Communications in Combinatorics and Optimization
Vol. ??, No. ?? (????), pp. 1-37
DOI:10.22049/CCO.2023.28681.1669

CCO
Commun. Comb. Optim.

Research Article

On the Sombor Index of Sierpiński and Mycielskian Graphs

Surabhi Chanda* **and Radha R. Iyer**†

Department of Mathematics, Amrita School of Physical Sciences, Coimbatore, Amrita Vishwa Vidyapeetham, India
†r_radhaiyer@cb.amrita.edu

Received: 26 May 2023; Accepted: 13 August 2023
Published Online: 20 August 2023

Abstract: In 2020, mathematical chemist, Ivan Gutman, introduced a new vertex-degree-based topological index called the Sombor Index, denoted by $SO(G)$, where G is a simple, connected, finite, graph. This paper aims to present some novel formulas, along with some upper and lower bounds on the Sombor Index of generalized Sierpiński graphs; originally defined by Klavžar and Milutinović by replacing the complete graph appearing in $S(n,k)$ with any graph and exactly replicating the same graph, yielding self-similar graphs of fractal nature; and on the Sombor Index of the m-Mycielskian or the generalized Mycielski graph; formed from an interesting construction given by Jan Mycielski (1955); of some simple graphs such as K_n, C_n^2, C_n, and P_n. We also provide Python codes to verify the results for the $SO(S(n,K_m))$ and $SO(\mu_m(K_n))$.

Keywords: topological index, Sombor index, bounds, Sierpiński graphs, Mycielskian graphs

AMS Subject classification: 05C07, 05C09, 05C76, 05C90, 05C92

1. Introduction

In the *Quantitative Structure-Activity Relationship (QSAR)* or *Quantitative Structure-Property Relationship (QSPR)* study, the structure of a molecule is related to a biological activity or a property using a statistical tool. Such relationships can be codified as [13]:

$$QSAR\ or\ QSPR = f\left(molecular\ structure\right) = f\left(molecular\ descriptor(s)\right)\right).$$

* *The author carried out the research work for this paper while pursuing a Student Exchange Programme at the University of Twente, The Netherlands.*
† *Corresponding Author*

The topological representation of a chemical (molecule or compound), that is a connected graph where vertices and edges are interpreted as atoms and covalent bonds between these atoms respectively, is called a *molecular graph* [6]. A *molecular descriptor* is the result of a logical, mathematical, and well-specified algorithm, applied to the corresponding molecular graph, which captures and converts the chemical information encoded within the molecular structure into a useful number; or it is the result of some standardized experiment. A *graph invariant* is a mathematical quantity; like a characteristic polynomial, a sequence of numbers, or a single numerical index (also called the *topological index*); derived from a molecular graph by the application of algebraic operators to graph-theoretical matrices, such that it is independent of vertex labeling and hence represent graph-theoretical properties that are preserved by isomorphism.

Topological indices include information on the molecular constitution (atom-connectivity), the natures of atoms, and the bond multiplicity; and are mainly applied to model physicochemical properties (like normal boiling point, molecular refractivity, molecular volume, the heat of formation, octane number, entropy, etc.) and biological activities. They play a key role in checking the stability of the structures of newly obtained products under chemical reactions [28]. They are either distance-based [15], degree-based, or eigenvalue-based. Among these types, vertex-degree-based topological indices play a vital role in chemical graph theory, particularly in theoretical chemistry.

Let $G = (V, E)$ be a connected, finite, simple graph with $V(G) = \{v_1, v_2, ..., v_n\}$ as the vertex set and $E(G)$, a collection of unordered pairs $\{u, v\}$ of distinct elements $u, v \in V(G)$, as the edge set of G. $|V(G)| = n$ is the order and $|E(G)| = m$ is the size of G. We denote $\{v_i, v_j\} \in E(G)$ as $e_{ij} = v_i v_j \in E(G)$ and say that v_i and v_j are adjacent and that e_{ij} and v_i (or v_j) are incident. The *degree* of a vertex v in G, denoted by $d_G(v)$, is the number of vertices adjacent with v in G. The maximum vertex degree in G is defined as $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$ and the minimum vertex degree in G is defined as $\delta(G) = \min\{d_G(v) : v \in V(G)\}$. Let $N_G(v)$ denote the set of vertices adjacent to v in G. The general formula for a vertex-degree-based topological index is [18]:

$$TI = TI(G) = \sum_{\{v_i, v_j\} \in E(G)} F(d_G(v_i), d_G(v_j))$$

where $F(x, y)$ is some function with the property $F(x, y) = F(y, x)$. Gutman listed 26 such vertex-degree-based topological indices in his paper [18] including the different types of Zagreb indices [1], Randić indices, sum-connectivity indices [10], atom-bond connectivity (ABC) index, etc. In the same paper [18], he introduced a new vertex-degree-based topological index of (molecular) graphs and called it the *Sombor Index.*

1.1. The Sombor Index

This index was motivated by the geometric interpretation of the *degree radius of an edge* $e_{ij} = \{v_i, v_j\} \in E(G)$, which is the distance from the origin to the *degree-*

coordinate $(d_G(v_i), d_G(v_j))$, where $d_G(v_i) \le d_G(v_j)$. It is defined as [9, 18]:

$$SO = SO(G) = \sum_{\{v_i, v_j\} \in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2}. \tag{1}$$

In [24], for a real number α, the graph invariant $SO_{V,\alpha}$ is defined as follows:

$$SO_{V,\alpha}(G) = \sum_{v_i \in V(G)} \sqrt{(d_G(v_i) + 1)^2 + \alpha^2}. \tag{2}$$

Substantial research has been carried out since its establishment in 2020 and more studies can be found in [7, 19, 27, 29].

2. Sierpiński Graph

Let t be an integer and G be a graph on a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *generalized Sierpiński graph* of G, denoted by $S(t, G)$, is the graph with the vertex set $V(G)^t$, the set of all words $x_1 x_2 \ldots x_r \ldots x_t$ of length t where $x_r \in V(G)$, $1 \le r \le t$ and the edge set $E(S(t, G))$ defined as: given $x = x_1 x_2 \cdots x_t, y = y_1 y_2 \ldots y_t \in V(G)^t$, $\{x, y\} \in E(S(t, G))$ if and only if there exists $i \in \{1, 2, \ldots, t\}$ [16]:

i. $x_j = y_j$, if $j < i$;

ii. $x_i \ne y_i$ and $\{x_i, y_i\} \in E(G)$;

iii. $x_j = y_i$ and $y_j = x_i$ if $j > i$.

