Communications in Combinatorics and Optimization
Vol. 10, No. 1 (2025), pp. 207-217
https://doi.org/10.22049/cco.2023.28704.1679

CCO
Commun. Comb. Optim.

Research Article

On the distance-transitivity of the folded hypercube

Seyed Morteza Mirafzal

Department of Mathematics, Faculty of Basic Science, Lorestan University,
Khorramabad, Iran
mirafzal.m@lu.ac.ir

Received: 1 June 2023; Accepted: 27 October 2023
Published Online: 3 November 2023

Abstract: The folded hypercube FQ_n is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, S)$, where $S = \{e_1, e_2, \ldots, e_n\} \cup \{u = e_1 + e_2 + \cdots + e_n\}$, and $e_i = (0, \ldots, 0, 1, 0, \ \ldots, 0)$, with 1 at the ith position, $1 \leq i \leq n$. In this paper, we show that the folded hypercube FQ_n is a distance-transitive graph. Then, we study some properties of this graph. In particular, we show that if $n \geq 4$ is an even integer, then the folded hypercube FQ_n is an *automorphic* graph, that is, FQ_n is a distance-transitive primitive graph which is not a complete or a line graph.

Keywords: distance-transitive graph, folded hypercube, distance regular graph, primitive graph, automorphic graph.

AMS Subject classification: 05C25, 94C15

1. Introduction

In this paper, a graph $\Gamma = (V, E)$ is considered as an undirected simple graph where $V = V(\Gamma)$ is the vertex-set and $E = E(\Gamma)$ is the edge-set. For all the terminology and notation not defined here, we follow [1, 3, 6].

Let $n \geq 3$ be an integer. The hypercube Q_n of dimension n is the graph with the vertex-set $\{(x_1, x_2, \ldots, x_n) \mid x_i \in \{0, 1\}\}$, two vertices $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are adjacent if and only if $x_i = y_i$ for all but one i. As a topology for an interconnection network of a multiprocessor system, the hypercube is a widely used and well-known model. The hypercube Q_n possesses many interesting properties, for example, its regularity, diameter and connectivity all are n. Also, it is bipartite and thus Q_n is 2-colorable. Moreover it is highly semmetric, that is, Q_n is vertex and edge-transitive [1, 6, 22]. There are many invariants of Q_n, for instance, generalized hypercube, folded hypercube, twisted hypercube, augmented hypercube and enhanced hypercube [2, 8, 22].

As a variant of the hypercube, the n-dimensional folded hypercube proposed first in [4]. The folded hypercube FQ_n of dimension n, is the graph obtained from the hypercube Q_n by adding edges, called complementary edges, between any two vertices $x = (x_1, x_2, \ldots, x_n)$, $y = (\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n})$, where $\bar{1} = 0$ and $\bar{0} = 1$. The folded hypercube FQ_n has some interesting properties, for example although it is regular of degree $n+1$ (while the hypercube Q_n is regular of degree n), its diameter is almost half of the hypercube Q_n, that is, $\lceil \frac{n}{2} \rceil$ [4]. It can be shown that the hypercube Q_n is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, B)$, where $B = \{e_1, e_2, \ldots, e_n\}$, e_i is the element of $\mathbb{Z}_2^n$ with 1 in the ith position and 0 in the other positions for, $1 \le i \le n$. Also, the folded hypercube FQ_n is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, S)$, where $S = B \cup \{u = e_1 + e_2 + \cdots + e_n\}$. Hence the hypercube Q_n and the folded hypercube FQ_n are vertex-transitive graphs. Since Q_n is Hamiltonian [9, 23] and it is a spanning subgraph of FQ_n, so FQ_n is Hamiltonian. Some properties of the folded hypercube FQ_n are discussed in [5, 9, 12, 21, 24].
The graphs shown in Figure 1. are the folded hypercubes FQ_3 and FQ_4

