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Research Article

NP-completeness of some generalized hop and step domination parameters in graphs

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Abstract: Let $r \geq 2$. A subset S of vertices of a graph G is a r-hop independent dominating set if every vertex outside S is at distance r from a vertex of S, and for any pair $v, w \in S$, $d(v, w) \neq r$. A r-hop Roman dominating function (rHRDF) is a function f on $V(G)$ with values $0, 1$ and 2 having the property that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u with $f(u) = 2$ and $d(u, v) = r$. A r-step Roman dominating function (rSRDF) is a function f on $V(G)$ with values $0, 1$ and 2 having the property that for every vertex v with $f(v) = 0$ or 2, there is a vertex u with $f(u) = 2$ and $d(u, v) = r$. A rHRDF f is a r-hop Roman independent dominating function if for any pair v, w with non-zero labels under f, $d(v, w) \neq r$. We show that the decision problem associated with each of r-hop independent domination, r-hop Roman domination, r-hop Roman independent domination and r-step Roman domination is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Keywords: dominating set, hop dominating set, step dominating set, hop independent set, hop Roman dominating function, hop Roman independent dominating function, complexity.

AMS Subject classification: 05C69

1. Introduction

For a graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$, the *order* of G is $n(G) = n_G = |V(G)|$ and the *size* of G is $m(G) = m_G = |E(G)|$. The *open neighborhood* of a vertex v is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. The *degree*

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of v, denoted by $\deg(v)$, is $|N_G(v)|$, and the *open neighborhood* of a subset $S \subseteq V$, is $N_G(S) = \bigcup_{v \in S} N_G(v)$. The *distance* between two vertices u and v in G, denoted by $d(u,v)$, is the minimum length of a (u,v)-path in G. A *bipartite graph* is a graph whose vertices can *chordal graph* is a graph that does not contain an induced cycle of length greater than 3. A *planar graph* is a graph which can be drawn in the plane without any edges crossing. A *vertex cover* of a graph is a set of vertices such that each edge of the graph is incident with at least one vertex of the set. A subset S of vertices of a graph G is a *dominating set* of G if every vertex in $V(G) - S$ has a neighbor in S. For notation and graph theory terminology not given here, we refer to [12].

Chartrand, Harary, Hossain, and Schultz [5] introduced the concept of r-step domination in graphs. For an integer $r \geq 1$, two vertices in a graph G are said to *r-step dominate* each other if they are at distance exactly r apart in G. A set S of vertices in G is a *r-step dominating set* of G if every vertex in $V(G)$ is r-step dominated by some vertex of S. The *r-step domination number*, $\gamma_{rstep}(G)$ of G, is the minimum cardinality of a r-step dominating set of G. The concept of r-step was further studied, for example in [4, 11, 14, 25]. Ayyaswamy et al. [3, 20] introduced the a similar concept, namely, hop domination in graphs. A subset S of vertices of a graph G is a *hop dominating set* (HDS) if every vertex outside S is at distance two from a vertex of S. The *hop domination number*, $\gamma_h(G)$ of G, is the minimum cardinality of an HDS of G. A subset S of vertices of a graph G is a *hop independent dominating set* (HIDS) if S is a HDS and for any pair $v, w \in S$, $d(v,w) \neq 2$. The *hop independent domination number* of G is the minimum cardinality of an HIDS of G. The concept of hop domination was further studied, for example, in [2, 13, 17]. A generalized version of hop domination, namely r-hop domination, (for any $r \geq 2$) is studied in [17]. For $r \geq 2$, a subset S of vertices of G is a *r-hop dominating set* (rHDS) if every vertex outside S is at distance r from a vertex of S. The *r-hop domination number* of G, is the minimum cardinality of a rHDS of G. For a subset $S \subseteq V(G)$ and a vertex $v \in V(G)$, we say that v is r-hop dominated by S (or S r-hop dominates v) if either $v \in S$ or $v \notin S$ and $d(u,v) = r$ for some vertex $u \in S$. Likewise, a subset S of vertices of G is a *r-hop independent dominating set* (rHIDS) if every vertex outside S is at distance r from a vertex of S, and for any pair $v, w \in S$, $d(v,w) \neq r$.

A function $f : V \longrightarrow \{0,1,2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N(v)$ with $f(u) = 2$, is called a *Roman dominating function* or just an RDF. The *weight* of an RDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. For an RDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus an RDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$. The mathematical concept of Roman domination, was defined and discussed by Stewart [24], and ReVelle and Rosing [21], and was subsequently developed by Cockayne et al. [10]. Many variations, generalizations and applications of Roman dominations parameters have been studied, and to see the latest progress until 2020 see [6–9].