If $\{x, y\}$ is an edge of $S(t, G)$, there is an edge $\{x_i, y_i\}$ of G and a word $w = x_1 x_2 \ldots x_{i-1}$ such that $x = w x_i y_i \ldots y_i$ and $y = w y_i x_i \ldots x_i$. The degrees of $x = w x_i y_i \ldots y_i$ and $y = w y_i x_i \ldots x_i$ in $S(t, G)$ given $\{x_i, y_i\}$ is an edge of G are $d_G(y_i) + 1$ and $d_G(x_i) + 1$, respectively. We then say that edge $\{x, y\}$ of $S(t, G)$ is using edge $\{x_i, y_i\}$ of G [31].
Given graph $G = (V, E)$, the copies of a specific edge $\{x, y\} \in E$ in $S(t, G)$ are edges $\{wxy^r, wyx^r\}$, where $r \in \{0, 1, \cdots, t-1\}$ and $w \in V^{t-1-r}$. Let the number of copies of $\{x, y\} \in E$ in $S(t, G)$, where vertex wyx^r has degree $d_G(x) + l$ and where vertex wxy^r has degree $d_G(y) + l'$ be denoted by $f_{S(t,G)}(d_G(x) + l, d_G(y) + l')$, where $l, l' \in \{0, 1\}$. We note that for any graph G of order n and any integer $t \ge 2$, $S(t, G)$ has n *extreme vertices* of the form $xx \cdots x \in V(G)^t \, \forall x \in V(G)$ and, if $x \in V(G)$ has degree $d_G(x)$, then the extreme vertex $xx \ldots x \in V(S(t, G))$ also has degree $d_G(x)$ [23, 31].
We construct $S(t, G)$ recursively from G as follows: $S(1, G)$ is isomorphic to G itself. To construct $S(t, G)$, $t \ge 2$, we copy $S(t-1, G)$ n times and add the letter x_i at the beginning of each label of the vertices belonging to the copy of $S(t-1, G)$ corresponding to x_i. Then for every edge $\{x, y\}$ of G, add an edge between vertex

$xyy\dots y$ and vertex $yxx\dots x$. The degrees of these two vertices $xyy\dots y$ and $yxx\dots x$, which connect two copies of $S(t-1,G)$, are equal to $d_G(y)+1$ and $d_G(x)+1$, respectively [16].

2.1. Bounds on the Sombor Index of generalized Sierpiński graphs

The set of neighbours that $x \in V$ has in G is given by $N(x) = \{z \in V : \{x,z\} \in E\}$. Given two vertices $x, y \in V$, the number of triangles of G containing x and y is denoted by $\tau(x,y)$. For any pair of adjacent vertices x, y we have $|N(x) \cap N(y) = \tau(x,y)|$, $|N(x) \cup N(y) = d_G(x) + d_G(y) - \tau(x,y)|$, and $|N(x) - N(y) = d_G(x) - \tau(x,y)|$. Given a graph of order n, from now on we will use the function $\psi_n(t) = 1 + n + n^2 + \dots + n^{t-1} = \frac{n^t-1}{n-1}$ [22, 23].

Proposition 1 ([14]). *For any integer $t \geq 2$ and any edge $\{x,y\}$ of a graph G of order n,*

i. $f_{S(t,G)}(d_G(x), d_G(y)) = n^{t-2}(n - d_G(x) - d_G(y) + \tau(x,y))$.

ii. $f_{S(t,G)}(d_G(x), d_G(y)+1) = n^{t-2}(d_G(y) - \tau(x,y)) - \psi_n(t-2)\, d_G(x)$.

iii. $f_{S(t,G)}(d_G(x)+1, d_G(y)) = n^{t-2}(d_G(x) - \tau(x,y)) - \psi_n(t-2)\, d_G(y)$.

iv. $f_{S(t,G)}(d_G(x)+1, d_G(y)+1) = n^{t-2}(\tau(x,y)+1) + \psi_n(t-2)(d_G(x) + d_G(y) + 1)$.

From this result, we can deduce that the number of copies of the edge $\{x,y\} \in E$ in $S(t,G)$ is $\sum_{l,l' \in \{0,1\}} f_{S(t,G)}(d_G(x)+l, d_G(y)+l') = \psi_n(t)$. In the following theorem, an upper bound for the Sombor index for generalized Sierpiński graphs, in terms of the Sombor index of the respective graph G, is presented.

Theorem 1. *For any graph G of order $n \geq 3$, $\delta(G) \geq 2$ and any integer $t \geq 2$,*

$$SO(S(t,G)) > \psi_n(t)\, SO(G). \tag{3}$$

Proof. The Sombor Index of $S(t,G)$ can be expressed as

$$\begin{aligned} SO(S(t,G)) &= \sum_{x\sim y} \sum_{i=0}^{1} \sum_{j=0}^{1} f_{S(t,G)}(d_G(x)+i, d_G(y)+j) \sqrt{(d_G(x)+i)^2 + (d_G(y)+j)^2} \\ &= \sum_{x\sim y} \{ f_{S(t,G)}(d_G(x), d_G(y)) \sqrt{d_G(x)^2 + d_G(y)^2} \\ &+ f_{S(t,G)}(d_G(x)+1, d_G(y)) \sqrt{(d_G(x)+1)^2 + d_G(y)^2} \\ &+ f_{S(t,G)}(d_G(x), d_G(y)+1) \sqrt{d_G(x)^2 + (d_G(y)+1)^2} \\ &+ f_{S(t,G)}(d_G(x)+1, d_G(y)+1) \sqrt{(d_G(x)+1)^2 + (d_G(y)+1)^2} \} \end{aligned}$$

and by using Proposition 1,

$$SO\left(S\left(t,G\right)\right)=\sum_{x\sim y}\{n^{t-2}\left(n-d_G\left(x\right)-d_G\left(y\right)+\tau\left(x,y\right)\right)\sqrt{d_G\left(x\right)^2+d_G\left(y\right)^2}$$
$$+\left(n^{t-2}\left(d_G\left(x\right)-\tau\left(x,y\right)\right)-\psi_n\left(t-2\right)d_G\left(y\right)\right)\sqrt{\left(d_G\left(x\right)+1\right)^2+d_G\left(y\right)^2}$$
$$+\left(n^{t-2}\left(d_G\left(y\right)-\tau\left(x,y\right)\right)-\psi_n\left(t-2\right)d_G\left(x\right)\right)\sqrt{d_G\left(x\right)^2+\left(d_G\left(y\right)+1\right)^2}$$
$$+\left(n^{t-2}\left(\tau\left(x,y\right)+1\right)+\psi_n\left(t-2\right)\left(d_G\left(x\right)+d_G\left(y\right)+1\right)\right)\sqrt{\left(d_G\left(x\right)+1\right)^2+\left(d_G\left(y\right)+1\right)^2}\}.$$

We have for $\delta\left(G\right)\geq 2$,

$$\sqrt{d_G\left(x\right)^2+d_G\left(y\right)^2}<\sqrt{\left(d_G\left(x\right)+1\right)^2+\left(d_G\left(y\right)+1\right)^2},$$
$$\sqrt{d_G\left(x\right)^2+d_G\left(y\right)^2}<\sqrt{\left(d_G\left(x\right)+1\right)^2+d_G\left(y\right)^2},$$
$$\sqrt{d_G\left(x\right)^2+d_G\left(y\right)^2}<\sqrt{d_G\left(x\right)^2+\left(d_G\left(y\right)+1\right)^2}.$$

