Communications in Combinatorics and Optimization
Vol. 10, No. 1 (2025), pp. 69-78
https://doi.org/10.22049/cco.2023.28186.1469

CCO
Commun. Comb. Optim.

Research Article

Well *ve*-covered graphs

Razika Boutrig[1], Mustapha Chellali[2,*], Nacéra Meddah[2,†]

[1]Faculty of Economic Sciences and Management, University of Boumerdes, Algeria
r.bourtig@yahoo.fr

[2]LAMDA-RO Laboratory, Department of Mathematics, University of Blida, B.P. 270, Blida, Algeria
*m_chellali@yahoo.com
†meddahn11@yahoo.fr

Received: 23 December 2022; Accepted: 6 September 2023
Published Online: 10 September 2023

Abstract: A vertex u of a graph $G = (V, E)$ ve-dominates every edge incident to u as well as every edge adjacent to these incident edges. A set $S \subseteq V$ is a vertex-edge dominating set (or a *ved*-set for short) if every edge of E is ve-dominated by at least one vertex in S. A *ved*-set is independent if its vertices are pairwise non-adjacent. The independent ve-domination number $i_{ve}(G)$ is the minimum cardinality of an independent *ved*-set and the upper independent ve-domination number $\beta_{ve}(G)$ is the maximum cardinality of a minimal independent *ved*-set of G. In this paper, we are interesting in graphs G such that $i_{ve}(G) = \beta_{ve}(G)$, which we call well ve-covered graphs. We show that recognizing well ve-covered graphs is co-NP-complete, and we present a constructive characterization of well ve-covered trees.

Keywords: vertex-edge domination, independent vertex-edge domination, well ve-covered graphs, trees.

AMS Subject classification: 05C69

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. The *order* of a graph G, denoted by n, is the number of its vertices $|V|$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ of vertices is $N(S) = \bigcup_{v \in S} N(v)$, while the *closed neighborhood* of a set S is

* *Corresponding Author*

the set $N[S] = \bigcup_{v \in S} N[v]$. Next, an *S-external private neighbor* of a vertex $v \in S$ is a vertex $u \in V - S$ which is adjacent to v but to no other vertex of S. The set of all S-external private neighbors of $v \in S$ is called the *S-external private neighbor set* of v and is denoted by $\text{epn}(v, S)$. In particular, if $|\text{epn}(v, S)| = 1$, then v is said to be *S-bad.* The *degree* of a vertex v of G is $d_G(v) = |N_G(v)|$. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex.* A *star* of order $n \geq 2$, denoted by $K_{1,n-1}$, is a tree with at least $n - 1$ leaves. A *trivial graph* is a graph containing only vertex. Throughout this paper, we only consider nontrivial connected graphs, called *ntc graphs.*

A set $S \subseteq V$ of a graph G is a *dominating set* if every vertex in $V - S$ has at least a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. As defined in Bange et al. [1], a dominating set S for which $|N[v] \cap S| = 1$ for all $v \in V$ is an *efficient dominating set.* Equivalently, a set S is an efficient dominating set if S is a dominating set and the vertices in S are pairwise at distance at least 3 apart in G. As shown in [1], if a graph G has an efficient dominating set S, then $|S| = \gamma(G)$, that is, every efficient dominating set is a minimum dominating set.

A set $S \subseteq V$ is independent if no two vertices in S are adjacent. A graph G is called *well-covered* if all maximal independent sets of G have the same size. Well-covered graphs were introduced by Plummer [8] and are widely studied. We refer the reader to [9] for a comprehensive survey on well-covered graphs and their properties.