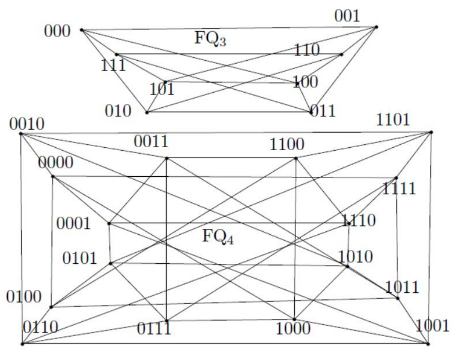

Figure 1. **The folded hypercubes FQ_3 and FQ_4**

We say that the graph Γ is *distance-transitive* if for all vertices u, v, x, y of Γ such that $d(u, v) = d(x, y)$, where $d(u, v)$ denotes the distance between the vertices u and v in Γ, there is an automorphism π in $\mathrm{Aut}(\Gamma)$ such that $\pi(u) = x$ and $\pi(v) = y$. The class of distance-transitive graphs contains many of interesting and important graphs. It is easy to see that the complete graphs K_n and the complete bipartite graph $K_{n,n}$ are distance-transitive. Also, it is not hard to check that the cycle C_n is distance-transitive. A more interesting example is the Petersen graph [6]. Another interesting example is the crown graph [13, 14, 17]. The class of Johnson graphs is one the important subclass of distance-transitive graphs [3, 14, 15, 18]. Another family of examples is the hypercube Q_n [1, 3, 6]. Distance-transitive graphs have been extensively studied from various aspects, by various authors and some of the works include [7, 10, 15, 16].
The fact that the folded hypercube is an edge-transitive graph, is one of the main results that has been shown in [9]. The result has been generalized in [12] by showing that the folded hypercube is in fact an arc-transitive graph.

In this paper we show, by an elementary and self-contained method, that the folded hypercube is in fact distance-transitive and hence distance-regular. Then, we study some properties of this graph. In particular, we show that if $n \geq 4$ is an even integer, then the hypercube FQ_n is an *automorphic* graph, that is, FQ_n is a distance-transitive primitive graph which is not a complete or a line graph.

2. Preliminaries

The graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are called *isomorphic*, if there is a bijection $\alpha : V_1 \longrightarrow V_2$ such that $\{a, b\} \in E_1$ if and only if $\{\alpha(a), \alpha(b)\} \in E_2$ for all $a, b \in V_1$. In such a case the bijection α is called an *isomorphism*. An *automorphism* of a graph Γ is an isomorphism of Γ with itself. The set of automorphisms of Γ with the operation of composition of functions is a group called the *automorphism group* of Γ and denoted by $\mathrm{Aut}(\Gamma)$.

The group of all permutations of a set V is denoted by $\mathrm{Sym}(V)$ or just $\mathrm{Sym}(n)$ when $|V| = n$. A *permutation group* G on V is a subgroup of $\mathrm{Sym}(V)$. In this case we say that G *acts* on V. If G acts on V we say that G is *transitive* on V (or G acts *transitively* on V) if given any two elements u and v of V, there is an element β of G such that $\beta(u) = v$. If Γ is a graph with vertex-set V then we can view each automorphism of Γ as a permutation on V and so $\mathrm{Aut}(\Gamma) = G$ is a permutation group on V.

A graph Γ is called *vertex-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. We say that Γ is *edge-transitive* if the group $\mathrm{Aut}(\Gamma)$ acts transitively on the edge-set E, namely, for any $\{x, y\}, \{v, w\} \in E(\Gamma)$, there is some π in $\mathrm{Aut}(\Gamma)$, such that $\pi(\{x, y\}) = \{v, w\}$. We say that Γ is *symmetric* (or *arc-transitive*) if for all vertices u, v, x, y of Γ such that u and v are adjacent, and also, x and y are adjacent, there is an automorphism π in $\mathrm{Aut}(\Gamma)$ such that $\pi(u) = x$ and $\pi(v) = y$. Note that if Γ is arc-transitive, then it is edge-transitive. Also, it is not hard to see that every distance-transitive graph is an arc-transitive graph. The automorphism group of a graph and its action on the vertex and edge or arc sets of a graph have crucial roles in finding some topological properties of the graph. Some recent works in this field include [11, 12, 15, 17, 19].