So, we deduce,

$$SO\left(S\left(t,G\right)\right)>\sum_{x\sim y}\{n^{t-2}\left(n-d_G\left(x\right)-d_G\left(y\right)+\tau\left(x,y\right)\right)\sqrt{d_G\left(x\right)^2+d_G\left(y\right)^2}$$
$$+\left(n^{t-2}\left(d_G\left(y\right)-\tau\left(x,y\right)\right)-\psi_n\left(t-2\right)d_G\left(x\right)\right)\sqrt{d_G\left(x\right)^2+d_G\left(y\right)^2}$$
$$+\left(n^{t-2}\left(d_G\left(x\right)-\tau\left(x,y\right)\right)-\psi_n\left(t-2\right)d_G\left(y\right)\right)\sqrt{d_G\left(x\right))^2+d_G\left(y\right)^2}$$
$$+\left(n^{t-2}\left(\tau\left(x,y\right)+1\right)+\psi_n\left(t-2\right)\left(d_G\left(x\right)+d_G\left(y\right)+1\right)\right)\sqrt{d_G\left(x\right)^2+d_G\left(y\right)^2}\}$$
$$=\sum_{x\sim y}\{n^{t-2}\left(n-d_G\left(x\right)-d_G\left(y\right)+\tau\left(x,y\right)\right)+n^{t-2}\left(d_G\left(y\right)-\tau\left(x,y\right)\right)$$
$$-\psi_n\left(t-2\right)d_G\left(x\right)+n^{t-2}\left(d_G\left(x\right)-\tau\left(x,y\right)\right)-\psi_n\left(t-2\right)d_G\left(y\right)$$
$$+n^{t-2}\left(\tau\left(x,y\right)+1\right)+\psi_n\left(t-2\right)\left(d_G\left(x\right)+d_G\left(y\right)+1\right)\}\sqrt{d_G\left(x\right)^2+d_G\left(y\right)^2}$$
$$=\sum_{x\sim y}\{n^{t-1}+n^{t-2}+\psi_n\left(t-2\right)\}\sqrt{d_G\left(x\right)^2+d_G\left(y\right)^2}$$
$$=\sum_{x\sim y}\psi_n\left(t\right)\sqrt{d_G\left(x\right)^2+d_G\left(y\right)^2}.$$

Thus, we get the required result,

$$SO\left(S\left(t,G\right)\right)>\psi_n\left(t\right)SO\left(G\right).$$

□

Remark 1. The proof of Theorem 1 shows that the given lower bound is strict and holds for $t\geq 2$ in accordance with Proposition 1. It is obvious to see that when $t=1$, $SO\left(S\left(t,G\right)\right)=SO\left(G\right)=\psi_n\left(t\right)SO\left(G\right)$.

Table 1. Edge partition of $S(n, K_m)$ Sierpiński network based on degrees of end vertices of each edge.

n	$\lvert E\left(S\left(n,K_m\right)\right)\rvert$	$f_{S(n,K_m)}\left(m-1,m\right)$	$f_{S(n,K_m)}\left(m,m\right)$
1	$\frac{m(m-1)}{2}$	0	$\frac{m(m-1)}{2}$
2	$\frac{m(m^2-1)}{2}$	$m\left(m-1\right)$	$\frac{m(m^2-2m+1)}{2}$
3	$\frac{m(m^3-1)}{2}$	$m\left(m-1\right)$	$\frac{m(m^3-2m+1)}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{m(m^n-1)}{2}$	$m\left(m-1\right)$	$\frac{m(m^n-2m+1)}{2}$

$f_{S(n,K_m)}\left(x,y\right)$ denotes the number of edges in $S\left(n,K_m\right)$ that have x and y as the degrees of their end vertices.

2.2. Sombor Index of the generalised Sierpiński of the complete graph, K_m

Theorem 2. *The Sombor Index of $S\left(n,K_m\right)$ is given by*

$$SO\left(S\left(n,K_m\right)\right) = m\left(m-1\right)\sqrt{\left(m-1\right)^2+m^2} + \frac{m^2\left(m^n-2m+1\right)}{\sqrt{2}} \tag{4}$$

Proof. It is clear that the total number of edges in $S\left(1,K_m\right) \cong K_m$ is

$$\lvert E\left(S\left(1,K_m\right)\right)\rvert = \frac{m\left(m-1\right)}{2}$$

In $S\left(2,K_m\right)$, $S\left(1,K_m\right)$ is copied m times and another set of $\lvert E\left(K_m\right)\rvert$ edges are added. So, the total number of edges in $S\left(2,K_m\right)$ is

$$\begin{aligned}\lvert E\left(S\left(2,K_m\right)\right)\rvert &= m\lvert E\left(S\left(1,K_m\right)\right)\rvert + \lvert E\left(K_m\right)\rvert) \\ &= m\left(\frac{m\left(m-1\right)}{2}\right) + \frac{m\left(m-1\right)}{2} \\ &= \frac{\left(m+1\right)m\left(m-1\right)}{2} \\ &= \frac{m\left(m^2-1\right)}{2}\end{aligned}$$

Similarly, in $S\left(3,K_m\right)$, $S\left(2,K_m\right)$ is copied m times and another set of $\lvert E\left(K_m\right)\rvert$ edges are added. So, the total number of edges in $S\left(3,K_m\right)$ is

$$\begin{aligned}\lvert E\left(S\left(3,K_m\right)\right)\rvert &= m\lvert E\left(S\left(2,K_m\right)\right)\rvert + \lvert E\left(K_m\right)\rvert) \\ &= m\left(m\left(\frac{m\left(m-1\right)}{2}\right) + \frac{m\left(m-1\right)}{2}\right) + \frac{m\left(m-1\right)}{2} \\ &= m^2\left(\frac{m\left(m-1\right)}{2}\right) + m\left(\frac{m\left(m-1\right)}{2}\right) + \frac{m\left(m-1\right)}{2} \\ &= \frac{\left(m^2+m+1\right)m\left(m-1\right)}{2} \\ &= \frac{m\left(m^3-1\right)}{2}\end{aligned}$$

Thus, as explained in the beginning of section 2, $S(n, K_m)$ is constructed by copying $S(n-1, K_m)$ m times and adding another set of $|E(K_m)|$ edges. Hence, the total number of edges in $S(n, K_m)$ is

$$\begin{aligned} |E(S(n, K_m))| &= m|E(S(n-1, K_m))| + |E(K_m)|) \\ &= \left(m^{n-1} + m^{n-2} + \cdots + m + 1\right) \frac{m(m-1)}{2} \\ &= \frac{m(m^n - 1)}{2}. \end{aligned} \tag{5}$$

Recall that $f_{S(n,K_m)}(x, y)$ denotes the number of edges in $S(n, K_m)$ that have x and y as the degrees of their end vertices. Now, in $S(1, K_m)$, $d_{S(1,K_m)}(v) = m - 1$ $\forall v \in V(S(1, K_m))$. Then,

$$f_{S(1,K_m)}(m-1, m-1) = \frac{m(m-1)}{2}.$$

So by using equation (1),

$$\begin{aligned} SO(S(1, K_m)) &= f_{S(1,K_m)}(m-1, m-1)\sqrt{(m-1)^2 + (m-1)^2} \\ &= \frac{m(m-1)}{2}\sqrt{(m-1)^2 + (m-1)^2} \\ &= \frac{m(m-1)^2}{\sqrt{2}} \end{aligned}$$