A vertex $u \in V$ is said to *ve-dominate* an edge $vw \in E$ if either (i) $u = v$ or $u = w$, that is, u is incident to vw or (ii) uv or uw is an edge in G, that is u is incident to an edge that is adjacent to vw. In other words, a vertex u *ve*-dominates the edges incident to vertices in $N[u]$. A set $S \subseteq V$ is a *vertex-edge dominating set* (or simply a *ved*-set) if for every edge $e \in E$, there exists a vertex $v \in S$ such that v *ve*-dominates e. Clearly, the property for a subset of V to be *ved*-set is superhereditary, and hence a *ved*-set S is minimal if, for every vertex $v \in S$, $S - \{v\}$ is not a *ved*-set in G. The minimum cardinality of a *ved-set* of G is called the *vertex-edge domination number* (or simply the *ve-domination number*) and is denoted by $\gamma_{ve}(G)$. The concept of vertex-edge domination was introduced by Peters [7], and further studied in [2, 4–6, 10].

A set $S \subseteq V$ is an *independent vertex-edge dominating set* (or simply an *independent ved-set*) if S is both independent and *ve*-dominating. The *independent vertex-edge domination number* (or simply the *independent ve-domination number*) $i_{ve}(G)$ of G is the minimum cardinality of an independent *ved*-set and the *upper independent vertex-edge domination number* (or simply the *upper independent ve-domination number*) $\beta_{ve}(G)$ is the maximum cardinality of a minimal independent *ved*-set of G. An independent *ved*-set of G with cardinality $\beta_{ve}(G)$ is called a $\beta_{ve}(G)$-set, and an $i_{ve}(G)$-set is defined similarly.

In this paper, by similarity to well covered graphs, we introduce the well *ve*-covered graphs G, i.e. those graphs G such that $i_{ve}(G) = \beta_{ve}(G)$. We show in the last section that recognizing well *ve*-covered graphs is co-NP-complete. Furthermore, a constructive characterization of well *ve*-covered trees is provided in Section 3.

2. Preliminary results

In this section, we gather some known results as well as some additional definitions which will be useful in the sequel of our work. Since any dominating set of an ntc graph is a *ved*-set, we have the following.

Observation 1. For every ntc graph G, $\gamma_{ve}(G) \leq \gamma(G)$.

Lewis et al. [5] were interested in the characterization of trees T such that $\gamma_{ve}(T) = \gamma(T)$ where both a descriptive and a constructive characterizations of those trees were given. Let $\mathcal{T}$ be the family of trees that can be obtained from r disjoint stars, each of order at least three, by first adding $r-1$ edges so that they are incident only with leaves of the stars and the resulting graph is a tree in which every center vertex of a star remains a support vertex.

Theorem 2 ([5]). *For any tree T of order at least three, the following statements are equivalent:*

1) $\gamma_{ve}(T) = \gamma(T)$.

2) T has an efficient dominating set S where each vertex in S is a support vertex in T.

3) $T \in \mathcal{T}$.

Restricted to the class of trees, the bound in Observation 1 has been improved by Boutrig et al. [2] who have shown that the domination number is an upper bound for the independent *ve*-domination number.

Theorem 3 ([2]). *For every nontrivial tree T, $\gamma_{ve}(T) \leq i_{ve}(T) \leq \gamma(T)$.*

It is worth noting that the domination number may be smaller or larger than the upper independent *ve*-domination number even for trees. Indeed, for the path P_9, one can easily see that $\gamma(P_9) = 3 < \beta_{ve}(P_9) = 4$, while for the tree T illustrated in Figure 1, we have $\gamma(T) = 8 > \beta_{ve}(T) = 7$. However, in the next section we shall show that for well *ve*-covered trees T, $\gamma(T) = \beta_{ve}(T)$.
A vertex $v \in S \subseteq V$ has a *private edge* $e = uw \in E$ with respect to a set S, if (i) v is incident to e or v is adjacent to either u or w, and (ii) for all vertices $x \in S - \{v\}$, x is not incident to e and x is not adjacent to either u or w. In other words, v *ve*-dominates the edge e and no other vertex in S *ve*-dominates e. Let $\text{pe}(v, S)$ denote the set of private edges of v with respect to S.
The following result on the minimality of *ved*-sets was given in [2].