Let G be any abstract finite group with identity 1 and suppose Ω is a subset of G with the properties:
(i) $x \in \Omega \Longrightarrow x^{-1} \in \Omega$, (ii) $1 \notin \Omega$.
The *Cayley graph* Γ=Cay(G, Ω) is the (simple) graph whose vertex-set and edge-set are defined as follows: $V(\Gamma) = G$, $E(\Gamma) = \{\{g, h\} \mid g^{-1}h \in \Omega\}$.
It can be shown that the Cayley graph Γ=Cay(G, Ω) is connected if and only if the set Ω is a generating set in the group G [1].

The group G is called a semidirect product of N by Q, denoted by $G = N \rtimes Q$, if G contains subgroups N and Q such that: (i)$N \trianglelefteq G$ (N is a normal subgroup of G); (ii) $NQ = G$; and (iii) $N \cap Q = 1$ [20].
It has been shown in [12] that if $n > 3$, then $\mathrm{Aut}(FQ_n)$ is a semidirect product of N

by M, where N is isomorphic to the Abelian group $\mathbb{Z}_2^n$ and M is isomorphic to the group $Sym(n+1)$.

3. Main results

Let $\Gamma = (V, E)$ be a graph with diameter D. For each vertex v of Γ we let $\Gamma_i(v) = \{x \in V \mid d(x, v) = i\}$, $0 \leq i \leq D$. In other words $\Gamma_i(v)$ is the set of vertices of Γ which are at distance i from the vertex v. The stabilizer subgroup of v in A=Aut(Γ) denoted by A_v is defined to be the subgroup of automorphisms g of Γ such that $g(v) = v$. We have the following result.

Proposition 1. *[1, 6] Let $\Gamma = (V, E)$ be a vertex-transitive graph with diameter D and v be an arbitrary vertex of Γ. Then Γ is a distance-transitive graph if and only if there is a subgroup H of $Aut(\Gamma)_v$=A_v such that H acts transitively on every Γ_i, $0 \leq i \leq D$.*

One of the interesting properties in the folded hypercube, concerning the distances between vertices, is shown in the following result.

Proposition 2. *Let $\Gamma = FQ_n$. If $1 \leq i \leq \lceil \frac{n}{2} \rceil$, then $\Gamma_i(0)$=$\{v \mid w(v) = i\} \cup \{x \mid w(x) = n - i + 1\}$=$\{v \mid w(v) = i\} \cup \{v + u \mid w(v) = i - 1\}$,where $w(v)$ is the number of 1s in the n-tuple v ($u = e_1 + \cdots + e_n$).*

Proof. Let v be a vertex in the hypercube Q_n. Let $w(v)$ denote the weight of v, that is, the number of 1s in the n-tuple v. Let $0 = (0, 0, \ldots, 0)$ be the zero n-tuple in Q_n. It is easy to see that $d_{Q_n}(0, v) = w(v)$. Thus in the hypercube Q_n we have $Q_{n_i}(0) = \{y \in V(Q_n) \mid w(y) = i\}$. We know that the diameter of the folded hypercube FQ_n is $\lceil \frac{n}{2} \rceil$. Now it is easy to check that if $1 \leq i \leq \lceil \frac{n}{2} \rceil$, and $w(v) = i$ or $w(v) = n - i + 1$, then the distance between the zero vertex and v in FQ_n is i. In fact we can check that if $\Gamma = FQ_n$, then $\Gamma_i(0)$=$\{v \mid w(v) = i\} \cup \{v + u \mid w(v) = i - 1\}$, where $u = e_1 + e_2 + \cdots + e_n$, e_j is the element of $\mathbb{Z}_2^n$ with 1 in the jth position and 0 in the other positions for $1 \leq j \leq n$. Note that if $w(x) = j - 1$, $1 \leq j \leq \lceil \frac{n}{2} \rceil$, then $w(u + x) = n - (j - 1) = n - j + 1$, but $d_{FQ_n}(0, u + x) = j$. □

We now are ready to prove the following important theorem.

