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Research Article

Independence number and connectivity of maximal connected domination vertex critical graphs

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Abstract: A k-CEC graph is a graph G which has connected domination number $\gamma_c(G) = k$ and $\gamma_c(G+uv) < k$ for every $uv \in E(\overline{G})$. A k-CVC graph G is a 2-connected graph with $\gamma_c(G) = k$ and $\gamma_c(G - v) < k$ for any $v \in V(G)$. A graph is said to be maximal k-CVC if it is both k-CEC and k-CVC. Let δ, κ, and α be the minimum degree, connectivity, and independence number of G, respectively. In this work, we prove that for a maximal 3-CVC graph, if $\alpha = \kappa$, then $\kappa = \delta$. We additionally consider the class of maximal 3-CVC graphs with $\alpha < \kappa$ and $\kappa < \delta$, and prove that every 3-connected maximal 3-CVC graph when $\kappa < \delta$ is Hamiltonian connected.

Keywords: connected domination, independence number, connectivity.

AMS Subject classification: 05C69, 05C40

1. Introduction

The basic graph theoretic terminology throughout this paper follow that of Bondy and Murty [3], and all graphs in this paper are simple and connected. Let G be a finite graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G *induced* by S. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of vertices that is adjacent to v. The *closed neighborhood* $N_G[v]$ of a vertex v in G is $\{v\} \cup N_G(v)$. The *degree* $deg_G(v)$ of a vertex v in G is $|N_G(v)|$. Let $\delta(G)$

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be the *minimum degree* of a graph G. $N_G(v) \cap S$ is denoted by $N_S(v)$ where S is a vertex subset of G. A connected graph without cycles is a *tree*. A tree with n vertices of degree 1 and exactly one vertex of degree n is a *star* $K_{1,n}$. An *independent set* is a set whose all pairs of vertices are non-adjacent. The *independence number* of G, $\alpha(G)$, is the maximum cardinality of an independent set of G.

For a connected graph G, a *cut set* is a vertex subset $S \subseteq V(G)$ such that $G - S$ is disconnected. The *connectivity* $\kappa(G)$ is the minimum cardinality of a vertex cut set of a graph G. If $S = \{a\}$ is a minimum cut set of G, then G has a *cut vertex* a and $\kappa(G) = 1$. A graph G is said to be *s-connected* if $\kappa(G) \geq s$. When there is no ambiguity, we shorten $\delta(G), \alpha(G)$, and $\kappa(G)$ to δ, α, and κ, respectively.

A path that visits every vertex of a graph exactly once is called a *Hamiltonian path*. If every pair of vertices of a graph are joined by a Hamiltonian path, then the graph is *Hamiltonian-connected*. It is an exercise to check that Hamiltonian connectivity exists only when the graphs are ℓ-connected for $\ell \geq 3$. For a graph G, the *Mycielskian* $\mu(G)$ of G is the graph with vertex set $V(G) \cup V' \cup \{x\}$, where $V' = \{u' | u \in V(G)\}$ and with edge set $E(G) \cup \{uv' | uv \in E(G)\} \cup \{v'x | v' \in V'\}$.

Let D and X be subsets of $V(G)$, then we say that D *dominates* X, or $D \succ X$, if every vertex in $X \setminus D$ is adjacent to a vertex in D. Furthermore, we write $a \succ X$ when $D = \{a\}$. In particular, if $X = V(G)$, then D is called a *dominating set* of G and we write $D \succ G$ instead of $D \succ V(G)$. A dominating set D of a graph G is called a *connected dominating set* of G if $G[D]$ is connected. A connected dominating set D of G is denoted by $D \succ_c G$. Let γ_c*-set* denote a smallest connected dominating set. The *connected domination number* of G is the cardinality of a γ_c-set of G and it is denoted by $\gamma_c(G)$. Let D be a subset of $V(G)$, then D is called a *total dominating set* of a graph G if every vertex in G is adjacent to a vertex in D. The *total domination number* is the minimum cardinality of a total dominating set of G and is denoted by $\gamma_t(G)$.

A graph G is *k-connected domination edge critical*, *k-CEC*, if $\gamma_c(G) = k$ but $\gamma_c(G + xy) < k$ for any $xy \notin E(G)$. If $\gamma_c(G) = k$ but $\gamma_c(G - x) < k$ for any $x \in V(G)$, then G is *k-connected domination vertex critical*, *k-CVC*. A *maximal k-CVC* graph is a k-CVC graph having largest possible number of edges. Thus, a maximal k-CVC graph is both edge and vertex critical. It can be observed that connected domination is defined on connected graph. From here on, we assume that k-CVC graphs are 2-connected. A *k-total domination edge critical*, *k-TEC*, graph can be defined similarly.

The aim of this paper is to study how the connectivity and the independence number are related if the graphs are maximal 3-CVC. For related results in the graphs whose domination number decreases after adding any edge (k-DEC graphs), Zhang and Tian [11] proved that every 3-DEC graph satisfies $\alpha \leq \kappa + 2$ and proved further that $\kappa = \delta$ if the equality holds. Kaemawichanurat [8] showed that every 3-CEC graph satisfies $\alpha \leq \kappa + 2$. Furthermore, for any 3-CEC graph, if $\kappa + 1 \leq \alpha \leq \kappa + 2$, then $\kappa = \delta$ with only one exception.
In this paper, we prove that if G is a maximal 3-CVC graph with the condition $\alpha = \kappa$, then $\kappa = \delta$. We provide a class of maximal 3-CVC graphs with $\alpha < \kappa < \delta$ so that the condition $\alpha = \kappa$ is needed. We finish by showing that all 3-connected maximal

3-CVC graphs are Hamiltonian-connected if $\kappa < \delta$.

2. Preliminaries

We state the results that used in establishing our theorems. The first theorem was proved by Chvátal and Erdös [5] which is Hamiltonian property of graphs when independence number and connectivity are given.