Proposition 1 ([2]). *Let S be a ved-set of an ntc graph G. Then S is a minimal ved-set if and only if for every vertex $v \in S$, we have $pe(v, S) \neq \emptyset$.*

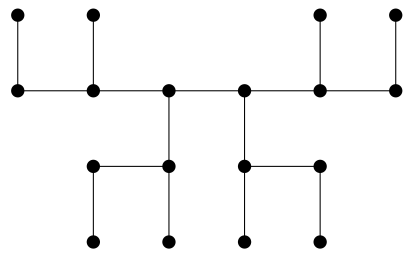

Figure 1. **A tree T with $\gamma(T) = 8$ and $\beta_{ve}(T) = 7$**

Since a private edge of any vertex v of a minimal *ved*-set S may be incident with either v or a vertex adjacent to v, we can consider further the set $\text{pe}_1(v, S)$ as the set of all private edges of v with respect to S that are incident with v. Next, let $\text{pe}_2(v, S) = \text{pe}(v, S) - \text{pe}_1(v, S)$, and let $A_S(v)$ be the set of vertices in $N(v)$ that are incident with edges in $\text{pe}_2(v, S)$. Notice that $A_S(v) \subseteq \text{epn}(v, S)$. In addition, if G is a triangle-free, let $B_S(v)$ be the set of vertices that are incident with edges in $\text{pe}_2(v, S)$ chosen so that each vertex in $A_S(v)$ is adjacent to exactly one vertex in $B_S(v)$. Hence $|A_S(v)| = |B_S(v)|$.
Now, we are ready to state the following result which will be very helpful in the sequel.

Proposition 2. *If T is well ve-covered tree, then there is a $\beta_{ve}(T)$-set D such that $\text{pe}(v, D) = \text{pe}_1(v, D)$ for every $v \in D$.*

Proof. Let T be a tree such that $i_{ve}(T) = \beta_{ve}(T)$. Suppose to the contrary that for every $\beta_{ve}(T)$-set D, there exists a vertex $v \in D$ such that $|\text{pe}_2(v, D)| \geq 1$. Among all $\beta_{ve}(T)$-sets, let D be one chosen so that $\varphi(D) = \sum_{t \in D} |\text{pe}_2(t, D)|$ is as small as possible.
Let us make the following three remarks. First, D is also an $i_{ve}(T)$-set, and since D is independent, $|\text{pe}_1(v, D)| \geq 1$ for every vertex $v \in D$. Also, for two vertices $u, v \in D$, an edge in $\text{pe}_2(u, D)$ may be adjacent to an edge in $\text{pe}_2(v, D)$, and clearly in that case we may have $B_D(u) \cap B_D(v) \neq \emptyset$. For the third remark, if x and y are two D-bad vertices of D with unique private neighbors x' and y', respectively, then $x'y' \notin E(T)$, for otherwise $\{x'\} \cup D - \{x, y\}$ is an independent *ve*-dominating set smaller than D which leads to a contradiction. Hence the set of all private neighbors of D-bad vertices is an independent set. Before going further, we need to show the following useful claim.

Claim: Let $v \in D$. Then
(i)- no vertex of $B_D(v)$ is adjacent to the unique private neighbor of a D-bad vertex.
(ii)- no vertex of $\text{epn}(v, D)$ is adjacent to the unique private neighbor of a D-bad vertex u with $A_D(u) \neq \emptyset$.

(iii)- if $A_D(v) \neq \emptyset$ and t is a vertex of $\mathrm{epn}(v, D) - A_D(v)$, then t is adjacent to some unique private neighbor of a D-bad vertex.

Proof of the claim: For the proof of the first two items, let u be a D-bad vertex with $\mathrm{epn}(u, D) = \{x\}$. Note that $|A_D(u)| \leq 1$.