Similarly, in $S(2, K_m)$, $d_{S(2,K_m)}(v) \in \{m-1, m\}$ $\forall v \in V(S(2, K_m))$. So we have,

$$f_{S(2,K_m)}(m-1, m) = m(m-1)$$

and

$$\begin{aligned} f_{S(2,K_m)}(m, m) &= |E(S(2, K_m))| - f_{S(2,K_m)}(m-1, m) \\ &= \frac{m\left(m^2 - 1\right)}{2} - m(m-1) \\ &= \frac{m^{2+1} - 2m^2 + m}{2} \\ &= \frac{m\left(m^2 - 2m + 1\right)}{2}. \end{aligned}$$

Therefore, by equation (1),

$$\begin{aligned} SO(S(2, K_m)) &= f_{S(2,K_m)}(m-1, m)\sqrt{(m-1)^2 + m^2} + f_{S(2,K_m)}(m, m)\sqrt{m^2 + m^2} \\ &= m(m-1)\sqrt{(m-1)^2 + m^2} + \frac{m\left(m^2 - 2m + 1\right)}{2}\sqrt{m^2 + m^2} \\ &= m(m-1)\sqrt{(m-1)^2 + m^2} + \frac{m\left(m^2 - 2m + 1\right)}{2}\left(m\sqrt{2}\right). \end{aligned}$$

Again, in $S(3, K_m)$, $d_{S(3,K_m)}(v) \in \{m-1, m\}$ $\forall v \in V(S(3, K_m))$. We have,

$$f_{S(3,K_m)}(m-1, m) = m(m-1)$$

and

$$\begin{aligned} f_{S(3,K_m)}(m, m) &= |E(S(3, K_m))| - f_{S(3,K_m)}(m-1, m) \\ &= \frac{m(m^3-1)}{2} - m(m-1) \\ &= \frac{m^{3+1} - 2m^2 + m}{2} \\ &= \frac{m(m^3 - 2m + 1)}{2} \end{aligned}$$

Therefore by equation (1),

$$\begin{aligned} SO(S(3, K_m)) &= f_{S(3,K_m)}(m-1, m)\sqrt{(m-1)^2 + m^2} + f_{S(3,K_m)}(m, m)\sqrt{m^2 + m^2} \\ &= m(m-1)\sqrt{(m-1)^2 + m^2} + \frac{m(m^3 - 2m + 1)}{2}\sqrt{m^2 + m^2} \\ &= m(m-1)\sqrt{(m-1)^2 + m^2} + \frac{m(m^3 - 2m + 1)}{2}\left(m\sqrt{2}\right) \end{aligned}$$

Thus proceeding as above, we observe that in $S(n, K_m)$, $d_{S(n,K_m)}(v) \in \{m-1, m\}$ $\forall v \in V(S(n, K_m))$ and $\forall n \geq 2$,

$$f_{S(n,K_m)}(m-1, m) = m(m-1) \tag{6}$$

and

$$\begin{aligned} f_{S(n,K_m)}(m, m) &= |E(S(n, K_m))| - f_{S(n,K_m)}(m-1, m) \\ &= \frac{m(m^n-1)}{2} - m(m-1) \quad \text{(using equations 5 and 6)} \\ &= \frac{m^{n+1} - 2m^2 + m}{2} \\ &= \frac{m(m^n - 2m + 1)}{2} \end{aligned}$$

(A partition of edges in $S(n, K_m)$ based on the degrees of their end vertices is outlined in Table 1). Thus, using equation (1), we derive

$$\begin{aligned} SO(S(n, K_m)) &= f_{S(n,K_m)}(m-1, m)\sqrt{(m-1)^2 + m^2} + f_{S(n,K_m)}(m, m)\sqrt{m^2 + m^2} \\ &= m(m-1)\sqrt{(m-1)^2 + m^2} + \frac{m(m^n - 2m + 1)}{2}\sqrt{m^2 + m^2} \end{aligned}$$

Therefore,

$$SO(S(n, K_m)) = m(m-1)\sqrt{(m-1)^2 + m^2} + \frac{m^2(m^n - 2m + 1)}{\sqrt{2}}.$$

□

The Python code to verify the above-derived result is given in section 5.

Remark 2. ***Generalization of Sierpiński-like graphs***
Motivated by the self-similar structures of the Sierpińki graphs, the *subdivided-line* graph operation and the *n-iterated subdivided-line* graph of a graph G is introduced in [20] cited in [32]. Let $G = (V, E)$ be a graph that may have self-loops but not multiple edges. If $v \in V(G)$ has a self-loop, then we count it one in $d_G(v)$ instead of two. The *line graph* $L(G)$ of G is the graph whose vertex set is $E(G)$ and in which two distinct vertices $\{u, v\}$ and $\{x, y\}$ are adjacent if and only if they are adjacent in G, i.e., $\{u, v\} \cap \{x, y\} \neq \phi$. Besides a vertex $\{w, w\}$ corresponding to a self-loop in G also has a self-loop in $L(G)$. Let $e = \{x, y\} \in E(G)$. Then, let G_e be the graph with $V(G_e) = V(G) \cup \{v_e\}$, where $v_e \notin V(G)$, and $E(G_e) = (E(G) - \{\{x, y\}\}) \cup \{\{x, v_e\}, \{v_e, y\}\}$. We say that G_e is obtained from G by *elementary subdividing* the edge e. The *barycentric subdivision* $B(G)$ of G is the graph obtained from G by elementary subdividing every edge of G except for self-loops.
The subdivided-line graph $\Gamma(G)$ of G is defined to be the line graph of the barycentric subdivision of G i.e., $\Gamma(G) = L(B(G))$. We call Γ the subdivided-line graph operation.
The n-iterated subdivided-line graph $\Gamma^n(G)$ of G is the graph obtained from G by iteratively applying the subdivided-line graph operation n times.
Another regularized variation called the *extended Sierpińki graph* $S^{++}(n, K_m)$ in the class of *Sierpińki-like graphs* is introduced by Klavžar and Mohar in [21] and is obtained from $m+1$ copies of $S(n-1, K_m)$ by connecting the extreme vertices in a fashion of the complete graph with $m+1$ vertices. Let $S^\circ(n, K_m)$ be the graph obtained from the Sierpińki graph $S(n, K_m)$ by adding a self-loop to each extreme vertex. Then, we have the following lemma, where K_1^m is the graph with one vertex and m self-loops, and K_m° is the complete graph with m vertices and a self-loop at each vertex.

Lemma 1. *Let $n \geq 1$.*

i. $\Gamma^n(K_1^m) = \Gamma^{n-1}(K_m^\circ) \cong S^\circ(n, K_m)$

ii. $\Gamma^{n-1}(K_{m+1}) \cong S^{++}(n, K_m)$

In Figure 1, $\Gamma^i\left(K_1^3\right) = \Gamma^{i-1}(K_3^\circ) \cong S^\circ(i, K_3)$ for $i \leq 3$ and $\Gamma^{i-1}(K_4) \cong S^{++}(i, K_3)$ for $i \leq 2$.