Theorem 1. *[5] Let G be an ℓ-connected graph with the independence number α. If $\alpha < \ell$, then G is Hamiltonian-connected.*

Chen et al. [4] provided properties of 3-CEC graphs as detailed in Lemmas 1 and 2.

Lemma 1. *[4] Let G be a 3-CEC graph and $ab \in E(\bar{G})$. If D_{ab} is a γ_c-set of $G + ab$. Then*

(1) $|D_{ab}| = 2$,

(2) $\{a, b\} \cap D_{ab} \neq \emptyset$,

(3) if $a \in D_{ab}$ and $b \notin D_{ab}$, then $D_{ab} \cap N_G(b) = \emptyset$.

Lemma 2. *[4] Let G be a 3-CEC graph having A an independent set containing $|A| = m \geq 3$ vertices. Then we can rename the vertices in A as $v_1, v_2, \ldots, v_m$ in which there is a corresponding path $u_1, u_2, \ldots, u_{m-1}$ in $G - A$ so that, for all $1 \leq i \leq m - 1$, $\{v_i, u_i\} \succ_c G + v_i v_{i+1}$.*

In Lemma 3, Ananchuen et al. [2] gave basic properties of 3-CVC graphs.

Lemma 3. *[2] Let G be a 3-CVC graph containing a vertex x. If D_x is a γ_c-set of $G - x$, then*

(1) $|D_x| = 2$ and

(2) $D_x \cap N_G[x] = \emptyset$.

Simmons [10] showed that 3-TEC graphs have $\alpha \leq \delta + 2$. Ananchuen [1] observed that a 3-CEC graph is also 3-TEC and vice versa. Thus every 3-CEC graph satisfies $\alpha \leq \delta + 2$. For 3-CEC graphs, the result that $\alpha = \delta + 2$ was established by Kaemawichanurat et al. [9]. These results can be combined into the following theorem.

Theorem 2. *[10] If G is a 3-CEC graph with $\delta \geq 2$, then $\alpha \leq \delta + 2$. Furthermore, if $\alpha = \delta + 2$, then there is the unique vertex $a \in V(G)$ so that $deg(a) = \delta$ and the subgraph $G[N[a]]$ is complete.*

We previously established [7] some results on maximal 3-CVC graphs.

Lemma 4. *[7] Suppose that G is a maximal 3-CVC graph having a cut set $S \subseteq V(G)$ and let $C_1, C_2, \ldots, C_r$ be the components that are obtained from $G - S$. Further, we let $x \in V(G)$. If $x \in V(C_i) \cup S$ which $|V(C_i)| > 1$ or $r \geq 3$, then*

(1) $D_x \cap S \neq \emptyset$ and

(2) S is not dominated by x.

Lemma 5. *[7] Suppose that G is a maximal 3-CVC graph having a cut set $S \subseteq V(G)$ and let $C_1, C_2, \ldots, C_r$ be the components that are obtained from $G - S$. Further, for some $i \in \{1, 2, \ldots, r\}$, we let $x \in V(C_i)$. Then*

(1) Let $y \in V(C_j)$ for some $j \in \{1, 2, \ldots, r\}$ such that $\{x, y\}$ does not dominate G. If $r \geq 3$ or $|V(C_i)|, |V(C_j)| > 1$, then $|D_{xy} \cap \{x, y\}| = 1$ and $|D_{xy} \cap S| = 1$.

(2) If $c \in D_x$ is an isolated vertex in S, then $r = 2$ and $\{u\} = V(C_j)$ for some $j \in \{1, 2\}$, where $\{u\} = D_x - \{c\}$.

In [7], we further characterized all maximal 3-CVC graphs whose smallest cut set contains no edges.

Theorem 3. *[7] If G is a maximal 3-CVC graph having a smallest cut set S. If S is independent, then G is isomorphic to $G_3 = \mu(K_s)$.*

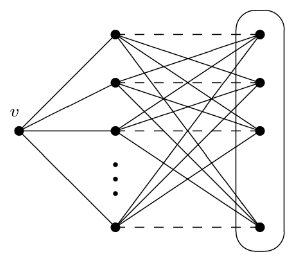

Figure 1. **A graph** $G_3 = \mu(K_s)$

In previous work [6], we established an upper bound for the independence number of maximal 3-CVC graphs in terms of the minimum degree.

Theorem 4. *[6] Let G be a maximal 3-CVC graph. Then $\alpha \leq \delta$.*

3. Connectivity of Maximal 3-CVC Graphs

In this section, we use Theorem 4 to prove that every maximal 3-CVC graph satisfies $\alpha \leq \kappa$. We further construct examples of such graphs for which $\alpha = \kappa$. In [7], we completely characterized all maximal 3-CVC graphs having connectivity at most three. Thus, we focus on $|S| = \kappa \geq 4$. Let $C_1, \ldots, C_m$ be the component of $G - S$. In particular, we let $H_1 = \cup_{i=1}^{\lfloor \frac{m}{2} \rfloor} V(C_i)$ and $H_2 = \cup_{i=\lfloor \frac{m}{2} \rfloor+1}^{m} V(C_i)$. Let I be a maximum independent set of G, $I_i = I \cap H_i$ and $|I_i| = \alpha_i$ for $i \in \{1, 2\}$. Then $I = I_1 \cup I_2 \cup (S \cap I)$. Let $|I_1 \cup I_2| = p$.

Theorem 5. *If G is a 3-CVC graph having independence number α and connectivity κ, then $\alpha \leq \kappa$*

Proof. For contradiction, assume that $\kappa + 1 \leq \alpha$. So $|S| + 1 \leq \alpha_1 + \alpha_2 + |S \cap I|$. Hence

$$|S - I| + 1 = |S| - |S \cap I| + 1 \leq \alpha_1 + \alpha_2 \tag{3.1}$$

Claim 1. $|V(C_i)| > 1$ for all $1 \leq i \leq r$, and $|H_i| > 1$.