Shabani at al. [23] introduced the concept of hop Roman dominating functions. A *hop Roman dominating function* (HRDF) is a function $f : V \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u with $f(u) = 2$ and $d(u, v) = 2$. The weight of an HRDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an HRDF on G is called the *hop Roman domination number* of G and is denoted $\gamma_{hR}(G)$. For an HRDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus an HRDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$. For a function $f = (V_0, V_1, V_2)$ and a vertex $v \in V(G)$, we say that v is hop Roman dominated by f (or f hop Roman dominates v), if either $v \in V_1 \cup V_2$ or there exist $u \in V_2$, such that $d(v, u) = 2$. An HRDF $f = (V_0, V_1, V_2)$ is a *hop Roman independent dominating function*(HRIDF) if for any pair $v, w \in V_1 \cup V_2$, $d(v, w) \neq 2$. The minimum weight of an HRIDF on G is called the *hop Roman independent domination number* of G. The concept of hop Roman domination was further studied, for example in [1, 15, 22].

We consider a generalized version of hop Roman domination. For $r \geq 2$, a *r-hop Roman dominating function* (rHRDF) is a function $f : V \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u with $f(u) = 2$ and $d(u, v) = r$. The weight of a rHRDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a rHRDF on G is called the *r-hop Roman domination number* of G and is denoted $\gamma_{rhR}(G)$. For a function $f = (V_0, V_1, V_2)$ and a vertex $v \in V(G)$, we say that v is r-hop Roman dominated by f (or f r-hop Roman dominates v), if either $v \in V_1 \cup V_2$ or there exist $u \in V_2$, such that $d(v, u) = r$. A rHRDF $f = (V_0, V_1, V_2)$ is a *r-hop Roman independent dominating function*(rHRIDF) if for any pair $v, w \in V_1 \cup V_2$, $d(v, w) \neq r$. The minimum weight of a rHRIDF on G is called the *r-hop Roman independent domination number* of G. Likewise, a *r-step Roman dominating function* (rSRDF) is a function $f : V \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V_0 \cup V_2$ there is a vertex $u \in V_2$ such that $d(u, v) = r$. The weight of a rSRDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a rSRDF on G is called the *r-step Roman domination number* of G.

Farhadi et al. [17] proved that for $r \geq 2$, the decision problems associated with both r-step domination and r-hop domination are NP-complete for planar bipartite graphs and planar chordal graphs. Jafari Rad et al. [16] proved that the decision problems associated with hop independent domination, r-hop Roman domination and the hop Roman independent domination are NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

In this paper we study the complexity of decision problems associated with the r-hop independent domination, r-hop Roman domination, r-hop Roman independent domination and r-step Roman domination. We show that the decision problem associated to each of these problems is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs. We use a transformation of the Vertex Cover Problem which was one of Karp's 21 NP-complete problems [19] (see also [18]). The Vertex Cover Problem is the following decision problem.

Vertex Cover Problem (VCP).

Instance: A non-empty graph G, and a positive integer k.

Question: Does G have a vertex cover of size at most k?

2. r-Hop Independent Domination

Consider the following decision problem:

r-Hop Independent Dominating Problem (rHIDP).
Instance: A non-empty graph G and two positive integers $r \geq 2$ and $k \geq 1$.
Question: Does G have a r-hop independent dominating set of size at most k?
We show that the decision problem for rHIDP is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Theorem 1. *r-HIDP is NP-complete for planar bipartite graphs.*

Proof. Clearly, the rHIDP is NP, since it is easy to verify a "yes" instance of the rHIDP in polynomial time. Now we transform the vertex cover problem to the rHIDP so that one of them has a solution if and only if the other has a solution. Let G be a connected planar bipartite graph of order n_G and size $m_G \geq 2$. Let H be the graph obtained from G as follows. For each edge $e = uv \in E(G)$, we subdivide the edge e, $2r-1$ times. Let $x_e^1, x_e^2, \ldots, x_e^{2r-1}$ be the subdivided vertices that are produced by subdividing e, where x_e^i is adjacent to x_e^{i+1}, for $i = 1, 2, \ldots, 2r-2$, u is adjacent to x_e^1, and v is adjacent to x_e^{2r-1}. For every vertex $v \in V(G) \cup \{x_e^1, x_e^2, \ldots, x_e^{2r-1}\}$, we add a P_{2r+1}-path $P_{2r+1}^v : v_1 v_2 \ldots v_{2r+1}$, and join v_{r+1} to v, and then subdivide the edge $v_{r+1}v$ $2r-2$ times. Let y_v^1, y_v^2, $\ldots$, y_v^{2r-2} be the subdivided vertices that were produced by subdividing the edge $v_{r+1}v$, where y_v^1 is adjacent v_{r+1} and y_v^{2r-2} is adjacent to v. For every vertex $v \in \{x_e^r \mid e \in E(G)\}$ we subdivide the edge vy_v^{2r-2}, and let z_v be the subdivided vertex, where z_v is adjacent to both v and y_v^{2r-2}. Finally, for every vertex $v \in \{x_e^r \mid e \in E(G)\}$, add a vertex v' and join v' to both x_e^1 and x_e^{2r-1} and then subdivide each edge $v'x_e^1$ and $v'x_e^{2r-1}$, $r-2$ times. The resulting graph H has order $n_H = 4rn_G + (8r^2 - 2r - 2)m_G$ and size $m_H = (4r-1)n_G + (8r^2 - 2r - 1)m_G$. Figure 1 illustrates the graph H if G is a path P_3 and $r = 2$.

