possible color we may apply for the related vertex.
3) The red dot on a vertex means that we can not continue the coloring process, since we do not have choice of color to be assigned on the vertex.
4) The boldface colors are fixed colors of the initial combination. In this case the boldface colors are **1**, **3**, and **7**.

Consider Figure 11. The process of coloring will be started from vertex a, b, then

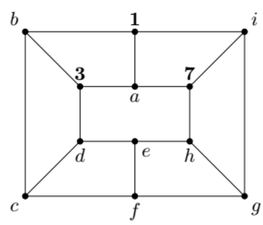

Figure 11. **Order of vertices assigned for coloring $C_6 \times P_2$.**

vertex c or d, e or f, g or h, and vertex i, if necessary. Following this coloring order, for combination a. we have nine cases in total (see Figure 12), and these

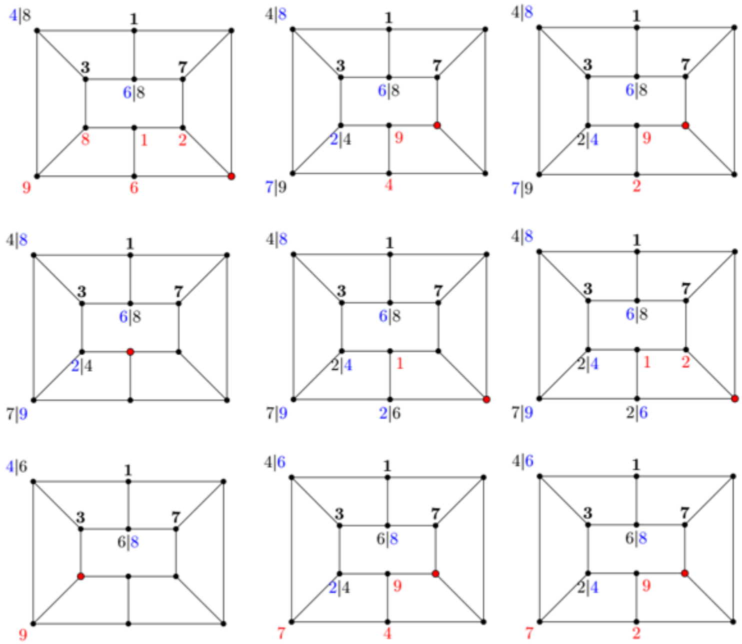

Figure 12. Nine coloring cases totally implied by combination of odd colors 1, 3, and 7, which are impossible as an odd-graceful coloring for $C_6 \times P_2$.

all are impossible for being an odd-graceful coloring. Now we may conclude that $\mathcal{X}_{og}(C_6 \times P_2) \geq 10$. Then, by considering the diagram of colored graph $C_6 \times P_2$ in Figure 13 we may conclude that $\mathcal{X}_{og}(C_6 \times P_2) = 10$.

For the remaining case, $n \equiv 2 \pmod 4, n \geq 10$, we have the following theorem

Theorem 12. *For every integer* $n \geq 10, n \equiv 2 \pmod 4$*, we have* $9 \leq \mathcal{X}_{og}(C_n \times P_2) \leq 10$.

Proof. We will introduce a terminology called as an *open prism*, which is in fact a ladder. Let the vertex set and edge set of $C_n \times P_2$ be $V = \{x_i, y_i : i = 1, 2, \ldots, n\}$ and $E = \{x_i x_{i+1}, y_i y_{i+1}, x_1 x_n, y_1 y_n : i = 1, 2, \ldots, n-1\} \cup \{x_i y_i : i = 1, 2, \ldots, n\}$, respectively. If we slice $C_n \times P_2$ on an edge $x_i y_i$ for some $1 \leq i \leq n$, then we will get a ladder $P_{n+1} \times P_2$ on $2(n+1)$ vertices. After renaming the vertices of this $P_{n+1} \times P_2$, we may consider the vertex and edge set of $P_{n+1} \times P_2$ as $V : \{x_i, y_i : i = 1, 2, \ldots, n, n+1\}$ and $E = \{x_i x_{i+1}, y_i y_{i+1} : i = 1, 2, \ldots, n\} \cup \{x_i y_i : i = 1, 2, \ldots, n, n+1\}$, respectively, where vertices x_{n+1}, y_{n+1}, and edge $x_{n+1} y_{n+1}$ are the duplicates of vertices x_1, y_1 and edge $x_1 y_1$, respectively. So, if c is an odd-graceful coloring for prism $C_n \times P_2$ and the prism is sliced through the edge $x_1 y_1$ in the above way, then we have: $c(x_1) = c(x_{n+1}), c(y_1) = c(y_{n+1})$, and $c(x_1 y_1) = c(x_{n+1} y_{n+1})$.
This ladder will be denoted by $L_{C_n \times P_2}$. Then, we can immediately see that $\mathcal{X}_{og}(C_n \times$