Theorem 1. *Let $n \geq 4$ be an integer. Then the folded hypercube FQ_n is a distance-transitive graph.*

Proof. Let $\Gamma = FQ_n$ and A=Aut(Γ). Let $v = 0$. In the rest of the proof we need some information about A_0, the stabilizer subgroup of the vertex 0 in the group A, and its action on the vertex-set of Γ explicitly. Note that the Abelian group Z_2^n is also a vector space over the field $F = \{0, 1\}$ and $B = \{e_1, e_2, \ldots, e_n\}$ is a basis of this vector space. It is easy to check that any n-subset of the set $S = B \cup \{u = e_1 + e_2 + \cdots + e_n\}$

is linearly independent over F and hence it is a basis of the vector space $\mathbb{Z}_2^n$. Let T be a subset of S with n elements and $f : B \longrightarrow T$ be a one to one function. We can extend f over $\mathbb{Z}_2^n$ linearly to a mapping $e(f)$, that is, if $v = a_1e_1 + a_2e_2 + \cdots + a_ne_n$, then $e(f)(v) = a_1f(e_1) + a_2f(e_2) + \cdots + a_nf(e_n)$. Thus $e(f)$ is a non-singular linear mapping of the vector space $\mathbb{Z}_2^n$ into itself such that $e(f)|_B = f$. Since B and T are bases of the vector space $\mathbb{Z}_2^n$, hence $e(f)$ is a permutation of $\mathbb{Z}_2^n$. Since $e(f)$ is an automorphism of the group $\mathbb{Z}_2^n$ which fixes the generating set S of the Cayley graph FQ_n, hence it is an automorphism of the folded hypercube FQ_n. Now it is easy to check that, $H = \{e(f) \mid f : B \longrightarrow T,\ T \subset S, |T| = n,\ f \textit{ is a one to one mapping}\}$, is a subgroup of the stabilizer group of the vertex $v = 0$. (In fact, it is not hard to show that H=A_0.) The graph FQ_n is a Cayley graph, thus it is a vertex-transitive graph, hence by Proposition 1, it is sufficient to show that the action of H on the set $\Gamma_i(0) = \Gamma_i$ is transitive, where $\Gamma_i(0)$ is the set of vertices at distance i from the vertex $v = 0$. Let x and y be two vertices in Γ_i. Then either $w(x) = w(y)$ or $w(x) \neq w(y)$.
First suppose that $w(x) = w(y)$. Let $x = e_{k_1} + \cdots + e_{k_i}$ and $y = e_{j_1} + \cdots + e_{j_i}$. There are vertices $e_{x_1}, \ldots, e_{x_{n-i}}$ and $e_{y_1}, \ldots, e_{y_{n-i}}$ in FQ_n such that $\{e_{k_1}, \ldots, e_{k_i}, e_{x_1}, \ldots, e_{x_{n-i}}\}$=$B$=$\{e_1, e_2, \ldots, e_n\}$=$\{e_{j_1}, \ldots, e_{j_i}, e_{y_1}, \ldots, e_{y_{n-i}}\}$. Let f be the permutation on the set B which is defined by the rule, $f(e_{k_r}) = e_{j_r}, 1 \leq r \leq i$, and $f(e_{x_l}) = e_{y_l}, 1 \leq l \leq n - i$. We now can see that $e(f)(x) = y$, where $e(f)$ is the linear extension of f to $\mathbb{Z}_2^n$. Note that $e(f) \in H$.
Now suppose that $w(x) \neq w(y)$. Without loss of generality we can assume that $w(x) = i$ and $w(y) = n - i + 1$. By Proposition 2, there is a vertex y_1 in Γ_{i-1} such that $w(y_1) = i - 1$ and $y = u + y_1$ (in fact $y_1 = y + u$).
Let $x = e_{k_1} + \cdots + e_{k_i}$ and $y_1 = e_{j_2} + \cdots + e_{j_i}$. There are vertices $e_{x_1}, \ldots, e_{x_{n-i}}$ and $e_{y_1}, \ldots, e_{y_{n-i}}$ in FQ_n such that $\{e_{k_1}, \ldots, e_{k_i}, e_{x_1}, \ldots, e_{x_{n-i}}\}$=$B$=$\{e_1, e_2, \ldots, e_n\}$ and $\{u, e_{j_2}, \ldots, e_{j_i}, e_{y_1}, \ldots, e_{y_{n-i}}\} = T$, $|T| = n$, $T \subset S$.
Let $f : B \longrightarrow T$ be a one to one function such that$f(e_{k_1}) = u$, $f(e_{k_r}) = e_{y_r}$, $2 \leq r \leq i$, $f(e_{x_r}) = e_{y_r}$, $1 \leq r \leq n - i$.
Now it is clear that for the automorphism $e(f)$ we have $e(f)(x) = y$. Now, since $e(f) \in H$, the result follows.