We show that G has a vertex cover of size at most k if and only if H has an rHIDS of size at most $k + rn_G + rm_G(2r-1)$. Assume S_G is a vertex cover of size at most k. Let

$$
\begin{aligned}
S_H &= S_G \cup \{v_{r+1}, v_{r+2}, \ldots, v_{2r} \mid v \in S_G\} \\
&\cup \ \{v_{r+1}, y_v^1, y_v^2, \ldots, y_v^{r-1} \mid v \in \big((V(G) - S_G) \cup \{x_e^1, x_e^2, \ldots, x_e^{2r-1} \mid e \in E(G)\}\big)\}.
\end{aligned}
$$

Clearly $d(a,b) \neq r$ for any pair $a, b \in S_H$. We show S_H is a rHIDS of size at most $k + rn_G + rm_G(2r-1)$. For each $e \in E(G)$, the vertices x_e^r and $x_e^{r\prime}$ are r-hop dominated by S_G, any vertex on the path from $x_e^{r\prime}$ to x_e^1 is r-hop dominated by $\{x_e^1{}_{r+1}, y_{x_e^1}^1, y_{x_e^1}^2, \ldots, y_{x_e^1}^{r-1}\}$, and any vertex on the path from $x_e^{r\prime}$ to x_e^{2r-1} is r-hop dominated by $\{x_e^{2r-1}{}_{r+1}, y_{x_e^{2r-1}}^1, y_{x_e^{2r-1}}^2, \ldots, y_{x_e^{2r-1}}^{r-1}\}$. For any vertex $v \in S_G$, any vertex in $\{v_1, v_2, \ldots, v_{2r+1}\} \cup \{y_v^1, y_v^2, \ldots, y_v^{2r-2}\}$ is hop dominated by $\{v_{r+1}, v_{r+2}, \ldots, v_{2r}\}$. For any vertex $v \in V(G) - S_G$, any vertex in $\{v_1, v_2, \ldots, v_{2r+1}\} \cup \{y_v^1, y_v^2, \ldots, y_v^{2r-2}\}$ is hop dominated

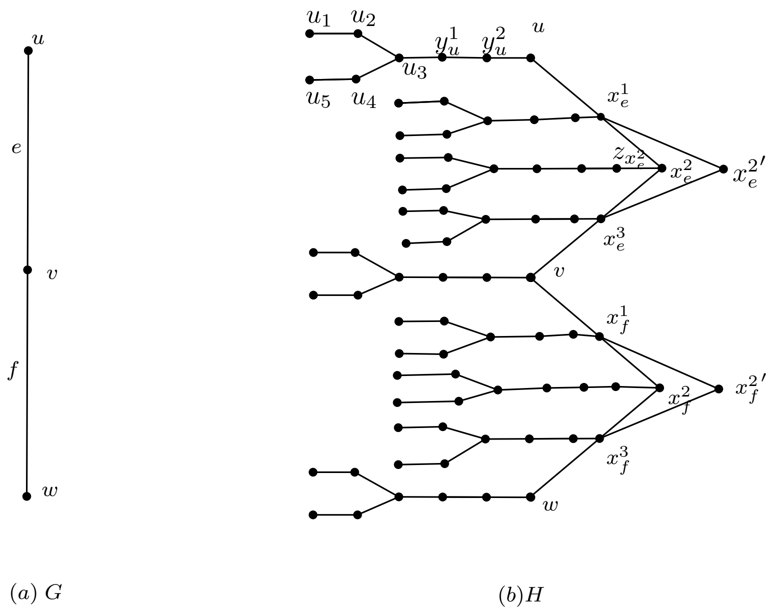


Figure 1. **The graphs G and H in the proof of Theorem 1**

by $\{v_{r+1}, y_v^1, y_v^2, \ldots, y_v^{r-1}\}$. For any edge $e \in E(G)$, any vertex in