Suppose that $V(C_i) = \{c\}$ for some $i \in \{1, 2, \ldots, r\}$. So by Theorem 4, $N_G(c) \subseteq S$. Then we have

$$\delta \leq \deg_G(c) < |S| + 1 = \kappa + 1 \leq \alpha \leq \delta,$$

a contradiction, thus establishing Claim 1.

Let $p = \alpha_1 + \alpha_2$ and $\{a_1, a_2, \ldots, a_p\} = \cup_{i=1}^{2} I_i$. If $p = 1$, then, by (3.1), $|S - I| = 0$. This implies that $S \cap I = S$ which implies that the set S is independent. Note that G is G_3 by Theorem 3. Hence, $N_{G_3}(x)$ in the graph G_3 is a minimum cut set which $G_3 - N_{G_3}(x)$ has a component containing exactly one vertex x. This contradicts Claim 1. Thus, $p > 1$.

Claim 2. $|D_{ab} \cap \{a, b\}| = 1$ and $|D_{ab} \cap (S - I)| = 1$ for any $a, b \in \cup_{i=1}^{2} I_i$.

Since $|S| \geq 4$ and $2 \leq p = \alpha_1 + \alpha_2$, if $p \geq 3$, then $\cup_{i=1}^{2} I_i - \{a, b\} \neq \emptyset$. If $p = 2$, then, by (3.1), $|S| - |S \cap I| + 1 \leq 2$. Because $|S| \geq 4$, we get $|S \cap I| \geq 3$, specifically, $S \cap I \neq \emptyset$. Thus $(S \cap I) \cup (\cup_{i=1}^{2} I_i - \{a, b\}) \neq \emptyset$ inplying that $\{a, b\}$ does not dominate G. By Lemma 5(1) and Claim 1, $|D_{ab} \cap \{a, b\}| = 1$ and $|D_{ab} \cap S| = 1$. Renaming vertices if necessary, we let $a \in D_{ab}$ and $\{a'\} = D_{ab} \cap S$. Since $(G + ab)[D_{ab}]$ is connected, $a' \in S - I$. This proves Claim 2.

Assume that $p = 2$. We consider the graph $G + a_1a_2$. By Claim 2, $|D_{a_1a_2} \cap (S - I)| = 1$. Since $D_{a_1a_2} \cap (S - I) \subseteq S - I$, by (3.1),

$$1 \leq |S - I| \leq \alpha_1 + \alpha_2 - 1 = p - 1 = 1.$$

Therefore, $D_{a_1a_2} \cap (S - I) = S - I$. If $p \geq 3$, then Lemma 2 yields that the vertices $a_1, a_2, \ldots, a_p$ can be renamed as $x_1, x_2, \ldots, x_p$ and there is a corresponding path $y_1, y_2, \ldots, y_{p-1}$ for which $\{x_i, y_i\} \succ_c G + x_ix_{i+1}$ for $i \in \{1, 2, \ldots, p - 1\}$. Since

$\{x_1, x_2, \ldots, x_p\} \subseteq \cup_{i=1}^{2} I_i$, it follows by Claim 2 that $\{y_1, y_2, \ldots, y_{p-1}\} \subseteq S - I$. So, the equation (3.1) gives $p - 1 \le |S - I| \le \alpha_1 + \alpha_2 - 1 = p - 1$. In both cases $p = 2$ and $p \ge 3$, we have that $\{y_1, y_2, \ldots, y_{p-1}\} = S - I$.

When $p = 2$, then it can be checked that the subgraph $G[\{y_1\}]$ is complete. When $p \ge 3$. Consider $G + x_i x_j$ for $2 \le i \ne j \le p$. By Claim 2, $|D_{x_i x_j} \cap \{x_i, x_j\}| = 1$ and $|D_{x_i x_j} \cap (S - I)| = 1$. Renaming vertices if necessary, w let $x_i \in D_{x_i x_j}$. As $S - I = \{y_1, y_2, \ldots, y_{p-1}\}$, by Lemma 1(3), $D_{x_i x_j} \cap (S - I) = \{y_{j-1}\}$. Since $x_i y_{i-1} \notin E(G)$, $y_{i-1} y_{j-1} \in E(G)$. Therefore, $G[\{y_1, y_2, \ldots, y_{p-1}\}]$ is a clique. Since $\{x_1, x_2, \ldots, x_p\} \subseteq I$, $y_i \succ (S \cap I)$ for $1 \le i \le p - 1$. Hence $y_i \succ S$. This contradicts Lemma 4(2). Therefore, $\alpha \le \kappa$. □

By Theorem 3, the graph $G_3 = \mu(K_s)$ has $N_{G_3}(x)$ as a minimum cut set as well as a maximum independent set. Therefore $\alpha(G_3) = \kappa(G_3)$. Hence, the bound in Theorem 5 is sharp. In particular, for maximal 3-CVC graphs satisfying $\alpha = \kappa$, we have that $|S - I| + |S \cap I| = |S| = \alpha_1 + \alpha_2 + |S \cap I|$. So

$$|S - I| = \alpha_1 + \alpha_2 = p. \tag{3.2}$$

Renaming if necessary, we let $\alpha_1 \le \alpha_2$. We will prove that if a maximal 3-CVC graph G satisfies $\alpha = \kappa$, then, any minimum cut set S, the graph $G - S$ has a component containing exactly one vertex. We may assume with a contradiction that $G - S$ has no singleton component. Thus, $|H_i| > 1$ for all $1 \le i \le 2$.