□

A block B, in the action of a group G on a set X, is a subset of X such that $B \cap g(B) \in \{B, \emptyset\}$, for each g in G. If G is transitive on X, then we say that the permutation group (X, G) is primitive if the only blocks are the trivial blocks, that is, those with cardinality 0,1 or $|X|$. In the case of an imprimitive permutation group (X, G), the set X is partitioned into a disjoint union of non-trivial blocks, which are permuted by G. We refer to this partition as a block system. A graph Γ is said to be primitive or imprimitive according to the group $\mathrm{Aut}(\Gamma)$ acting on $V(\Gamma)$ has the corresponding property. In the sequel, we need the following definition.

Definition 1. A graph $\Gamma = (V, E)$ of diameter D is said to be *antipodal* if for any $x, v, w \in V$ such that $d(x, v) = d(x, w) = D$, then we have $d(v, w) = D$ or $v = w$.

Let $\Gamma_i(x)$ denote the set of vertices of Γ at distance i from the vertex x. Let Γ be a distance-transitive graph. From Definition 1, it follows that if $\Gamma_D(x)$ is a singleton set, then the graph Γ is antipodal. It is easy to see that the hypercube Q_n is antipodal, since every vertex u has a unique vertex at maximum distance from it. Note that this graph is at the same time bipartite. We have the following result [1].

Proposition 3. *A distance-transitive graph Γ of diameter D has a block $X = \{v\} \cup \Gamma_D(v)$ if and only if Γ is antipodal, where $\Gamma_D(v)$ is the set of vertices of Γ at distance D from the vertex v.*

Also, we have the following important result [1].

Theorem 2. *An imprimitive distance-transitive graph is either bipartite or antipodal. (Both possibilities can occur in the same graph.)*

We have the following result.

Proposition 4. *[23] The folded hypercube FQ_n is a bipartite graph if and only if n is an odd integer.*

We now can state and prove the following fact concerning the folded hypercube FQ_n.

Theorem 3. *Let $n \geq 4$ be an integer. Then, the folded hypercube FQ_n is a primitive distance-transitive graph if and only if n is an even integer.*

Proof. By Theorem 1, the folded hypercube FQ_n is a distance-transitive graph. If n is an odd integer, then by Proposition 4, the folded hypercube FQ_n is a bipartite graph, thus by Theorem 2, it is imprimitive.
Let n be an even integer. Therefore, by Proposition 4, FQ_n is not bipartite. Let $n = 2m$. Thus the diameter of the FQ_n is m. Let v be a vertex in FQ_n such that $w(v) = m$. Let $t = u + v$, where $u = e_1 + e_2 + \cdots + e_n$. Hence $w(t) = m$. This follows that $d(0, v) = d(0, t) = m$, but $d(v, t) = 1 \neq m$. Hence FQ_{2m} is not antipodal. Thus, by Theorem 2, FQ_{2m} is primitive.