$$\{x_{e1}^r, x_{e2}^r, \ldots, x_{e2r+1}^r\} \cup \{y_{x_e^r}^1, y_{x_e^r}^2, \ldots, y_{x_e^r}^{2r-2}\}$$

is r-hop dominated by $\{x_{er+1}^r, y_{x_e^r}^1, y_{x_e^r}^2, \ldots, y_{x_e^r}^{r-1}\}$. Similarly, for any edge $e \in E(G)$, any vertex in $\{x_e^i, x_{e1}^i, x_{e2}^i, \ldots, x_{e2r+1}^i\} \cup \{y_{x_e^i}^1, y_{x_e^i}^2, \ldots, y_{x_e^i}^{2r-2}\}$, where $i \neq r$, is r-hop dominated by $\{x_{er+1}^i, y_{x_e^i}^1, y_{x_e^i}^2, \ldots, y_{x_e^i}^{r-1}\}$. Consequently, S_H is a rHIDS of size at most $k + rn_G + rm_G(2r-1)$.

Assume next that H has a rHIDS, S_H, of size at most $k + rn_G + rm_G(2r-1)$. It is evident that for any vertex $v \in V(G) \cup \{x_e^1, x_e^2, \ldots, x_e^{2r-1} \mid e \in E(G)\}$,

$$|S_H \cap \{v_1, v_2, \ldots, v_{2r+1}, y_v^1, y_v^2, \ldots, y_v^{2r-2}\}| \geq r.$$

Let

$$A = S_H \cap \bigcup_{v \in V(G) \cup \{x_e^1, x_e^2, \ldots, x_e^{2r-1} | e \in E(G)\}} (\{v_1, v_2, \ldots, v_{2r+1}, y_v^1, y_v^2, \ldots, y_v^{2r-2}\}).$$

Then $|A| \geq rn_G + rm_G(2r-1)$, and so $|S_H - A| \leq k$. For any edge $e = uv$, since $x_e^{r\prime}$ is r-hop dominated by S_H, either $x_e^{r\prime} \in S_H$ or $S_H \cap \{u, v\} \neq \emptyset$. If for an edge $e = uv$, $S_H \cap \{u, v\} = \emptyset$, then $x_e^{r\prime} \in S_H$, and we replace S_H by $(S_H - \{x_e^{r\prime}\}) \cup \{u\}$. Thus we assume that for any edge $e = uv$, $S_H \cap \{u, v\} \neq \emptyset$. Thus $S_H \cap V(G)$ is a vertex cover for G of size at most k. Therefore G has a vertex cover of size at most k, as desired. □

We next prove the NP-completeness of rHIDP for planar chordal graphs.

Theorem 2. *rHIDP is NP-complete for planar chordal graphs.*

Proof. Let G be a planar chordal graph of order n_G and size $m_G \geq 2$, and let H be the graph presented in the proof of Theorem 1. For any edge $e \in E(G)$, let $x_e^1, x_e^{2'}, \ldots, x_e^{r-1'}, x_e^{r'}$ be vertices on the path from x_e^1 to $x_e^{r'}$, and $x_e^{r'}, x_e^{r+1'}, \ldots, x_e^{2r-1}$ be the vertices on the path from $x_e^{r'}$ to x_e^{2r-1}. We join x_e^i to both $x_e^{i'}$ and $x_e^{i+1'}$ for each $i = 2, 3, \ldots, 2r-3$, and join x_e^{2r-2} to $x_e^{2r-2'}$. Let H' be the constructed graph. Clearly H' is a planar chordal graph. Now with the same argument given in the proof of Theorem 1, we can see that G has a vertex cover of size at most k if and only if H' has an rHIDS of size at most $k + rn_G + rm_G(2r-1)$. □

3. r-Hop Roman Domination

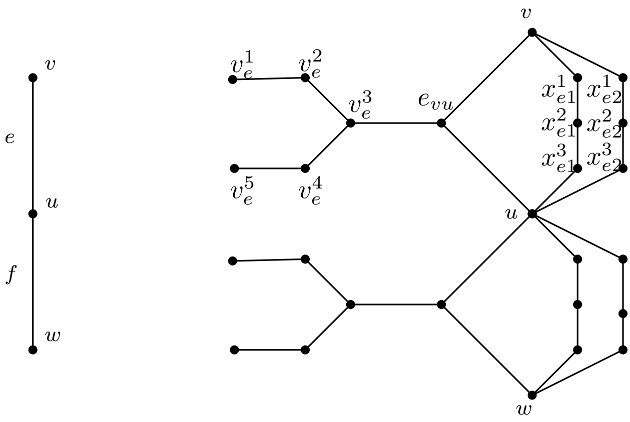
Consider the following decision problem:
r-Hop Roman Dominating Function Problem (rHRDFP).
Instance: A non-empty graph G, and two positive integers $r \geq 2$ and $k \geq 1$.
Question: Does G have a r-hop Roman dominating function of weight at most k?