(i)- Assume that there is a vertex $y \in B_D(v)$ adjacent to x. Consider the set $D^* = \{x\} \cup D - \{u\}$ and notice that D^* is an $i_{ve}(T)$-set. Observe that the edge $xy \in \mathrm{pe}_2(x, D)$ while $\mathrm{pe}_2(x, D^*) = \emptyset$. Moreover, since no new private edges are produced for $\varphi(D^*)$, we have $\varphi(D^*) < \varphi(D)$, contradicting our choice of D.

(ii)- If $A_D(u) \neq \emptyset$, then $\mathrm{pe}_2(u, D) \neq \emptyset$ and thus the set $D^* = \{x\} \cup D - \{u\}$ is an $i_{ve}(T)$-set in which the edges in $\mathrm{pe}_2(u, D)$ are no longer included in the calculation of $\varphi(D^*)$, and since no new private edges are produced for $\varphi(D^*)$, we conclude that $\varphi(D^*) < \varphi(D)$, contradicting our choice of D.

(iii)- If t is a leaf, then by item (i), the set $(D - \{v\}) \cup B_D(v) \cup \{t\}$ is a minimal independent ve-dominating set of T larger than D, a contradiction. Thus t is not a leaf. Now assume that t is not adjacent to any unique private neighbor of a D-bad vertex. Then as before, $(D - \{v\}) \cup B_D(v) \cup \{t\}$ is a minimal independent ve-dominating set of T larger than D, a contradiction. This completes the proof of the claim.

Now let v be a vertex of D such that $\mathrm{pe}_2(v, D) \neq \emptyset$. Hence $|A_D(v)| \geq 1$ and thus $|B_D(v)| \geq 1$. By item (i) of the previous claim with T being a tree, we conclude that every vertex $z \in D - \{v\}$ has at least one edge in $\mathrm{pe}_1(z, D)$ which is not ve-dominated by any vertex of $B_D(v)$. Consider the following two cases.

Case 1. No private edge in $\mathrm{pe}_2(v, D)$ is adjacent to a private edge of any other vertex in $D - \{v\}$.

Accordingly, $B_D(v) \cap B_D(u) = \emptyset$ for every $u \in D - \{v\}$. Assume first that $\mathrm{epn}(v, D) - A_D(v) = \emptyset$. If $|B_D(v)| \geq 2$, then as remarked above, the set $D' = (D - \{v\}) \cup B_D(v)$ is a minimal independent ved-set of T larger than D, a contradiction. Hence assume $|B_D(v)| = 1$, and thus $|A_D(v)| = 1$. Let $A_D(v) = \{x\}$, and consider the set $D' = (D - \{v\}) \cup \{x\}$. Then D' is an independent ve-dominating set of T and thus an $i_{ve}(T)$-set in which $\mathrm{pe}_2(x, D') = \emptyset$ yielding $\varphi(D') < \varphi(D)$, a contradiction.

In the sequel, we can assume that $\mathrm{epn}(v, D) - A_D(v) \neq \emptyset$. Let $\mathrm{epn}(v, D) - A_D(v) = \{v_1, v_2, \ldots, v_k\}$. By item (iii) of the Claim, each v_i is adjacent to some unique private neighbor x_i of a D-bad vertex u_i. Note that by item (ii) of the Claim, $A_D(u_i) = \emptyset$ for each i. Also, since T is a tree, for any two distinct vertices v_i and v_j we have $x_i \neq x_j$. Now, consider the set $D' = (D - \{v, u_1, u_2, \ldots, u_k\}) \cup \{x_1, x_2, \ldots, x_k\} \cup B_D(v)$. Observe that D' is a minimal ve-dominating set of T, and thus to avoid D' becoming larger than D we must have $|B_D(v)| = 1$, and thus $|A_D(v)| = 1$. Let $A_D(v) = \{h\}$. Notice that h is not adjacent to any private neighbor of a D-bad vertex, for otherwise replacing in D such a D-bad vertex and v with h provides an independent ve-dominating set smaller than D, leading to a contradiction. Also, if v_i is adjacent to another unique private neighbor of some D-bad vertex, say y, then $(D - \{u_i, y, v\}) \cup \{v_i, h\}$ is an independent ve-dominating set of T smaller than D, a contradiction. Hence x_i is the unique private neighbor of a D-bad vertex adjacent to v_i. Now, the set $D^* = (D - \{v, u_1, u_2, \ldots, u_k\}) \cup \{h, x_1, x_2, \ldots, x_k\}$ is an $i_{ve}(T)$ set for which $\varphi(D') < \varphi(D)$, since all edges in $\text{pe}_2(v, D)$ are no longer included in the calculation of $\varphi(D^*)$ and no new private edges are produced for $\varphi(D^*)$, a contradiction. This concludes Case 1.