Lemma 6. *For a maximal 3-CVC graph G, if $|V(C_i)| > 1$ for all $1 \le i \le m$ and $\alpha = \kappa$, then $p \ge 3$.*

Proof. Suppose that $|H_i| > 1$ for all $1 \le i \le 2$. Firstly, assume that $p = 0$. So $S = S \cap I$. Theorem 3 implies that G is G_3. hence, G_3 has $N_{G_3}(x)$ as a minimum cut set and $G - N_{G_3}(x)$ has x as a singleton component, a contradiction. We discuss 2 cases.

Case 1. $p = 1$.

By (3.2), $|S - I| = 1$. We let $\{a_1\} = \cup_{i=1}^{2} I_i$, $\{v\} = S - I$, and $\{a_2, a_3, \ldots, a_\alpha\} = S \cap I$. Therefore $\alpha_1 = 0$ and $\alpha_2 = 1$. Therefore $a_1 \in H_2$. As $|S| \ge 4$, we have that $|S \cap I| \ge 3$. By Lemma 2, we can rename the vertices in $\{a_2, a_3, \ldots, a_\alpha\}$ as $x_1, x_2, \ldots, x_{\alpha-1}$ for which there is a corresponding path $P = y_1, y_2, \ldots, y_{\alpha-2}$ such that $\{x_i, y_i\} \succ_c G + x_i x_{i+1}$ for $i \in \{1, \ldots, \alpha - 2\}$. Note that $y_i \ne a_1$ because every vertex y_i is adjacent to a vertex of I for $1 \le i \le \alpha - 2$. To dominate a_1, $y_i \in H_2 \cup \{v\}$. We consider 2 subcases.

Subcase 1.1. The vertex v is not in the path P.

Thus $V(P) \subseteq H_2$, and hence $x_i \succ H_1$ for $1 \le i \le \alpha - 2$. Because $N_{H_1}(v) \ne \emptyset$, it follows that S is a minimum cut set. Let $u \in N_{H_1}(v)$. Thus $u \succ \{x_1, x_2, \ldots, x_{\alpha-2}, v\}$. By Lemma 4(2) we get that $u x_{\alpha-1} \notin E(G)$. For $G + u y_{\alpha-2}$. Since $u x_{\alpha-1}, y_{\alpha-2} x_{\alpha-1} \notin E(G)$. Lemma 5(1) implies that $|D_{u y_{\alpha-2}} \cap \{u, y_{\alpha-2}\}| = 1$ and $|D_{u y_{\alpha-2}} \cap S| = 1$. Hence, $y_{\alpha-2} \in D_{u y_{\alpha-2}}$ or $u \in D_{u y_{\alpha-2}}$. When $y_{\alpha-2} \in D_{u y_{\alpha-2}}$, by Lemma 1(3),

$\{x_1, x_2, \ldots, x_{\alpha-2}, v\} \cap D_{uy_{\alpha-2}} = \emptyset$. Hence $x_{\alpha-1} \in D_{uy_{\alpha-2}}$. But note that $G[D_{uy_{\alpha-2}}]$ is not connected. Hence $u \in D_{uy_{\alpha-2}}$. Since $(G+uy_{\alpha-2})[D_{uy_{\alpha-2}}]$ is connected, $x_{\alpha-1} \notin D_{uy_{\alpha-2}}$. If $x_i \in D_{uy_{\alpha-2}}$ for all $1 \leq i \leq \alpha-2$, then no vertex in $D_{uy_{\alpha-2}}$ is adjacent to $x_{\alpha-1}$. Thus $v \in D_{uy_{\alpha-2}}$, and therefore $va_1 \in E(G)$. Consider $G + ua_1$. Since $ux_{\alpha-1}, a_1x_{\alpha-1} \notin E(G)$, by Lemma 5(1), $|D_{ua_1} \cap \{u, a_1\}| = 1$ and $|D_{ua_1} \cap S| = 1$. Hence either $u \in D_{ua_1}$ or $a_1 \in D_{ua_1}$. In the case $u \in D_{ua_1}$, $v \notin D_{ua_1}$ because of Lemma 1(3). Since $(G + ua_1)[D_{ua_1}]$ is connected, $x_{\alpha-1} \notin D_{ua_1}$. To dominate $x_{\alpha-1}$, $D_{ua_1} \cap \{x_1, x_2, \ldots, x_{\alpha-2}\} \neq \emptyset$. So $D_{ua_1} \cap S = \emptyset$, a contradiction. Hence $a_1 \in D_{ua_1}$. Lemma 1(3) implies that $v \notin D_{ua_1}$. Since $(G + ua_1)[D_{ua_1}]$ is connected, $\{x_1, x_2, \ldots, x_{\alpha-1}\} \cap D_{ua_1} = \emptyset$. Note that $D_{ua_1} \cap S = \emptyset$, a contradiction. Therefore, Subcase 1.1 cannot occur.

Subcase 1.2. The vertex v is in the path P.

In this case, $y_j = v$ for some $j \in \{1, 2, \ldots, \alpha-2\}$. Hence $x_i \succ H_1$ for $i \neq j$, and $\alpha-1$ and $va_1 \in E(G)$. Because $a_1, x_{\alpha-1} \in I$, it follows that a_1 is not adjacent to $x_{\alpha-1}$. If $x_{\alpha-1}$ is not adjacent to the vertex $w \in H_1$, then consider $G+wa_1$. Lemma 5(1) yields that $|D_{wa_1} \cap \{w, a_1\}| = 1$ and $|D_{wa_1} \cap S| = 1$. Thus either $w \in D_{wa_1}$ or $a_1 \in D_{wa_1}$. In both cases, $x_{\alpha-1} \notin D_{wa_1}$ because $(G + wa_1)[D_{wa_1}]$ is connected. If $w \in D_{wa_1}$, then Lemma 1(3) gives $v \notin D_{wa_1}$. To dominate $x_{\alpha-1}$, $\{x_1, x_2, \ldots, x_{\alpha-2}\} \cap D_{wa_1} = \emptyset$. So $D_{wa_1} \cap S = \emptyset$, a contradiction. Hence $a_1 \in D_{wa_1}$. By the connectedness of $(G + wa_1)[D_{wa_1}]$, $D_{wa_1} \cap \{x_1, x_2, \ldots, x_{\alpha-1}\} = \emptyset$. To dominate x_{j+1}, $v \notin D_{wa_1}$. We then have $D_{wa_1} \cap S = \emptyset$, a contradiction. Thus $x_{\alpha-1} \succ H_1$. Clearly $x_i \succ H_1$ for $i \neq j$. Note that S is a minimum cut set. Thus $N_{H_1}(v) \neq \emptyset$. Let $u' \in N_{H_1}(v)$. Lemma 4(2) implies that $u' \succ S - \{x_j\}$. For $G + u'a_1$. By using the same arguments of $G + ua_1$, we get a contradiction. Therefore Case 1 cannot exist.