$P_2) \geq \mathcal{X}_{og}(L_{C_n \times P_2})$.
Furthermore, by Theorem 10 we may conclude that $\mathcal{X}_{og}(C_n \times P_2) \geq \mathcal{X}_{og}(L_{C_n \times P_2}) \geq 9$, for every $n \geq 10$.
Now, let us consider the odd-graceful colored graph $C_6 \times P_2$ in Figure 13. Slice

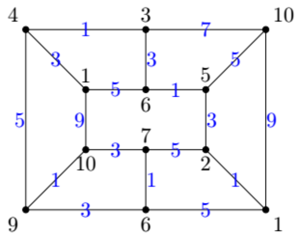

Figure 13. **An odd-graceful coloring for $C_6 \times P_2$.**

the colored $C_6 \times P_2$ through the edge having end vertex colors 1 and 2. On this, we obtain the colored open prism $L_{C_6 \times P_2}$ as depicted in Figure 14. Note that by

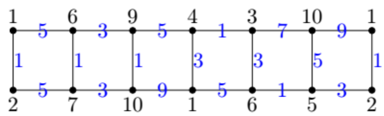

Figure 14. **An odd-graceful coloring for $L_{C_6 \times P_2}$.**

re-glueing(re-identifying) the edge and end vertices of $L_{C_6 \times P_2}$ which was sliced from $C_6 \times P_2$, we will get back the original $C_6 \times P_2$.

To obtain an odd-graceful coloring for $C_n \times P_2$ for $n \equiv 2 (\mathrm{mod}\ 4), n \geq 10$, we need to introduce a ladder of length $4k+1$, L_{4k+1}, for some integer $k \geq 1$. Let the vertex set and edge set of L_{4k+1} are $V = \{u_i, v_i : i = 1, 2, \ldots, 4k+1\}$ and $E = \{u_i u_{i+1}, v_i v_{i+1} : i = 1, 2, \ldots, 4k\} \cup \{u_i v_i : i = 1, 2, \ldots, 4k+1\}$, respectively. Then, define an odd-graceful coloring c' for L_{4k+1} similar with Eq. (2) as the following.

$$c'(u_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 4, \\ 6 & \text{if } i \equiv 2 \pmod 4, \\ 3 & \text{if } i \equiv 3 \pmod 4, \\ 8 & \text{if } i \equiv 0 \pmod 4, \end{cases} \text{ and } c'(v_i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod 4, \\ 7 & \text{if } i \equiv 2 \pmod 4, \\ 4 & \text{if } i \equiv 3 \pmod 4, \\ 9 & \text{if } i \equiv 0 \pmod 4. \end{cases} \tag{3}$$

Observe that $c'(u_1) = 1 = c'(u_{4k+1}), c'(v_1) = 2 = c'(v_{4k+1})$, and therefore, $c'(u_1 v_1) = 1 = c'(u_{4k+1} v_{4k+1})$, for every integer $k \geq 1$.
We have also $c(x_7) = c'(u_1)$ and $c(y_7) = c'(v_1)$, and therefore $c(x_7 y_7) = c'(u_1 v_1)$. For producing an odd-graceful coloring for $C_n \times P_2$ with $n \equiv 2 (\mathrm{mod}\ 4), n \geq 10$, we proceed by identifying the vertex u_1 and x_7, v_1 and y_7 and of course the edge $x_7 y_7$