We show that the decision problem for the rHRDFP is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Theorem 3. *For $r \geq 2$, rHRDFP is NP-complete for planar bipartite graphs.*

Proof. Clearly, the rHRDFP is in NP. We transform the vertex cover problem to the rHRDFP so that one of them has a solution if and only if the other one has a solution. Let G be a connected planar bipartite graph of order n_G and size $m_G \geq 2$, and let H be the graph obtained from G as follows: We convert each edge $e = vu \in E(G)$ into a double edge $e_1 = vu$, and $e_2 = vu$, and then subdivide each of edges e_1 and e_2, $2r-1$ times. Let the vertices $x_{e_i}^1, x_{e_i}^2, \ldots, x_{e_i}^{2r-2}$ be the vertices that were produced from subdividing the edge e_i, for $i = 1, 2$, where the vertex $x_{e_i}^1$ is adjacent to v, for $i = 1, 2$. For each edge $e = vu \in E(G)$, we add a new vertex e_{vu} and a P_{2r+1}-path $v_e^1 v_e^2 \ldots v_e^{2r+1}$, join the vertex e_{vu} to u, v and v_e^{r+1}. Finally, we subdivide the edge $e_{vu}v_e^{r+1}$, $r-2$ times. Let $y_v^1, \ldots, y_v^{r-2}$ be the subdivided vertices produced by subdivision of $e_{vu}v_e^{r+1}$, where y_v^1 is adjacent to v_e^{r+1} and y_v^{r-2} is adjacent to e_{uv}. The resulting graph H has order $n_H = n_G + (7r-2)m_G$ and size $m_H = (7r+1)m_G$. Figure 2 illustrates the graph H if G is a path P_3 and $r = 2$. We note that since G is connected and planar, so H is connected and planar. Further, by construction, H is bipartite. Thus, H is a connected planar bipartite graph.
We show that G has a vertex cover of size at most k if and only if H has a rHRDF of weight $2k + 2rm_G$. Assume that G has a vertex cover, S_G, of size at most k. Let

$$S_H = S_G \cup \bigcup_{e=uv \in E(G)} \{v_e^{r+1}, y_v^1, \ldots, y_v^{r-2}, e_{vu}\}.$$



$(a)G$ $(b)H$

Figure 2. **The graph G and H in the proof of Theorem 3**

We show that $f = (V(H) - S_H, \emptyset, S_H)$ is an rHRDF for H of weight at most $2k + 2rm_G$. For every edge $e = vu \in E(G)$, the vertex v_e^{r+1} r-hop Roman dominates the vertices v_e^1, v_e^{2r+1}, u and v in H, while the vertex y_v^i $(i = 1, 2, \ldots, r-2)$ r-hop dominates the vertices v_e^{i+1}, v_e^{2r+1-i}, $x_{e_1}^i$, $x_{e_2}^i$, $x_{e_1}^{2r-i}$ and $x_{e_2}^{2r-i}$. Furthermore, e_{vu} r-hop Roman dominates the vertices $x_{e_1}^{r+1}$ and $x_{e_2}^{r+1}$, since S_G is a vertex cover in G. Therefore, the function f is a rHRDF for H of weight at most $2k + 2rm_G$.

Assume next that $f = (V_0^f, V_1^f, V_2^f)$ is a rHRDF for H of weight $2k + 2rm_G$. Without loss of generality we assume that f has minimum weight. If for an edge $e \in E(G)$, $f(v_e^1) + \cdots + f(v_e^{2r+1}) + f(y_v^1) + \cdots + f(y_v^{r-2}) + f(e_{vu}) < 2r$, then there is a vertex in $\{v_e^1, \ldots, v_e^{2r+1}\}$ such that it is not r-hop Roman dominated by f, a contradiction. Therefore, $f(v_e^1) + \cdots + f(v_e^{2r+1}) + f(y_v^1) + \cdots + f(y_v^{r-2}) + f(e_{vu}) \geq 2r$ for every edge $e \in E(G)$. If for an edge $e \in E(G)$, $f(v_2^e) + f(v_4^e) + f(e_{vu}) \leq 1$, then v_2^e or v_4^e is not hop Roman dominated by f, a contradiction. Therefore, $f(v_2^e) + f(v_4^e) + f(e_{vu}) \geq 2$ for every edge $e \in E(G)$. Suppose that there exists an edge $e = uv \in E(G)$ such that $f(x_{e_i}^r) > 0$ for each $i = 1, 2$. Assume that $f(u) \geq f(v)$. Then the function g defined by $g(x_{e_1}^r) = g(x_{e_2}^r) = 0$, $g(u) = \max\{f(u), 2\}$ and $g(z) = f(z)$ otherwise, is an rHRDF. If $f(u) \neq 0$ then $g(V) < f(V)$, a contradiction by the choice of f. Thus, assume that $f(u) = 0$, and so g is a minimum rHRDF. Thus we may assume that $f(x_{e_1}^r) = f(x_{e_2}^r) = 0$ for any edge $e = uv \in E(G)$. Then either $f(u) = 2$ or $f(v) = 2$. Hence, $S_G = V_2^f \cap V(G)$ is a vertex cover of G of size at most $\frac{1}{2}\big(w(f) - 2rm_G\big)$. Thus, G has a vertex cover of size at most k. $\square$