Case 2. A private edge in $\text{pe}_2(v, D)$ is adjacent to a private edge in $\text{pe}_2(u, D)$ of some vertex $u \in D - \{v\}$.

Let $v_1v_2 \in \text{pe}_2(v, D)$ and $u_1v_2 \in \text{pe}_2(u, D)$ be two adjacent edges. We have $v_1 \in A_D(v), u_1 \in A_D(u)$ and thus we can assume that $v_2 \in B_D(v) \cap B_D(u)$. Then $u_1 \in \text{epn}(u, D)$ and by item (i) of the Claim, u is not a D-bad vertex, that is $|\text{epn}(u, D)| \geq 2$. Likewise, v is not a D-bad vertex, and consequently each of u and v has degree at least two.

Assume first that $\text{epn}(v, D) - A_D(v) \neq \emptyset$ and let $\text{epn}(v, D) - A_D(v) = \{t_1, t_2, \ldots, t_k\}$. By item (iii) of the Claim, t_i has a neighbor which is a unique private of a D-bad vertex. Thus, let x_i be the unique private neighbor of a D-bad vertex u_i which is adjacent to t_i, for $i \in \{1, 2, \ldots, k\}$. Note that by item (ii) of the Claim, $A_D(u_i) = \emptyset$. Also since T is a tree, all x_i's are distinct. Now, consider the set $D' = (D - \{v, u_1, u_2, \ldots, u_k\}) \cup \{x_1, x_2, \ldots, x_k\} \cup B_D(v)$. Observe that D' is a minimal *ve*-dominating set of T, and thus $|D'| = |D|$ yields that $|B_D(v)| = 1$. Hence $|A_D(v)| = 1$, and so $A_D(v) = \{v_1\}$. As in Case 1, it can be seen that v_1 is not adjacent to any private neighbor of a D-bad vertex. Now, the set $D^* = (D - \{v, u_1, u_2, \ldots, u_k\}) \cup \{v_1, x_1, x_2, \ldots, x_k\}$ is an $i_{ve}(T)$ set for which $\varphi(D^*) < \varphi(D)$, since all edges in $\text{pe}_2(v, D)$ are no longer included in the calculation of $\varphi(D^*)$ and no new private edges are produced for $\varphi(D^*)$.

Finally, assume that $\text{epn}(v, D) - A_D(v) = \emptyset$. By symmetry, we can also assume that $\text{epn}(u, D) - A_D(u) = \emptyset$. Recall that $|B_D(v)| \geq 2$ and $|B_D(u)| \geq 2$ since each of v and u has degree at least two. In this case, $(D - \{v, u\}) \cup B(v) \cup B(u)$ is a minimal independent *ved*-set of T larger than D, a contradiction. This complete the proof. □