Theorem 3. *Let $G = (V, E)$ be a graph that may have self-loops but not multiple edges. The first Zagreb Index and the Forgotten Index are defined as $M_1(G) = \sum_{v \in V(G)} d_G(v)^2$ and $M_1^3(G) = \sum_{v \in V(G)} d_G(v)^3$, respectively. Then we have*

$$SO(\Gamma(G)) = SO(G) + \frac{1}{\sqrt{2}}\left(M_1^3(G) - M_1(G)\right)$$

Proof. For each non-loop edge $\{u, v\}$ of G, there exist two corresponding vertices uv, vu in $\Gamma(G)$. For a self-loop $\{w, w\}$ of G, the only corresponding vertex in $\Gamma(G)$ is ww. For each $v \in V(G)$, we denote $K_{d_G(v)}$ as the complete graph induced by a set $\{\{vw, vw'\} : w, w' \in N_G(v), w \neq w'\}$ of generated edges. The subgraph of $\Gamma(G)$ induced by the set of generated edges is the disjoint union of complete graphs, i.e., $\bigcup_{v \in V(G)} K_{d_G(v)}$. Original edges of the form $\{vw, wv\}$, where $w \in N_G(v)$, are injectively incident to the vertices of $K_{d_G(v)}$. Thus, the degree of every vertex of

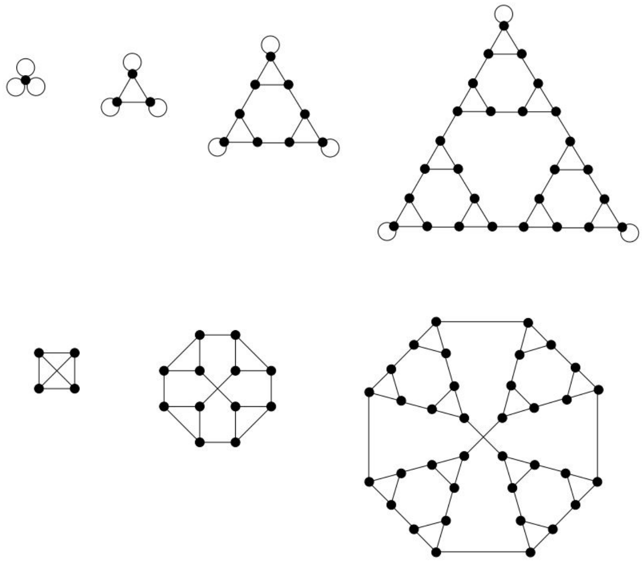

Figure 1. $\Gamma^i\left(K_1^3\right)$ **for** $i \leq 3$ **and** $\Gamma^i\left(K_4\right)$ **for** $i \leq 2$ [20]

$K_{d_G(v)}$ in $\Gamma(G)$ is equal to $d_G(v)$. So, the Sombor Index for $\Gamma(G)$ can be calculated as follows:

$$SO(\Gamma(G)) = SO\left(\bigcup_{v\in V(G)} K_{d_G(v)}\right) + \sum_{\{vw,wv\}\in\Gamma(G), w\in N_G(v)} \sqrt{d_{\Gamma(G)}(vw)^2 + d_{\Gamma(G)}(wv)^2}$$

or,

$$SO(\Gamma(G)) = \sum_{v\in V(G)} SO\left(K_{d_G(v)}\right) + \sum_{vw\in E(G)} \sqrt{d_G(v)^2 + d_G(w)^2}$$

or,

$$SO(\Gamma(G)) = \sum_{v\in V(G)} \frac{d_G(v)^2\left(d_G(v)-1\right)}{\sqrt{2}} + \sum_{vw\in E(G)} \sqrt{d_G(v)^2 + d_G(w)^2}$$

or,

$$SO(\Gamma(G)) = \frac{1}{\sqrt{2}}\left(\sum_{v\in V(G)} d_G(v)^3 - \sum_{v\in V(G)} d_G(v)^2\right) + SO(G)$$

that is,

$$SO(\Gamma(G)) = SO(G) + \frac{1}{\sqrt{2}}\left(M_1^3(G) - M_1(G)\right).$$

□

Theorem 4. *For $n \geq 1$,*

i. $SO\left(S^{\circ}\left(n, K_m\right)\right) = \frac{m^2\left(m^n+1\right)}{\sqrt{2}} = SO\left(\Gamma^{n-1}\left(K_m^{\circ}\right)\right) = SO\left(\Gamma^n\left(K_1^m\right)\right)$

ii. $SO\left(S^{++}\left(n, K_m\right)\right) = \frac{m^{n+1}(m+1)}{\sqrt{2}} = SO\left(\Gamma^{n-1}\left(K_{m+1}\right)\right)$

Proof. i. By the definition of $S^{\circ}\left(n, K_m\right)$,

$$\begin{aligned}
\left|E\left(S^{\circ}\left(n, K_m\right)\right)\right| &= \left|E\left(S\left(n, K_m\right)\right)\right| + m \text{ (self-loops at the extreme vertices)} \\
&= \frac{m\left(m^n-1\right)}{2} + m \quad \text{(refer equation 5)} \\
&= \frac{m^{n+1}+m}{2} \\
&= \frac{m\left(m^n+1\right)}{2}.
\end{aligned}$$

Also, since $S^{\circ}\left(n, K_m\right)$ is a regularized variation of $S\left(n, K_m\right)$, (see Figure 1, $\Gamma^3\left(K_1^3\right) = \Gamma^2\left(K_3^{\circ}\right) \cong S^{\circ}\left(3, K_3\right)$) $d_{S^{\circ}\left(n, K_m\right)}(v) = m \ \forall v \in V\left(S^{\circ}\left(n, K_m\right)\right)$. Therefore,

$$\begin{aligned}
SO\left(S^{\circ}\left(n, K_m\right)\right) &= \left|E\left(S^{\circ}\left(n, K_m\right)\right)\right| \sqrt{m^2+m^2} \quad \text{(using 1)} \\
&= \frac{m\left(m^n+1\right)}{2}\left(m\sqrt{2}\right) \\
&= \frac{m^2\left(m^n+1\right)}{\sqrt{2}}.
\end{aligned}$$

Hence, by Lemmma 1(i) we conclude

$$SO\left(S^{\circ}\left(n, K_m\right)\right) = \frac{m^2\left(m^n+1\right)}{\sqrt{2}} = SO\left(\Gamma^{n-1}\left(K_m^{\circ}\right)\right) = SO\left(\Gamma^n\left(K_1^m\right)\right).$$

ii. As explained earlier, (see Figure 1, $\Gamma^2\left(K_4\right) \cong S^{++}\left(3, K_3\right)$) $S^{++}\left(n, K_m\right)$ is constructed by copying $S\left(n-1, K_m\right)$ $m+1$ times and adding another set of $\left|E\left(K_{m+1}\right)\right|$ edges. Hence, the total number of edges in $S^{++}\left(n, K_m\right)$ is

$$\begin{aligned}
\left|E\left(S^{++}\left(n, K_m\right)\right)\right| &= (m+1)\left|E\left(S\left(n-1, K_m\right)\right)\right| + \left|E\left(K_{m+1}\right)\right|) \\
&= (m+1)\frac{m\left(m^{n-1}-1\right)}{2} + \frac{(m+1)m}{2} \quad \text{(refer equation 5)} \\
&= \frac{m^n(m+1)}{2}.
\end{aligned}$$