4. r-Hop Roman Independent Domination

We next study the complexity issue of the r-hop Roman independent domination. Consider the following decision problem:

r-Hop Roman Independent Dominating Function Problem (HRIDFP).
Instance: A non-empty graph G, and two positive integers $r \geq 2$ and $k \geq 1$.
Question: Does G have a r-hop Roman independent dominating function of weight at most k?

We show that the decision problem for rHRIDFP is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Theorem 4. *For $r \geq 2$, rHRIDFP is NP-complete for planar bipartite graphs.*

Proof. Let G be a graph of order n_G and size m_G, and let H be the connected planar bipartite graph constructed in the proof of Theorem 1. Note that H has order $n_H = 4rn_G + (8r^2 - 2r - 2)m_G$ and size $m_H = (4r-1)n_G + (8r^2 - 2r - 1)m_G$. We show that G has a vertex cover of size at most k if and only if H has an rHRIDF of weight at most $2k + 2rn_G + 2rm_G(2r-1)$. Assume first that G has a vertex cover, S_G, of size at most k. Let

$$\begin{aligned} S_H &= S_G \cup \{v_{r+1}, v_{r+2}, \ldots, v_{2r} \mid v \in S_G\} \\ &\cup \ \{v_{r+1}, y_v^1, y_v^2, \ldots, y_v^{r-1} \mid v \in \big((V(G) - S_G) \cup \{x_e^1, x_e^2, \ldots, x_e^{2r-1} \mid e \in E(G)\}\big)\}. \end{aligned}$$

Clearly $d(a,b) \neq r$ for any pair $a, b \in S_H$. We set $f = (V(H) - S_H, \emptyset, S_H)$. As it is proved in the proof of Theorem 1, that S_H is a rHIDS for H, we conclude that any vertex v with $f(v) = 0$ is r-hop dominated by a vertex u with $f(u) = 2$. Hence H has a rHRIDF of weight at most $2k + 2rn_G + 2rm_G(2r-1)$.

Assume now that H has a rHRIDF f, of weight at most $2k + 2rn_G + 2rm_G(2r-1)$. It is evident that for any vertex $v \in V(G) \cup \{x_e^1, x_e^2, \ldots, x_e^{2r-1} \mid e \in E(G)\}$,

$$\sum_{v \in \{v_1, v_2, \ldots, v_{2r+1}, y_v^1, y_v^2, \ldots, y_v^{2r-2}\}} f(v) \geq 2r.$$

Let

$$A = S_H \cap \bigcup_{v \in V(G) \cup \{x_e^1, x_e^2, \ldots, x_e^{2r-1} \mid e \in E(G)\}} (\{v_1, v_2, \ldots, v_{2r+1}, y_v^1, y_v^2, \ldots, y_v^{2r-2}\}).$$

Then $\sum_{v \in A} f(v) \geq 2rn_G + 2rm_G(2r-1)$. For any edge $e = uv$, since both x_e^r and $x_e^{r\prime}$ are r-hop dominated by f, either $f(x_e^r) \geq 1$ and $f(x_e^{r\prime}) \geq 1$, or $2 \in \{f(u), f(v)\}$. If

$2 \notin \{f(u), f(v)\}$, then we replace $f(u)$ by 2 and both $f(x_e^r)$ and $f(x_e^{r\prime})$ by 0. Thus we my assume that for any edge $e = uv$, $2 \in \{f(u), f(v)\}$. Then $\{v \in V(G) : f(v) = 2\}$ is a vertex cover for G of size at most $2k$. Therefore G has a vertex cover of size at most $2k$. □