Case 2. $p = 2$.

Suppose $\{a_1, a_2\} = \cup_{i=1}^{2} I_i$. By (3.2), we have that $|S - I| = p = 2$. As $|S| \geq 4$, we have $|S \cap I| \geq 2$, specifically, $S \cap I \neq \emptyset$ and $\{a_1, a_2\}$ does not dominate G. Consider $G+a_1a_2$. Lemma 5(1) gives that $|D_{a_1a_2} \cap \{a_1, a_2\}| = 1$ and $|D_{a_1a_2} \cap S| = 1$. Without loss of generality, assume $a_1 \in D_{a_1a_2}$. By the connectedness of $(G + a_1a_2)[D_{a_1a_2}]$, $|(S-I) \cap D_{a_1a_2}| = 1$. Let $\{u\} = (S-I) \cap D_{a_1a_2}$. Thus $ua_1 \in E(G)$, $ua_2 \notin E(G)$, and $u \succ S \cap I$. If we let $v \in S - (I \cup \{u\})$, then by Lemma 4(2), we have that $uv \notin E(G)$. Thus $a_1v \in E(G)$

Subcase 2.1. $\alpha_1 = 1$ and $\alpha_2 = 1$.

Renaming vertices if necessary, suppose that $a_1 \in I_1$ and $a_2 \in I_2$. Since $|S \cap I| \geq 2$, there exist $a_3, a_4 \in S \cap I$. Consider $G + a_3a_4$. Lemma 1(2) gives that $D_{a_3a_4} \cap \{a_3, a_4\} \neq \emptyset$. To dominate a_1, $D_{a_3a_4} \neq \{a_3, a_4\}$. Without loss of generality, let $a_3 \in D_{a_3a_4}$. Lemma 1(1) implies that $|D_{a_3a_4} - \{a_3\}| = 1$. Let $y \in D_{a_3a_4} - \{a_3\}$. To dominate $\{a_1, a_2\}$, $y \notin \cup_{i=1}^{2} H_i$. By the connectedness of $(G + a_3a_4)[D_{a_3a_4}]$, $y \in \{v, u\}$. Since $uv \notin E(G)$, then $a_3u, a_3v \in E(G)$. Consider $G - a_3$. Lemma 3(2) implies that $D_{a_3} \cap \{u, v\} = \emptyset$, and Lemma 4(1) yields that $D_{a_3} \cap S \neq \emptyset$. Hence there exists $z \in D_{a_3} \cap (S \cap I)$. Lemma 3(1) implies that $|D_{a_3} - \{z\}| = 1$. We may let $\{z'\} = D_{a_3} - \{z\}$. As $z \in S \cap I$, we have z is not adjacent to a_1. Hence $z' \in H_1$ to dominate a_1. Therefore D_{a_3} does not dominate a_2 contradicting D_{a_3} is a dominating set of $G - a_3$. Subcase 2.1 cannot occur.

Subcase 2.2. $\alpha_1 = 0$ and $\alpha_2 = 2$.

Hence $u \succ H_1$. Let $b_1 \in H_1$. Clearly $\{a_1, b_1\}$ does not dominate G. Consider $G + a_1b_1$. Lemma 5(1) gives that $|D_{a_1b_1} \cap S| = 1$ and either $b_1 \in D_{a_1b_1}$ or $a_1 \in D_{a_1b_1}$. In the first case, $\{u, v\} \cap D_{a_1b_1} = \emptyset$ by Lemma 1(3). To dominate a_2, $D_{a_1b_1} \cap (S \cap I) = \emptyset$. Hence, $D_{a_1b_1} \cap S = \emptyset$, a contradiction. Therefore, $a_1 \in D_{a_1b_1}$. To dominate $H_1 - b_1$ and by the connectedness of $(G + a_1b_1)[D_{a_1b_1}]$, $(D_{a_1b_1} - \{a_1\}) \subseteq \{u, v\}$. Lemma 1(3) implies that $v \in D_{a_1b_1}$. Thus $v \succ H_1 - b_1$. Let $b_2 \in H_1 - \{b_1\}$. Therefore $b_2 \succ \{u, v\}$. Consider $G + a_1b_2$. Lemma 5(1) implies that we have $|D_{a_1b_1} \cap S| = 1$ and either $a_1 \in D_{a_1b_2}$ or $b_2 \in D_{a_1b_2}$. In the first case, $\{u, v\} \cap D_{a_1b_2} = \emptyset$ by Lemma 1(3). By the connectedness of $(G + a_1b_2)[D_{a_1b_2}]$, $(S \cap I) \cap D_{a_1b_2} = \emptyset$. Thus $D_{a_1b_2} \cap S = \emptyset$, a contradiction. Therefore, $b_2 \in D_{a_1b_2}$. To dominate a_2, $(S \cap I) \cap D_{a_1b_2} = \emptyset$. Lemma 1(3) yields that $D_{a_1b_2} \cap \{u, v\} = \emptyset$. Therefore $D_{a_1b_2} \cap S = \emptyset$, a contradiction and so Case 2 cannot occur. Thus $p \geq 3$. □

By Lemma 6, we have that $p \geq 3$. By Lemma 2, the vertices in $\cup_{i=1}^{2} I_i$ can be ordered as $x_1, x_2, \ldots, x_p$ and there exists a path $y_1, y_2, \ldots, y_{p-1}$ with $\{x_i, y_i\} \succ_c G + x_ix_{i+1}$ for $i = 1, 2, \ldots, p-1$.