and u_1v_1 of graph L_{4k+1}, $k \geq 1$, and graph $L_{C_6 \times P_2}$ in Figure 14, respectively. The resulting graph is a ladder of length $6 + 4k + 1 = 4l + 3$, for some integer $l \geq 2$. This ladder can be considered as an open prism $L_{C_{4l+2} \times P_2}$ of $C_{4l+2} \times P_2$. Since, $|c(x_1) - c(x_2)| \neq |c'(u_{4k}) - c'(u_{4k+1})|$ and $|c(y_1) - c(y_2)| \neq |c'(v_{4k}) - c'(v_{4k+1})|$, by identifying vertex x_1 and vertex u_{4k+1}, y_1 and v_{4k+1}, and edge x_1y_1 and $u_{4k+1}v_{4k+1}$, we obtain an odd-graceful coloring of $C_n \times P_2$, for $n \geq 10, n \equiv 2 \pmod 4$ with odd-graceful chromatic number $9 \leq \mathcal{X}_{og}(C_n \times P_2) \leq 10$. □

3. Discussion and Open Problems

Odd-graceful coloring topic is introduced. Our current focus on this coloring is on undirected finite simple graphs. We found some results on this observation. The prospective future researches can be about some modification of odd-graceful coloring to directed graphs.

Particularly on undirected finite simple graphs, finding odd-graceful chromatic numbers of some classes of well-known graphs is also worthwhile research topics.

Based on partial results we discussed above, we propose the following open problems. We start to propose odd-graceful chromatic number of caterpillars.

Problem 1. Calculate the odd-graceful chromatic number for some classes of caterpillars.

In relation with graphs $T_{\Delta,h}, h \geq 3$, we have the following open problem.

Problem 2. Find the exact value of the odd-graceful chromatic number of graph $T_{\Delta,h}$ for every integer $h \geq 3$.

Theorem 11 did not give yet the exact value for $\mathcal{X}_{og}(C_n \times P_2)$. Here, we propose the following.

Problem 3. Find the odd-graceful chromatic number of prism graph $C_n \times P_2$, with $n \geq 10, n \equiv 2 (\text{mod } 4)$.

As for general cases, we may consider graph coloring problems based on d-graceful labeling introduced by Pasotti [7].

Problem 4. Let G be a graph of size $q = d \times m$, for some positive integers d and m. Find the d-graceful chromatic number for classes of basic graphs with $d \notin \{1, q\}$.

Acknowledgement: Authors express gratitude to anonymous referees for the invaluable comments, corrections, and suggestions. Especially, authors give high appreciation to a referee who brought their attention to the paper [7]. The first author would also like to express his gratitude to Universitas Pendidikan Ganesha for the support provided through Grant No.: 1166/UN48.16/LT/2023.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] R. Alfarisi, R.M. Prihandini, R. Adawiyah, E.R. Albirri, and I.H. Agustin, *Graceful chromatic number of unicyclic graphs*, Journal of Physics: Conference Series, vol. 1306, IOP Publishing, 2019, pp. Article ID: 012039.
https://doi.org/10.1088/1742-6596/1306/1/012039.

[2] A.D. Byers, *Graceful Colorings and Connection in Graphs*, Ph.D. thesis, Kalamazoo, Michigan, USA, 2018.

[3] S. English and P. Zhang, *On graceful colorings of trees*, Math. Bohem. **142** (2017), no. 1, 57–73.
http://doi.org/10.21136/MB.2017.0035-15.

[4] R.B. Gnanajothi, *Topics in Graph Theory*, Ph.D. thesis, Madurai, India, 1991.

[5] A.I. Kristiana, A. Aji, E. Wihardjo, and D. Setiawan, *on graceful chromatic number of vertex amalgamation of tree graph family*, CAUCHY: Jurnal Matematika Murni dan Aplikasi **7** (2022), no. 3, 432–444.
http://doi.org/10.18860/ca.v7i3.16334.

[6] R. Mincu, C. Obreja, and A. Popa, *The graceful chromatic number for some particular classes of graphs*, 2019 21st International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC), 2019, pp. 109–115.
https://doi.org/10.1109/SYNASC49474.2019.00024.

[7] A. Pasotti, *On d-graceful graphs*, Ars Combin. **111** (2013), 207–223.

[8] J. Su, H. Sun, and B. Yao, *Odd-graceful total colorings for constructing graphic lattice*, Mathematics **10** (2022), no. 1, Article ID: 109.
https://doi.org/10.3390/math10010109.

[9] H. Wang, J. Xu, and B. Yao, *Twin odd-graceful trees towards information security*, Procedia Computer Science **107** (2017), 15–20.
https://doi.org/10.1016/j.procs.2017.03.050.