Theorem 5. *For $r \geq 2$, rHRIDFP is NP-complete for planar chordal graphs.*

Proof. Let G be a graph of order n_G and size m_G, and let H' be the connected planar chordal graph constructed in the proof of Theorem 2. With a similar argument as it is given in proof of Theorem 4, we can see that G has a vertex cover of size at most k if and only if H' has an rHRIDS of weight at most $2k + 2rn_G + 2rm_G(2r-1)$. □

5. r-Step Roman domination

Consider the following decision problem:

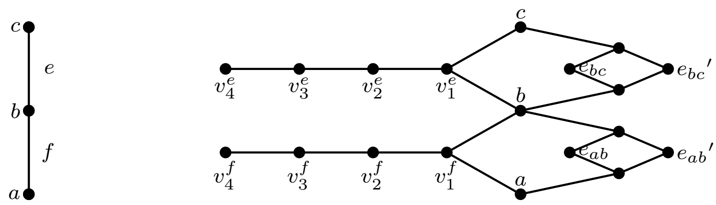
r-Step Roman Dominating Function Problem (rSRDFP).
Instance: A non-empty graph G, and two positive integers $r \geq 2$ and $k \geq 1$.
Question: Does G have a r-step Roman dominating function of weight at most k?

We show that the decision problem for rSRDFP is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Theorem 6. *For $r \geq 2$, rSRDFP is NP-complete for planar bipartite graphs.*

Proof. Clearly, the rSRDFP is in NP, since it is easy to verify a "yes" instance of rSRDFP in polynomial time. Now we transform the vertex cover problem to the rSRDFP so that one of them has a solution if and only if the other has a solution. Let G be a connected planar bipartite graph of order n_G and size $m_G \geq 2$. Let H be the graph obtained from G as follows. For each edge $e = uv \in E(G)$ we subdivide the edge e, $2r-1$ times, and add a path $v_1^e v_2^e \dots v_{2r}^e$, and join v_1^e to both u and v. For any edge $e = uv \in E(G)$, let e_{uv} be the subdivided vertex at distance r from both u and v in H that resulted from subdividing the edge e, $2r-1$ times. Then add a vertex $e_{uv}{}'$ and join it to both neighbors of e_{uv}. Let H be the resulted graph. Then H has order $n_H = n_G + 4rm_G$ and size $m_H = (4r+3)m_G$. The transformation can clearly be performed in polynomial time. We note that since G is connected and planar, so H is connected and planar. Further, by construction, H is bipartite. Thus, H is a connected planar bipartite graph. Figure 3 depicts the graph H if $r = 2$ and $G = P_3$.

We show that G has a vertex cover of size at most k if and only if H has a r-step Roman dominating function of weight at most $2k + 2rm_G$. Assume that G has a



(a) G (b) H

Figure 3. **The graphs G and H in the proof of Theorem 6 for $r = 2$**

vertex cover, namely S_G, of size at most k. Let

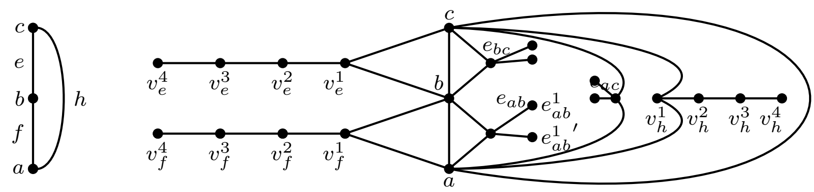
$$S_H = S_G \cup \bigcup_{e \in E(G)} \{v_e^1, v_e^2, \ldots, v_e^r\}.$$

We show that $f = (V(H) - S_H, \emptyset, S_H)$ is a r-step Roman dominating function. Clearly $S_G \neq \emptyset$, since $m_G \geq 2$. For every edge $e = uv \in E(G)$, the vertex v_r^e r-step dominates the vertices v_e^{2r}, u and v in H, while the vertex v_e^i $(i = 1, 2, \ldots, r-1)r$-step dominates the vertex v_e^{i+r} and the r-neighbors of u and v in H that belong to the (u, v)-path in H that resulted from subdividing the edge $e = uv$ of G. Since S_G is a vertex cover in G, every subdivided vertex that is not a neighbor of a vertex in $V(G)$ is r-step dominated by the set S_G in H. Further, the set S_G r-step dominates the vertex v_e^r for every edge $e \in E(G)$. Since G is connected and $m_G \geq 2$, for every two adjacent edges e and f in G the vertices v_e^i and v_f^j r-step dominate each other for $1 \leq i, j < r$, where $i + j = r$. Therefore, S_H is a r-step dominating set for H, and thus $f = (V(H) - S_H, \emptyset, S_H)$ is a r-step Roman dominating function for H of weight at most $2k + 2rm_G$ in H.