□

Let $\Gamma = (V, E)$ be a simple connected graph with diameter D. A *distance-regular* graph $\Gamma = (V, E)$, with diameter D, is a regular connected graph of valency k with the following property. There are positive integers

$$b_0 = k, b_1, \ldots, b_{D-1}; c_1 = 1, c_2, \ldots, c_D,$$

such that for each pair (u, v) of vertices satisfying $u \in \Gamma_i(v)$, we have
(1) the number of vertices in $\Gamma_{i-1}(v)$ adjacent to u is c_i, $1 \leq i \leq D$.

(2) the number of vertices in $\Gamma_{i+1}(v)$ adjacent to u is b_i, $0 \leq i \leq D-1$.
The intersection array of Γ is $i(\Gamma) = \{k, b_1, \ldots, b_{D-1}; 1, c_2, ..., c_D\}$.
It is easy to show that if Γ is a distance-transitive graph, then it is distance-regular [1]. Hence, the hypercube Q_n, $n > 2$ is a distance-regular graph. We can verify by an easy argument that the intersection array of Q_n is

$$\{n, n-1, n-2, \ldots, 1; 1, 2, 3, \ldots, n\}.$$

In other words, for hypercube Q_n, we have $b_i = n - i$, $c_i = i$, $1 \leq i \leq n-1$, and $b_0 = n$, $c_n = n$. In the following theorem, we determine the intersection array of the Folded hypercube FQ_n.

Proposition 5. *Let $n > 3$ be an integer and $\Gamma = FQ_n$ be the folded hypercube. Let D denote the diameter of FQ_n. Then for the intersection array of this graph we have b_i=$n+1-i$, $0 \leq i < D$. c_i=i, $1 \leq i \leq D$ (note that D=$\lceil \frac{n}{2} \rceil$).*

Proof. Nothing to what is stated in the proof of Proposition 2, the proof of the theorem is straightforward.

□

An *automorphic* graph is a distance-transitive graph whose automorphism group acts primitively on its vertices, and not a complete graph or a line graph [1].
Automorphic graphs are apparently very rare. For instance, there are exactly three cubic automorphic graphs [1]. It is clear that for $n \geq 3$, the graph FQ_n is not a complete graph. In the sequel, we show that if $n \geq 4$ is an even integer, then the graph FQ_n is an automorphic graph. In the first step, we show that FQ_n is not a line graph. In the rest of our paper , we need some information about the eigenvalues of FQ_n. We do not need the spectrum of FQ_n, that is, all the eigenvalues of FQ_n.
Let Γ be a graph with vertex set $V(\Gamma) = V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(\Gamma)$. The adjacency matrix $A = A(\Gamma) = [a_{ij}]$ of Γ is an $n \times n$ symmetric matrix of $0s$ and $1s$ with $a_{ij} = 1$ if and only if v_i and v_j are adjacent. The characteristic polynomial of Γ is the polynomial $P(G) = P(G, x) = \det(xI_n - A)$, where I_n denotes the $n \times n$ identity matrix. The spectrum of $A(\Gamma)$ is also called the spectrum of Γ. If the distinct eigenvalues are ordered by $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, and their multiplicities are $m_1, m_2, \ldots, m_r$, respectively, then we write,

$$Spec(\Gamma) = \binom{\lambda_1, \lambda_2, \ldots, \lambda_r}{m_1, m_2, \ldots, m_r} \text{ or } Spec(\Gamma) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, ..., \lambda_r^{m_r}\}.$$

Let Γ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and adjacency matrix A, and the rows and columns of A are labeled by the set V. Let π be a permutation of the set V. We know that π can be represented by a permutation matrix $P_\pi = P = (p_{ij})$, where $p_{ij} = 1$ if $v_i = \pi(v_j)$, and $p_{ij} = 0$ otherwise. It is a well known fact that π is an automorphism of the graph Γ if and only if $AP = PA$ [1].