3. Well *ve*-covered trees

In this section we provide a constructive characterization of well *ve*-covered trees. For this purpose, we define the family $\mathcal{F}$ of trees that can be obtained from r disjoint stars, each of order at least three, by adding $r - 1$ edges so that they are incident with exactly one leaf of each star and the resulting graph is a tree. Clearly, $\mathcal{F}$ is a subfamily of the family $\mathcal{T}$ defined in Theorem 2. However, here is an example of a tree belonging to $\mathcal{T} - \mathcal{F}$ which is not well *ve*-covered. Let T be a tree obtained from three disjoint stars $K_{1,3}$, the first one with center a and leaves a_1, a_2, a_3, the second with center b and leaves b_1, b_2, b_3 and the last one with center c and leaves c_1, c_2, c_3 by adding two edges a_1b_1 and a_2c_1. Then $\{a, b, c\}$ is both a minimum dominating set of T and a minimum (independent) *ved*-set of T, and thus $\gamma(T) = \gamma_{ve}(T) = i_{ve}(T)$. But the set $\{a_1, a_2, b_3, c_3\}$ is a minimal independent *ved*-set of T, thus yielding $\beta_{ve}(T) \geq 4 > i_{ve}(T)$. In the remainder of this section, we shall prove the following.

Theorem 4. *A nontrivial tree T is well ve-covered if and only if $T = P_2$ or $T \in \mathcal{F}$.*

Proof. Clearly, if T is a path P_2, then $\beta_{ve}(P_2) = i_{ve}(P_2) = 1$, and thus T is well ve-covered. Hence assume that $T \in \mathcal{F}$. Then T is obtained from r disjoint stars, each of order at least three, by adding $r-1$ edges so that they are incident with exactly one leaf of each star and the resulting graph is a tree. Notice that from the construction, each center vertex of a star remains a support vertex whose all neighbors but one are leaves. Since T belongs also to $\mathcal{T}$, by Theorem 2, $\gamma_{ve}(T) = \gamma(T) = r$. Moreover, any minimal independent ved-set of T can contain only one vertex of each star, and thus $\beta_{ve}(T) \leq r = \gamma_{ve}(T)$. By Theorem 3, the equality $\beta_{ve}(T) = i_{ve}(T)$ follows and therefore T is well ve-covered.

Conversely, assume that T is a well ve-covered tree. If T is of order 2, then $T = P_2$. So assume that $|V(T)| \geq 3$. By Proposition 2, let D be a $\beta_{ve}(T)$-set such that $\mathrm{pe}(v, D) = \mathrm{pe}_1(v, D)$ for every $v \in D$. To show that $T \in \mathcal{F}$, we adopt the following outline of the proof: we first show that D is an efficient dominating, then we show that every vertex of D is a support vertex, and finally we show that all neighbors but one of each vertex of D are leaves.

We start by showing that D is an efficient dominating set. Since D is independent, we only need to see that the vertices in D are pairwise at distance at least 3 apart in T and $V(T) = N[D]$, that is any edge of T has its endvertices in $N[D]$. Suppose there is a vertex $z \in N(D)$ with at least two neighbors in D. Since D is chosen such that $\mathrm{pe}_2(v, D) = \emptyset$ for every $v \in D$, the set $\{z\} \cup D - (N(z) \cap D)$ is an independent ved-set of T smaller than D, a contradiction. Hence no two vertices of D have a common neighbor. Suppose now that $V(T) \neq N[D]$, and let y be a vertex not dominated by D. Let x be a neighbor of y. Since the edge xy is ve-dominated by D, $x \in N(D)$. Also, from the above, x should have a single neighbor in D, say w. But then $xy \in \mathrm{pe}_2(w, D)$, contradicting the choice of D. Consequently, $V(T) = N[D]$ and therefore we conclude that D is an efficient dominating set of D. It follows that $\gamma(T) = |D| = i_{ve}(T) = \beta_{ve}(T)$.