Suppose next that H has a r-step Roman dominating function f of weight at most $2k + 2rm_G$. Without loss of generality we assume that f has minimum weight. Let $e = uv \in E(G)$. For $i = r+1, \ldots, 2r$, in order to r-step Roman dominate v_e^i in H, it is required that $\sum_{i=1}^{2r} f(v_e^i) \geq 2r$. If $2 \notin \{f(u), f(v)\}$, then $f(e_{uv}) \neq 0$ and $f(e_{uv}') \neq 0$. Let g be a function obtained by changing both $f(e_{uv})$ and $f(e_{uv}')$ to 0 and $f(u)$ to 2. Since f has minimum weight, we find that $w(g) = w(f)$. Thus we may assume that $2 \in \{f(u), f(v)\}$. Hence, $\{v \in V(G) : f(v) = 2\}$ is a vertex cover of G. Further, $|\{v \in V(G) : f(v) = 2\}| \leq k$, since $\sum_{i=1}^{2r} f(v_e^i) \geq 2r$ for every edge $e \in E(G)$. Thus, G has a vertex cover of size at most k. □

Theorem 7. *For $r \geq 2$, rSRDFP is NP-complete for planar chordal graphs.*

Proof. Let G be a connected planar chordal graph of order n_G and size $m_G \geq 2$. Let H be the graph obtained from G as follows. For each edge $e = uv \in E(G)$ we add a new vertex e_{uv} adjacent to both u and v in H and we add a P_{r-1}-path ${e^1}_{uv}{e^2}_{uv} \ldots {e^{r-1}}_{uv}$ and join e_{uv} to e_{uv}^1. Further, we add a P_{2r}-path $v_e^1 v_e^2 \ldots v_e^{2r}$, and join v_e^1 to u and v. Finally for each edge $e = uv \in E(G)$ add a new vertex ${e^{r-1}}_{uv}'$ and join it to the neighbor of ${e^{r-1}}_{uv}$. The resulting graph H has order $n_H = n_G + (3r+1)m_G$ and size $m_H = (3r+4)m_G$. The transformation can clearly be performed in polynomial time. We note that since H is a connected planar chordal graph.



(a) G (b) H

Figure 4. **The graphs G and H in the proof of Theorem 7 for $r = 2$**

We show that G has a vertex cover of size at most k if and only if H has a r-step Roman dominating function of weight at most $2k + 2rm_G$. Let S_G be a vertex cover of size at most k, and let

$$S_H = S_G \cup \bigcup_{e \in E(G)} \{v_e^1, v_e^2, \ldots, v_e^r\}.$$

Let $f = (V(H) - S_H, \emptyset, S_H)$. Note that $S_G \neq \emptyset$. For every edge $e = uv \in E(G)$, the vertex v_e^r r-step dominates the vertices v_e^{2r}, u and v in H, while the vertex v_e^i $(1 \leq i < r)$ r-step dominates the vertices v_e^{i+r} and ${e_{uv}}^{r-i-1}$, where ${e_{uv}}^0 =: e_{uv}$. Since S_G is a vertex cover in G, every vertex ${e_{uv}}^{r-1}$ is r-step dominated by S_G in H. Further, S_G r-step dominates v_e^r for every edge $e \in E(G)$. Since G is connected and $m_G \geq 2$, for every two adjacent edges e and f in G the vertices v_e^i and v_f^j r-step dominate each other for $1 \leq i, j < r$, where $i + j = r$. Therefore, f is a r-step Roman dominating function of weight at most $2k + 2rm_G$.

Suppose next that H has a r-step Roman dominating function f of weight at most $2k + 2rm_G$. Let $e = uv \in E(G)$. For $i = r+1, \ldots, 2r$, in order to r-step Roman dominate v_e^i in H, it is required that $\sum_{i=1}^{2r} f(v_e^i) \geq 2r$. If $2 \notin \{f(u), f(v)\}$, then $f({e^{r-1}}_{uv}') \neq 0$ and $f({e^{r-1}}_{uv}) \neq 0$. Let g be a function obtained by changing both $f({e^{r-1}}_{uv})$ and $f({e^{r-1}}_{uv}')$ to 0 and $f(u)$ to 2. Since f has minimum weight, we find that $w(g) = w(f)$. Thus we may assume that $2 \in \{f(u), f(v)\}$. Hence, $\{v \in V(G) : f(v) = 2\}$ is a vertex cover of G. Further, $|\{v \in V(G) : f(v) = 2\}| \leq k$, since $\sum_{i=1}^{2r} f(v_e^i) \geq 2r$ for every edge $e \in E(G)$. Thus, G has a vertex cover of size at most k. □

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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