Again, since $S^{++}\left(n, K_m\right)$ is a regularized variation of $S\left(n, K_m\right)$, $d_{S^{++}\left(n, K_m\right)}(v) = m \ \forall v \in V\left(S^{++}\left(n, K_m\right)\right)$. Therefore,

$$\begin{aligned}
SO\left(S^{++}\left(n, K_m\right)\right) &= \left|E\left(S^{++}\left(n, K_m\right)\right)\right| \sqrt{m^2+m^2} \quad \text{(using 1)} \\
&= \frac{m^n(m+1)}{2}\left(m\sqrt{2}\right) \\
&= \frac{m^{n+1}(m+1)}{\sqrt{2}}.
\end{aligned}$$

Hence, by Lemmma 1(ii) we conclude

$$SO\left(S^{++}\left(n,K_m\right)\right)=\frac{m^{n+1}\left(m+1\right)}{\sqrt{2}}=SO\left(\Gamma^{n-1}\left(K_{m+1}\right)\right).$$

□

We note that $|E\left(\Gamma^n\left(G\right)\right)|\leq|E\left(K_\lambda^\circ\right)|$, where $\lambda=|V\left(\Gamma^n\left(G\right)\right)|$. Hence by Lemma 1 (i) in [9], we can deduce the following proposition.

Proposition 2. *Let $G=(V,E)$ be a graph that may have self-loops but not multiple edges. Let $\lambda=|V\left(\Gamma^n\left(G\right)\right)|=\sum_{v\in V(G)}d_G\left(v\right)^n$. Then we have*

$$SO\left(\Gamma^n\left(G\right)\right)\leq SO\left(K_\lambda^\circ\right)=\frac{\lambda^2\left(\lambda+1\right)}{\sqrt{2}}.$$

3. Mycielskian Graph

In [25], Jan Mycielski introduced an interesting construction that transforms a given graph into a new larger one, preserving its triangle-free property and clique number but with a significantly increased chromatic number. For a given graph G with vertex set $V\left(G\right)=\{v_1,v_2,\cdots,v_n\}$ and edge set $E\left(G\right)$, the Mycielskian of G, denoted by $\mu\left(G\right)$, is the graph with the vertex set $V\left(\mu\left(G\right)\right)=V\cup V'\cup\{w\}$, where $V'=\{u_i:v_i\in V\left(G\right),1\leq i\leq n\}$, i.e., $V\left(\mu\left(G\right)\right)=\{v_1,\cdots,v_n,u_1,\cdots,u_n,w\}$ and the edge set $E\left(\mu\left(G\right)\right)=E\left(G\right)\cup\{u_iv_j,v_iv_j:v_iv_j\in E\left(G\right)\}\cup\{u_iw:1\leq i\leq n\}$. Figure 2 shows the Mycielskian of the C_5 graph also known as the *Grötzsch graph*, the smallest triangle-free 4-chromatic graph.

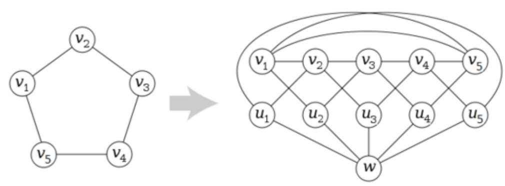

Figure 2. $\mu\left(C_5\right)$: **Grötzsch graph [5]**

We note that $\mu\left(G\right)$ contains G itself as an induced subgraph. The vertices $v_1,v_2,\ldots,v_n$ in $\mu\left(G\right)$ are referred to as *original* vertices and the vertices $u_1,u_2,\ldots,u_n$ as *shadow* vertices. The vertex w is called the *root*.

The generalized Mycielskian construction was introduced by Stiebitz in [34], which are natural generalizations of the Mycielskian graphs. Let G be a graph with vertex set $V\left(G\right)=\{v_1,v_2,\cdots,v_n\}$ and edge set $E\left(G\right)$. Let m be any positive integer. For each integer $k\left(0\leq k\leq m\right)$, let V^k be the k^{th} copy of vertices in V, that is,

$V^k = \{v_1^k, v_2^k, \cdots, v_n^k\}$. The *generalized Mycielskian* or the *m-Mycielskian* of G (also known as a *a cone over* G), denoted as $\mu_m(G)$, is the graph with the vertex set $V(\mu_m(G)) = V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^m \cup \{z\}$ and the edge set $E(\mu_m(G)) = \{v_i^0 v_j^0 : v_i v_j \in E\} \cup \left(\bigcup_{k=0}^{m-1} \{v_i^k v_j^{k+1} : v_i v_j \in E\}\right) \cup \left(\bigcup_{k=0}^{m-1} \{v_i^{k+1} v_j^k : v_i v_j \in E\}\right) \cup \{v_j^m z : \forall v_j^m \in V^m\}$. For $m = 0$, $\mu_0(G)$ is defined to be the graph obtained from G by adding a universal vertex z, and for $m = 1$, we observe that $\mu_1(G) = \mu(G)$ which is but the *Mycielskian* of G and so the subscript is omitted.
We say that the vertex v_i^k is at *level* k and so the vertices of V^k is called as vertices of *level* k. We can make the identification $v_i^0 = v_i$, for $1 \leq i \leq n$ and thus refer to the vertices at *level* 0 as *original* vertices and the vertices at *level* $k \geq 1$ as *shadow* vertices and the vertex w as the *root*. The root is only adjacent to the shadow vertices at *level* m. *Level* m is the *top* level [4]. Figure 3 illustrates the above concept.

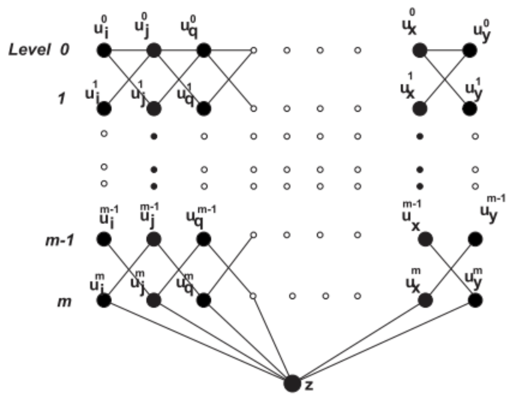

Figure 3. $\mu_m(P_n)$: demonstrating the levels from 0 to m-top level and the root vertex z [26]

3.1. Bounds on the Sombor Index of the Mycielskian graph

Understanding the degrees of the vertices of the m-Mycielskian graphs helps us understand the structure of the graph construction better.