Lemma 7. *$y_i \succ S \cap I$ and $y_i \in S - I$ for all $1 \leq i \leq p-1$.*

Proof. Since $\{x_i, y_i\} \succ_c G + x_ix_{i+1}$ for $i = 1, 2, \ldots, p-1$ and $x_i \in I$, $y_i \succ S \cap I$. By the connectedness of $(G + x_ix_{i+1})[D_{x_ix_{i+1}}]$ and by Lemma 5(1), $y_i \in S - I$. □

Lemma 7 implies that $\{y_1, y_2, ..., y_{p-1}\} \subseteq S - I$. By (3.2), $|(S - I) - \{y_1, y_2, \ldots, y_{p-1}\}| = 1$. Let $\{y_p\} = (S - I) - \{y_1, y_2, \ldots, y_{p-1}\}$.

Lemma 8. *For $i, j \in \{2, 3, \ldots, p\}$, if $y_px_i, y_px_j \in E(G)$, then $y_{i-1}y_{j-1} \in E(G)$.*

Proof. Consider $G + x_ix_j$. Lemma 5(1) yields that $|D_{x_ix_j} \cap \{x_i, x_j\}| = 1$ and $|D_{x_ix_j} \cap S| = 1$. Without loss of generality, let $x_i \in D_{x_ix_j}$ and $\{a\} = D_{x_ix_j} \cap S$. By the connectedness of $(G + x_ix_j)[D_{x_ix_j}]$, $a \in S - I$. Since $x_j \succ (S - I) - \{y_{j-1}\}$, it follows by Lemma 1(3) that $a = y_{j-1}$. Since $y_{i-1}x_i \notin E(G)$, $y_{j-1}y_{i-1} \in E(G)$. □

Lemma 9. *$\alpha_1, \alpha_2 > 0$.*

Proof. By the assumption that $\alpha_1 \leq \alpha_2$, we can suppose for contradiction that $\alpha_1 = 0$. Clearly $\{x_1, x_2, ..., x_p\} \subseteq H_2$ and $y_i \succ H_1$ for all $1 \leq i \leq p-1$. Note that S is a minimum cut set, so $N_{H_1}(y_p) \neq \emptyset$. Let $b \in N_{H_1}(y_p)$. Therefore $b \succ S - I$. Consider $G + x_1b$. Lemma 5(1) yields that $|D_{x_1b} \cap S| = 1$ and either $b \in D_{x_1b}$ or $x_1 \in D_{x_1b}$. Suppose that $b \in D_{x_1b}$. To dominate x_2, $D_{x_1b} \cap (S - I) \neq \emptyset$. Lemmas 2 and 1(3) then imply that $D_{x_1b} \cap (S - I) = \{y_p\}$. So $y_p \succ \{x_2, x_3, \ldots, x_p\}$. Lemma 8 gives, further, that $G[y_1, y_2, \ldots, y_{p-1}]$ is a clique. Lemma 7 then yields that $y_i \succ S \cap I$ for $i = 1, 2, \ldots, p-1$. By Lemma 4(2), $y_iy_p \notin E(G)$ for $i = 1, 2, ..., p-1$. Therefore

$y_1y_p \notin E(G)$. Because $\{x_1, y_1\} \succ_c G+x_1x_2$, $x_1y_p \in E(G)$, contradicting Lemma 1(3). Therefore $x_1 \in D_{x_1b}$. By the connectedness of $(G + x_1b)[D_{x_1b}]$, $D_{x_1b} \cap (S \cap I) = \emptyset$. Lemma 1(3) implies that $D_{x_1b} \cap (S - I) = \emptyset$. Thus $D_{x_1b} \cap S = \emptyset$, contradicting Lemma 5(1). □

Theorem 6. *Let G be a maximal 3-CVC graph having S a minimum cut set. If $\alpha = \kappa$, then $G - S$ has at least one component with exactly one vertex.*

Proof. Assume that G is a maximal 3-CVC graph with $\alpha = \kappa$. By (3.2), $|S - I| = \alpha_1 + \alpha_2$. Suppose that $G - S$ has no singleton component, specifically $|H_i| > 1$ for $i = 1, 2$. Let $\alpha_1 + \alpha_2 = p$. Lemma 6 implies that $p \geq 3$, and Lemma 9 gives that $0 < \alpha_1 \leq \alpha_2$. We also define $x_1, x_2, \ldots, x_p$, a path $y_1, y_2, \ldots, y_{p-1}$ and a vertex y_p as in the previous lemmas.