We now show that each vertex in D is a support vertex in T. To do this, we will use the same strategy as the one used in [5]. Assume that $D = \{x_1, x_2, \ldots, x_{\gamma(T)}\}$ and consider the partition $D_1, D_2, \ldots, D_{\gamma(T)}$ of $V(T)$, where $D_i = N[x_i]$ for every $i \in \{1, 2, \ldots, \gamma(T)\}$. Note that this partition is possible because D is an efficient dominating set of T. Suppose now that x_i is not a support vertex. Then each vertex $y \in N(x_i)$ has a neighbor in $D_j - \{x_j\}$ for some j. Hence for each vertex in $N(x_i)$, we select exactly one of its neighbors (of course other than x_i) that we put in a set we call H_i. Since T is a tree, no two vertices of $N(x_i)$ have the same selected vertex in H_i, and so $|H_i| = |N(x_i)|$. Moreover, since T is a tree, H_i is an independent set. Let $H_i^* = N(H_i) \cap D$ and note that $|H_i| = |H_i^*|$. Then one can observe that $H_i \cup D - (H_i^* \cup \{x_i\})$ is an independent ved-set of T of cardinality $\gamma(T) - 1$, a contradiction. Hence every $x_i \in D$ is a support vertex, and therefore D is an efficient dominating set D where each vertex in D is a support vertex in T.

Lastly, we show that all neighbors but one of each vertex of D are leaves. As before, let $D = \{x_1, x_2, \ldots, x_{\gamma(T)}\}$ and consider $D_1, D_2, \ldots, D_{\gamma(T)}$ the partition of $V(T)$, where $D_i = N[x_i]$ for every $i \in \{1, 2, \ldots, \gamma(T)\}$. Since each x_i is a support vertex, for each x_i, we select one of its leaf neighbors which we denote by z_i. Note that since T is a tree of order at least three, the degree of each x_i is at least two. Now, without loss of generality, assume that $N(x_1)$ contains at least two vertices that are not leaves, say x and y. Again, without loss of generality, assume that x has a (unique) neighbor in $D_i - \{x_i\}$, for $i = 2, 3, \ldots, j$, and y has a (unique) neighbor in $D_i - \{x_i\}$ for $i = j+1, \ldots, k$, for some $k \le \gamma(T)$. Now, it follows that $\{x, y, z_2, \ldots, z_k, x_{k+1}, \ldots, x_{\gamma(T)}\}$ is a minimal independent *ved*-set of T of cardinality $|D| + 1$, and therefore $\beta_{ve}(T) \ge |D| + 1 = i_{ve}(T) + 1$, a contradiction. Hence for every i, all vertices in $D_i - \{x_i\}$ except one are leaves in T. Consequently, $T \in \mathcal{F}$. □

4. Complexity result

Our aim in this section is to consider the complexity of the problem of deciding whether a graph G is well *ve*-covered. In other words, does there exist an efficient recognition algorithm for such graphs? We shall show that this problem is co-NP-complete, that is, the problem of deciding whether a graph G is not well *ve*-covered is NP-complete by transforming the 3-satisfiability problem (3-SAT problem) to it in polynomial time. Recall that the 3-SAT problem specified below was proven to be NP-complete in [3]. Moreover, it is worth noting that the problems of computing $\beta_{ve}(G)$ and $i_{ve}(G)$ have both been shown in [6] to be NP-complete even for bipartite graphs G.

3-SAT problem
Instance: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set U of variables such that $|C_j| = 3$ for $j = 1, 2, \ldots, m$.
Question: Is there a truth assignment for U that satisfies all the clauses in $\mathcal{C}$?