Let $\Gamma = (V, E)$ be a graph. The line graph $L(\Gamma)$ of the graph Γ is constructed by taking the edges of Γ as vertices of $L(\Gamma)$, and joining two vertices in $L(\Gamma)$ whenever the corresponding edges in Γ have a common vertex. Note that if $e = \{v, w\}$ is an edge of Γ, then its degree in the graph $L(\Gamma)$ is $\deg(v) + \deg(w) - 2$. Concerning the eigenvalues of the line graphs, we have the following fact [1].

Proposition 6. *If λ is an eigenvalue of a line graph $L(\Gamma)$, then $\lambda \geq -2$.*

Therefore, if $\lambda < -2$ is an eigenvalue of a graph Γ, then Γ is not a line graph.
In the proof of the following theorem, we need the following fact.

Proposition 7. *Let $\Gamma = FQ_n$. Then the mapping $\alpha : V(\Gamma) \to V(\Gamma)$, $\alpha(v) = v^c$, where v^c is the complement of v ($v^c = (\bar{x_1}, \bar{x_2}, ..., \bar{x_n})$, when $v = (x_1, x_2, \ldots, x_n)$, $\bar{1}$=0, $\bar{0}$=1), is an automorphism of Γ and the hypercube Q_n.*

Proof. The proof is straightforward.

□

Using this result we show that, without having the spectrum of the folded hypercube FQ_n in the hand, if $n \geq 4$, then FQ_n has an eigenvalue less than -2, hence it is not a line graph.

Theorem 4. *If $n \geq 4$, then FQ_n is not a line graph.*

Proof. If $\Gamma = FQ_n$, then by Proposition 7, the permutation $\alpha : V(\Gamma) \to V(\Gamma)$, $\alpha(v) = v^c$, where v^c is the complement of the set v, is an automorphism of the graph Γ and the hypercube Q_n. Thus, if P is the permutation matrix of α, then we have $MP = PM$ where M is the adjacency matrix of the graph FQ_n.
It is not hard to check that the adjacency matrix of FQ_n is of the form $M = A + P$, where A is the adjacency matrix of the hypercube Q_n. Since α is of order 2, then $P^2 = E$ where $E = I_h$ is the identity matrix of size h ($h = 2^n$). Hence if $p(x) = x^2 - 1$, then $p(P) = 0$. Thus, if μ is an eigenvalue of the matrix P, then $p(\mu) = 0$, namely, $\mu \in \{1, -1\}$. Since α is an automorphism of the graph Q_n, thus $AP = PA$. On the other hand, the matrices A and P are symmetric, hence the matrices A and P are diagonalizable, and therefore there is a basis $B = \{u_1, \ldots, u_h\}$ of $\mathbb{R}^h$ such that each u_i is an eigenvector of the matrices A and P [6]. Therefore, if $Au_i = \lambda_i u_i$, then $Mu_i = (A + P)u_i = \lambda_i u_i + t_i u_i = (\lambda_i + t_i)u_i$, where $t_i \in \{1, -1\}$. Every eigenvalue of the hypercube Q_n is of the form $n - 2i$, $0 \leq i \leq n$, [1]. Thus, for $i = n$, $n - 2n + 1 = -n + 1$, or $n - 2n - 1 = -n - 1$ is an eigenvalue of the folded hypercube FQ_n. Nothing that $n \geq 4$, FQ_n has an eigenvalue δ such that $\delta \leq -3$. Now, by Proposition 6, the hypercube FQ_n is not a line graph.

□

Theorem 5. *Let $n \geq 4$ be an integer. Then the folded hypercube FQ_n is an automorphic graph if and only if n is an even integer.*

Proof. By Theorem 3, the folded hypercube FQ_n is a primitive distance-transitive graph if and only if n is an even integer. By Theorem 4, FQ_n is not a line graph. It is clear that FQ_n is not a complete graph. We now conclude that the folded hypercube FQ_n is automorphic if and only if n is an even integer.

□

Acknowledgements: The author is thankful to the anonymous reviewers for their valuable comments and suggestions.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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