We may assume that there exist x_i, x_j for $i, j \in \{2, 3, \ldots, p\}$ such that $y_p \in D_{x_ix_j}$. Lemma 1(1) and 1(2) then imply that either $D_{x_ix_j} = \{x_i, y_p\}$ or $D_{x_ix_j} = \{x_j, y_p\}$. Without loss of generality, let $D_{x_ix_j} = \{x_j, y_p\}$. Thus $y_p \succ \{x_1, x_2, \ldots, x_p\} - \{x_i\}$. Since $\{x_i, y_i\} \succ_c G+x_ix_{i+1}$, $y_iy_p \in E(G)$. Lemma 8 yields that $G[\{y_1, y_2, \ldots, y_{p-1}\} - \{y_{i-1}\}]$ is a clique. Since $y_iy_{i-1} \in E(G)$, $y_i \succ S - I$. Lemma 7 implies that $y_i \succ S \cap I$. Therefore $y_i \succ S$, contradicting Lemma 4(2). Hence, $y_p \notin D_{x_ix_j}$ for any $i, j \in \{2, 3, \ldots, p\}$. By using the same arguments as in the proof of Lemma 8, the subgraph $G[\{y_1, y_2, \ldots, y_{p-1}\}]$ is complete. As $y_i \succ S \cap I$, by Lemma 4(2), we must have $y_iy_p \notin E(G)$ for $i \in \{1, 2, \ldots, p - 1\}$. Since $\{x_i, y_i\} \succ_c G + x_ix_{i+1}$ for $i \in \{1, 2, \ldots, p - 1\}$, $x_iy_p \in E(G)$. So $x_1 \succ S - I$. By Lemma 4(2), $S \cap I \neq \emptyset$, since otherwise $x_1 \succ S$. Let $x_1 \in H_i$ for some $i \in \{1, 2\}$. Then, we consider $G - x_1$. Since $|H_j| > 1$ for $j = 1, 2$, neither $D_{x_1} \subseteq H_1$ nor $D_{x_1} \subseteq H_2$. Lemma 4(1) gives, further, that $D_{x_1} \cap S \neq \emptyset$. Lemma 3(2) implies that $D_{x_1} \cap (S - I) = \emptyset$. Thus $D_{x_1} \cap (S \cap I) \neq \emptyset$. Let $u_1 \in D_{x_1} \cap (S \cap I)$. By Lemma 3(1), $|D_{x_1} - \{u_1\}| = 1$. Let $\{w\} = D_{x_1} - \{u_1\}$. If $w \in H_i$, then $u_1 \succ H_{3-i}$. Since $u_1 \in I$, $\alpha_{3-i} = 0$, contradicting Lemma 9. So $w \in H_{3-i}$ and $u_1 \succ H_i - x_1$. Since $u_1 \in I$, $I_i = \{x_1\}$. It follows that $\{x_2, x_3, \ldots, x_p\} \subseteq H_{3-i}$.

Claim 1. For all $u \in S \cap I$, u does not dominate $S - I$.

Assume that $u \succ S - I$. For $G - u$, Lemma 4(1) implies that $D_u \cap S \neq \emptyset$. By Lemma 3(2), we have that $D_u \cap (S - I) = \emptyset$. Hence there exists $u' \in D_u \cap (S \cap I)$. Lemma 3(1) gives that $|D_u - \{u'\}| = 1$. Let $\{z\} = D_u - \{u'\}$. To dominate x_1, $z \in H_i$. Clearly D_u does not dominate I_{3-i}, so we have a contradiction. This proves Claim 1.

Claim 1 and Lemma 7 imply that y_p is not adjacent to any vertex in $S \cap I$. Therefore, y_p is an isolated vertex in S.

Claim 2. $y_1 \succ H_i$.

Suppose y_1 is not adjacent to $b_1 \in H_i$. Consider $G + b_1x_2$. We see that $b_1y_1, x_2y_1 \notin E(G)$. Lemma 5(1) gives that $|D_{b_1x_2} \cap S| = 1$ and either $b_1 \in D_{b_1x_2}$ or $x_2 \in D_{b_1x_2}$. If $b_1 \in D_{b_1x_2}$, then $(S - \{y_1, y_p\}) \cap D_{b_1x_2} = \emptyset$ to dominate I_{3-i}. Since $y_px_2 \in E(G)$, by Lemma 1(3), $y_p \notin D_{b_1x_2}$. By the connectedness of $(G + b_1x_2)[D_{b_1x_2}]$,

$y_1 \notin D_{b_1x_2}$. Therefore $D_{b_1x_2} \cap S = \emptyset$, a contradiction. Hence $x_2 \in D_{b_1x_2}$. To dominate $I_{3-i} \cup (S \cap I)$, $D_{b_1x_2} \cap \{y_2, y_3, \ldots, y_p\} = \emptyset$. By the connectedness of $(G + b_1x_2)[D_{b_1x_2}]$, $((S \cap I) \cup \{y_1\}) \cap D_{b_1x_2} = \emptyset$. Therefore, $D_{b_1x_2} \cap S = \emptyset$, a contradiction, establishing Claim 2.

Let $b_1 \in H_i - \{x_1\}$. Recall that $u_1 \succ H_i - x_1$. Clearly $b_1u_1 \in E(G)$. By Claim 2 and Lemma 2, $b_1 \succ \{y_1, y_2, \ldots, y_{p-1}\} \cup \{u_1\}$. Consider $G - b_1$. Lemma 4(1) implies that $D_{b_1} \cap S \neq \emptyset$. Lemma 3(2) gives that $D_{b_1} \cap (\{y_1, y_2, \ldots, y_{p-1}\} \cup \{u_1\}) = \emptyset$. If there is $u_2 \in D_{b_1} \cap ((S \cap I) - \{u_1\})$, then, by Lemma 3(1), let $\{y'\} = D_{b_1} - \{u_2\}$. To dominate x_1, $y' \in H_i$. Thus D_{b_1} does not dominate x_2, a contradiction. Therefore, $\{y_p\} = D_{b_1} \cap S$. Note that y_p is an isolated vertex in S, so by Lemma 5(2), at least one of C_i is a singleton component, a contradiction. □

Theorem 6 leads to the following corollary.