Theorem 5. *Recognizing non well ve-covered graphs is NP-complete.*

Proof. Recognizing non well *ve*-covered graphs G is in $\mathcal{NP}$, since it is enough to exhibit two independent *ve*-dominating set of G having different sizes. For the next, let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-SAT problem, where it is assumed that no clause contains both a literal and its negation (for otherwise such a clause is obviously satisfiable and thus can be discarded). We construct in polynomial time, the graph $G = (V, E)$ as follows. First for each variable $u_i \in U$, we associate a path $P_6^i : x_i y_i u_i \overline{u_i} w_i z_i$, and for each clause $C_j = \{f_j, h_j, r_j\} \in \mathcal{C}$, we associate a single vertex c_j and add the edge set $E_j = \{c_j f_j, c_j h_j, c_j r_j\}$. Next, we add all edges between the c_j's, and we add a new vertex v attached to each c_j. Figure 2 provides an example of the resulting graph when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathcal{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \overline{u}_3\}$,

$C_2 = \{\overline{u}_1, u_2, u_4\}$ and $C_3 = \{\overline{u}_2, u_3, \overline{u}_4\}$.

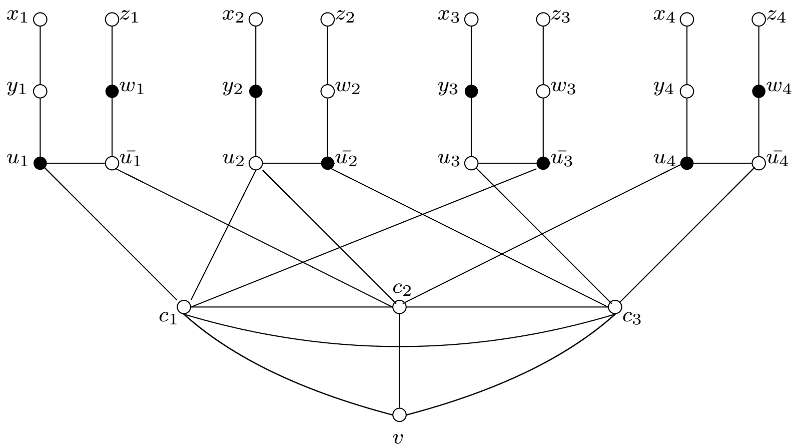

Figure 2. **A construction of the graph** G **for** $(u_1 \vee u_2 \vee \overline{u_3}) \wedge (\overline{u_1} \vee u_2 \vee u_4) \wedge (\overline{u_2} \vee u_3 \vee \overline{u_4})$

Note that for every independent *ve*-dominating set of G must contains two vertices from each path P_6 corresponding to variables, and so every independent *ve*-dominating set of G has size at least $2n$. Moreover, it is easy to construct a minimal independent *ve*-dominating set of G of size $2n+1$, for instance put in such a set an arbitrary vertex c_j and all y_i's and w_i's. In addition, one can see that the graph G does not admit independent *ve*-dominating sets of size greater than $2n + 1$.

Assume first that $\mathcal{C}$ is satisfiable and let $t : U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct an independent *ve*-dominating set D of G of size $2n$ as follows. For every i, if $t(u_i) = T$ then put u_i and w_i in D, and if $t(u_i) = F$ then put $\overline{u_i}$ and y_i in D. It is easy to check that D has size $2n$ and it is an independent *ve*-dominating set of G. Now since G has a minimal independent *ve*-dominating set of size $2n + 1$, we deduce that G is not a well *ve*-covered graph.

Conversely, assume that G is not well *ve*-covered. Hence, there is an independent *ve*-dominating set D of size $2n$. By our earlier observation that every independent *ve*-dominating set of G contains two vertices of each P_6^i, we deduce that D contains no c_j. Consequently, to *ve*-dominate all edges incident with vertex v, set D must contain at least one neighbor of each c_j corresponding to a literal. Moreover, using the fact that D cannot contain two vertices corresponding to a literal and its negation, we can then assign *true* to the literals corresponding to vertices in D to obtain a satisfying truth assignment for $\mathcal{C}$. □

As a future research topic, it would be interesting to characterize those well *ve*-covered graphs in other classes of graphs such as planar or cubic graphs. Also, design polynomial-time recognition algorithms for well *ve*-covered graphs.

Acknowledgements: The authors are grateful to anonymous referees for their constructive suggestions that improved the paper.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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