Proposition 3 ([2]). *Let* $\mu_m(G)$ *be the* m-Mycielskian *of a graph* G *with* n *vertices. Then for each* $v \in V(\mu_m(G))$ *we have*

$$d_{\mu_m(G)}(v) = \begin{cases} n & v = z \\ 1 + d_G(v_i) & v = v_i^m \\ 2d_G(v_i) & v = v_i^k \quad \textit{where} \quad k = 0, 1, \ldots, m-1. \end{cases} \tag{7}$$

Theorem 5. *Given G is a graph over n vertices with m edges,*

$$2SO(G) + SO_{V,n}(G) + 2m\sqrt{4\delta^2 + (1+\delta)^2} \leq SO(\mu(G)) \tag{8}$$

$$\leq 2SO(G) + SO_{V,n}(G) + 2m\sqrt{4\Delta^2 + (1+\Delta)^2}. \tag{9}$$

Equality (left and right) holds if and only if G is a regular graph, as $\Delta = \delta$.

Proof. We recall that Δ is the maximum degree of G and δ is the minimum degree of G. We know, the Mycielskian of $G = (V, E)$, where $|V| = n$ and $|E| = m$, is the graph with the vertex set $V(\mu(G)) = V^0 \cup V^1 \cup \{z\}$ and the edge set $E(\mu(G)) = \{v_i^0 v_j^0 : v_i v_j \in E\} \cup \{v_i^0 v_j^1 : v_i v_j \in E\} \cup \{v_i^1 v_j^0 : v_i v_j \in E\} \cup \{v_j^1 z : \forall v_j \in V\}$.
So, from Proposition 3, we have

$$d_{\mu(G)}(v) = \begin{cases} n & v = z \\ 1 + d_G(v_i) & v = v_i^1 \\ 2d_G(v_i) & v = v_i^0 \end{cases}$$

The Sombor Index of $\mu(G)$ is given by,

$$\begin{aligned} SO(\mu(G)) = &\sum_{\{v_i, v_j\} \in E(G)} \sqrt{d_{\mu(G)}(v_i^0)^2 + d_{\mu(G)}\left(v_j^0\right)^2} \\ &+ \sum_{\{v_i^0 v_j^1 : v_i v_j \in E(G)\}} \sqrt{d_{\mu(G)}(v_i^0)^2 + d_{\mu(G)}\left(v_j^1\right)^2} \\ &+ \sum_{\{v_i^1 v_j^0 : v_i v_j \in E(G)\}} \sqrt{d_{\mu(G)}(v_i^1)^2 + d_{\mu(G)}\left(v_j^0\right)^2} \\ &+ \sum_{v_i \in V(G)} \sqrt{d_{\mu(G)}(v_i^1)^2 + d_{\mu(G)}(z)^2} \end{aligned}$$

or,

$$\begin{aligned} SO(\mu(G)) = &\sum_{\{v_i, v_j\} \in E(G)} \sqrt{\{2d_G(v_i)\}^2 + \{2d_G(v_j)\}^2} \\ &+ \sum_{\{v_i^0 v_j^1 : v_i v_j \in E(G)\}} \sqrt{\{2d_G(v_i)\}^2 + \{1 + d_G(v_j)\}^2} \\ &+ \sum_{\{v_i^0 v_j^1 : v_i v_j \in E(G)\}} \sqrt{\{1 + d_G(v_i)\}^2 + \{2d_G(v_j)\}^2} \\ &+ \sum_{v_i \in V(G)} \sqrt{\{1 + d_G(v_i)\}^2 + n^2} \end{aligned}$$

or,

$$
\begin{aligned}
SO(\mu(G)) = 2 & \sum_{\{v_i,v_j\}\in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2} \\
& + \sum_{\{v_i^0 v_j^1 : v_i v_j \in E(G)\}} \sqrt{4d_G(v_i)^2 + \{1 + d_G(v_j)\}^2} \\
& + \sum_{\{v_i^1 v_j^0 : v_i v_j \in E(G)\}} \sqrt{\{1 + d_G(v_i)\}^2 + 4d_G(v_j)^2} \\
& + \sum_{v_i \in V(G)} \sqrt{\{1 + d_G(v_i)\}^2 + n^2}
\end{aligned}
$$

and so using equations (1) and (2), we have

$$
\begin{aligned}
SO(\mu(G)) = {} & 2SO(G) + SO_{V,n}(G) \\
& + \sum_{\{v_i^0 v_j^1 : v_i v_j \in E(G)\}} \sqrt{4d_G(v_i)^2 + \{1 + d_G(v_j)\}^2} \\
& + \sum_{\{v_i^1 v_j^0 : v_i v_j \in E(G)\}} \sqrt{\{1 + d_G(v_i)\}^2 + 4d_G(v_j)^2}.
\end{aligned} \tag{10}
$$

Clearly,

$$
\sum_{\{v_i^0 v_j^1 : v_i v_j \in E(G)\}} \sqrt{4d_G(v_i)^2 + \{1 + d_G(v_j)\}^2} \geq \sum_{\{v_i^0 v_j^1 : v_i v_j \in E(G)\}} \sqrt{4\delta^2 + \{1 + \delta\}^2}
$$

and,

$$
\sum_{\{v_i^1 v_j^0 : v_i v_j \in E(G)\}} \sqrt{\{1 + d_G(v_i)\}^2 + 4d_G(v_j)^2} \geq \sum_{\{v_i^1 v_j^0 : v_i v_j \in E(G)\}} \sqrt{\{1 + \delta\}^2 + 4\delta^2}.
$$

So,

$$
\begin{aligned}
\sum_{\{v_i^0 v_j^1 : v_i v_j \in E(G)\}} \sqrt{4d_G(v_i)^2 + \{1 + d_G(v_j)\}^2} + \sum_{\{v_i^1 v_j^0 : v_i v_j \in E(G)\}} \sqrt{\{1 + d_G(v_i)\}^2 + 4d_G(v_j)^2} \\
\geq 2m\sqrt{4\delta^2 + \{1 + \delta\}^2}.
\end{aligned} \tag{11}
$$

Similarly,

$$
\sum_{\{v_i^0 v_j^1 : v_i v_j \in E(G)\}} \sqrt{4d_G(v_i)^2 + \{1 + d_G(v_j)\}^2} \leq \sum_{\{v_i^0 v_j^1 : v_i v_j \in E(G)\}} \sqrt{4\Delta^2 + \{1 + \Delta\}^2}
$$

and,

$$\sum_{\{v_i^1 v_j^0 : v_i v_j \in E(G)\}} \sqrt{\{1+d_G(v_i)\}^2 + 4d_G(v_j)^2} \le \sum_{\{v_i^1 v_j^0 : v_i v_j \in E(G)\}} \sqrt{\{1+\Delta\}^2 + 4\Delta^2}.$$

So,

$$\sum_{\{v_i^0 v_j^1 : v_i v_j \in E(G)\}} \sqrt{4d_G(v_i)^2 + \{1+d_G(v_j)\}^2} + \sum_{\{v_i^1 v_j^0 : v_i v_j \in E(G)\}} \sqrt{\{1+d_G(v_i)\}^2 + 4d_G(v_j)^2} \le 2m\sqrt{4\Delta^2 + \{1+\Delta\}^2}. \tag{12}$$