Corollary 1. *If G is a maximal 3-CVC graph and $\alpha = \kappa$, then $\kappa = \delta$.*

Proof. Theorem 6 implies that $G - S$ has a component containning exactly one vertex. Renaming if necessary, we let $V(C_i) = \{c\}$. Hence $N_G(c) \subseteq S$. Thus, $\delta \leq \deg_G(c) \leq |S| = \kappa \leq \delta$. □

Now we give the construction of the class $\mathcal{G}_4(s)$ of maximal 3-CVC graphs with $\alpha < \kappa$ and $\kappa < \delta$ in order to show that the condition $\alpha = \kappa$ is needed in Corollary 1. We may let R, T, W, and Z be disjoint sets of vertices where $R = \{r_1, r_2, \ldots, r_s\}$, $T = \{t_1, t_2, \ldots, t_s\}$, $W = \{w_1, w_2, \ldots, w_s\}$, $Z = \{z_1, z_2, \ldots, z_s\}$, and $s \geq 3$. Note that we can construct a graph G in the class $\mathcal{G}_4(s)$ from R, T, W, and Z by adding edges depending on the join operations:

- for $1 \leq i \leq s$, $r_i \vee R \cup T \cup W - \{r_i, t_i\}$,
- $t_i \vee R \cup W \cup Z - \{w_i, r_i\}$,
- $w_i \vee R \cup T \cup Z - \{t_i, z_i\}$,
- $z_i \vee Z \cup T \cup W = \{z_i, w_i\}$ and
- adding edges so that the vertices in R and Z form cliques.

It can be checked that, for $1 \leq i \leq s$, $N_G(r_i) = R \cup T \cup W - \{r_i, t_i\}$, $N_G(t_i) = R \cup W \cup Z - \{w_i, r_i\}$, $N_G(w_i) = R \cup T \cup Z - \{t_i, z_i\}$, and $N_G(z_i) = Z \cup T \cup W = \{z_i, w_i\}$. Note that the sets T and W are independent. Figure 2 shows a graph G, where the double lines joining between two sets mean that every vertex in one set is joined to all vertices in the other set.

Lemma 10. *If $G \in \mathcal{G}_4(s)$, then G is a maximal 3-CVC graph.*

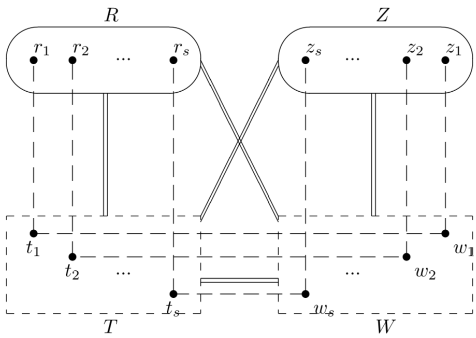

Figure 2. **A graph G in the class $\mathcal{G}_4(s)$**

Proof. Note that $\{r_1, t_2, w_2\} \succ_c G$. Thus $\gamma_c(G) \leq 3$. Let $u, v \in V(G)$ such that $\{u, v\} \succ_c G$. Suppose that $i \in \{1, ..., s\}$, and let $u = r_i$. To dominate the set Z, we have that $v \notin R$. For $v \in T$, we have, by connected, $v \neq t_i$. Hence $\{u, v\}$ does not dominate t_i. To dominate Z, we have that $v \notin W$. Hence $v \in Z$ implying that the subgraph $G[\{u, v\}]$ is disconnected, a contradiction. Thus, $\{u, v\} \cap R = \emptyset$. Note that, by symmetry, $\{u, v\} \cap Z = \emptyset$. Thus $\{u, v\} \subseteq T \cup W$. Renaming vertices if necessary, assume that $u = t_i$. Then, by connected, $v \in W - \{w_i\}$. Therefore $\{u, v\}$ does not dominate w_i. Thus $\gamma_c(G) = 3$.

To consider the criticality, we let $u, v \in V(G)$ such that $uv \notin E(G)$. For $1 \leq i \leq s$, if $\{u, v\} = \{r_i, t_i\}$, then $D_{uv} = \{r_i, t_i\}$. If $\{u, v\} = \{t_i, w_i\}$, then $D_{uv} = \{t_i, w_i\}$. If $\{u, v\} = \{w_i, z_i\}$, then $D_{uv} = \{w_i, z_i\}$. For $1 \leq i \neq j \leq s$, if $\{u, v\} = \{t_i, t_j\}$, then $D_{uv} = \{t_i, r_j\}$. If $\{u, v\} = \{w_i, w_j\}$, then $D_{uv} = \{w_i, z_j\}$. If $\{u, v\} = \{r_i, z_l\}$ where $l \in \{1, 2, \ldots, s\}$, then $D_{uv} = \{r_i, z_l\}$. Thus G is a 3-CEC graph. Let $v \in V(G)$. For $1 \leq i \neq j \leq s$, if $u = r_i$, then $D_v = \{t_i, z_j\}$. If $v = t_i$, then $D_v = \{t_j, r_i\}$. If $v = w_i$, then $D_v = \{z_i, w_j\}$. Finally, if $v = z_i$, then $D_v = \{w_i, r_j\}$. Therefore G is a maximal 3-CVC graph. □

Note that G has T as a maximum independent set and has $T \cup W$ as a minimum cut set. Hence $\alpha = s < 2s = \kappa$. Furthermore, for all $v \in V(G)$, G is a regular graph with $\deg_G(v) = 3s - 2$. Because $s \geq 3$, it follows that $\delta = 3s - 2 > 2s = \kappa$. Thus, $\alpha = \kappa$ is needed to prove Corollary 1.

Finally, we consider the Hamiltonian property of maximal 3-CVC graphs. Using Theorem 1, we obtain that:

Corollary 2. *Let G be a 3-connected maximal 3-CVC graph G. If $\kappa < \delta$, then G is Hamiltonian-connected.*

Proof. Let $\kappa < \delta$. Theorem 5 and Corollary 1 then yield that $\alpha < \kappa$. Hence Theorem 1 implies that G is Hamiltonian-connected. □

Therefore, to prove that every 3-connected maximal 3-CVC graph is Hamiltonian-connected, we need only prove the following conjecture.

Conjecture 7. For any 3-connected maximal 3-CVC graph G, if $\alpha = \kappa = \delta$, then G is Hamiltonian-connected.

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