Applying equations (11) and (12) in equation (10), we get the result

$$2SO(G) + SO_{V,n}(G) + 2m\sqrt{4\delta^2 + (1+\delta)^2} \le SO(\mu(G)) \le 2SO(G) + SO_{V,n}(G) + 2m\sqrt{4\Delta^2 + (1+\Delta)^2}.$$

When $\Delta = \delta = r$,

$$SO(\mu(G)) = 2SO(G) + SO_{V,n}(G) + 2m\sqrt{4r^2 + (1+r)^2}.$$

□

We can derive a similar result as above for the *generalized Mycielskian* of a graph G as follows:

Theorem 6. *Let t be any positive integer and G be a graph over n vertices with m edges,*

$$2tSO(G) + SO_{V,n}(G) + 2m\sqrt{4\delta^2 + (1+\delta)^2} \le SO(\mu_t(G)) \le 2tSO(G) + SO_{V,n}(G) + 2m\sqrt{4\Delta^2 + (1+\Delta)^2}. \tag{13}$$

Equality (left and right) holds if and only if G is a regular graph, as $\Delta = \delta$.

Proof. Again, we recall that Δ is the maximum degree of G and δ is the minimum degree of G. By the definition of *generalized Mycielskian* of $G = (V, E)$ described at the beginning of section 3 and by Proposition 3, we have

$$d_{\mu_m(G)}(v) = \begin{cases} n & v = z \\ 1 + d_G(v_i) & v = v_i^t \\ 2d_G(v_i) & v = v_i^k \quad where \quad k = 0, 1, \ldots, t-1. \end{cases}$$

The Sombor Index of $\mu_t(G)$ is given by,

$$\begin{aligned}
SO(\mu_t(G)) &= \sum_{\{v_i,v_j\}\in E(G)} \sqrt{d_{\mu_t(G)}\left(v_i^0\right)^2 + d_{\mu_t(G)}\left(v_j^0\right)^2} \\
&+ \sum_{k=0}^{t-2} \sum_{\{v_i^k v_j^{k+1}: v_i v_j \in E(G)\}} \sqrt{d_{\mu_t(G)}\left(v_i^k\right)^2 + d_{\mu_t(G)}\left(v_j^{k+1}\right)^2} \\
&+ \sum_{k=0}^{t-2} \sum_{\{v_i^{k+1} v_j^{k}: v_i v_j \in E(G)\}} \sqrt{d_{\mu_t(G)}\left(v_i^{k+1}\right)^2 + d_{\mu_t(G)}\left(v_j^{k}\right)^2} \\
&+ \sum_{\{v_i^{t-1} v_j^{t}: v_i v_j \in E(G)\}} \sqrt{d_{\mu_t(G)}\left(v_i^{t-1}\right)^2 + d_{\mu_t(G)}\left(v_j^{t}\right)^2} \\
&+ \sum_{\{v_i^{t} v_j^{t-1}: v_i v_j \in E(G)\}} \sqrt{d_{\mu_t(G)}\left(v_i^{t}\right)^2 + d_{\mu_t(G)}\left(v_j^{t-1}\right)^2} \\
&+ \sum_{v_i \in V(G)} \sqrt{d_{\mu(G)}\left(v_i^{t}\right)^2 + d_{\mu(G)}(z)^2}
\end{aligned}$$

or,

$$\begin{aligned}
SO(\mu_t(G)) &= \sum_{\{v_i,v_j\}\in E(G)} \sqrt{\{2d_G(v_i)\}^2 + \{2d_G(v_j)\}^2} \\
&+ \sum_{k=0}^{t-1} \sum_{\{v_i^k v_j^{k+1}: v_i v_j \in E(G)\}} \sqrt{\{2d_G(v_i)\}^2 + \{2d_G(v_j)\}^2} \\
&+ \sum_{k=0}^{t-1} \sum_{\{v_i^{k+1} v_j^{k}: v_i v_j \in E(G)\}} \sqrt{\{2d_G(v_i)\}^2 + \{2d_G(v_j)\}^2} \\
&+ \sum_{\{v_i^{t-1} v_j^{t}: v_i v_j \in E(G)\}} \sqrt{\{2d_G(v_i)\}^2 + \{1 + d_G(v_j)\}^2} \\
&+ \sum_{\{v_i^{t} v_j^{t-1}: v_i v_j \in E(G)\}} \sqrt{\{1 + d_G(v_i)\}^2 + \{2d_G(v_j)\}^2} \\
&+ \sum_{v_i \in V(G)} \sqrt{\{1 + d_G(v_i)\}^2 + n^2}
\end{aligned}$$

or,

$$\begin{aligned}
SO(\mu_t(G)) = 2 &\sum_{\{v_i,v_j\}\in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2} \\
&+ (t-1) \sum_{\{v_i,v_j\}\in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2} \\
&+ (t-1) \sum_{\{v_i,v_j\}\in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2}
\end{aligned}$$

$$+ \sum_{\{v_i^{t-1}v_j^t : v_iv_j \in E(G)\}} \sqrt{4d_G(v_i)^2 + \{1 + d_G(v_j)\}^2}$$
$$+ \sum_{\{v_i^t v_j^{t-1} : v_iv_j \in E(G)\}} \sqrt{\{1 + d_G(v_i)\}^2 + 4d_G(v_j)^2}$$
$$+ \sum_{v_i \in V(G)} \sqrt{\{1 + d_G(v_i)\}^2 + n^2}$$

and so using equations (1) and (2), we have

$$SO(\mu_t(G)) = 2tSO(G) + SO_{V,n}(G) + \sum_{\{v_i^{t-1}v_j^t : v_iv_j \in E(G)\}} \sqrt{4d_G(v_i)^2 + \{1 + d_G(v_j)\}^2}$$
$$+ \sum_{\{v_i^t v_j^{t-1} : v_iv_j \in E(G)\}} \sqrt{\{1 + d_G(v_i)\}^2 + 4d_G(v_j)^2}. \tag{14}$$

Clearly,

$$\sum_{\{v_i^{t-1}v_j^t : v_iv_j \in E(G)\}} \sqrt{4d_G(v_i)^2 + \{1 + d_G(v_j)\}^2} \geq \sum_{\{v_i^{t-1}v_j^t : v_iv_j \in E(G)\}} \sqrt{4\delta^2 + \{1 + \delta\}^2}$$

and,

$$\sum_{\{v_i^t v_j^{t-1} : v_iv_j \in E(G)\}} \sqrt{\{1 + d_G(v_i)\}^2 + 4d_G(v_j)^2} \geq \sum_{\{v_i^t v_j^{t-1} : v_iv_j \in E(G)\}} \sqrt{\{1 + \delta\}^2 + 4\delta^2}.$$

So,

$$\sum_{\{v_i^{t-1}v_j^t : v_iv_j \in E(G)\}} \sqrt{4d_G(v_i)^2 + \{1 + d_G(v_j)\}^2} + \sum_{\{v_i^t v_j^{t-1} : v_iv_j \in E(G)\}} \sqrt{\{1 + d_G(v_i)\}^2 + 4d_G(v_j)^2}$$
$$\geq 2m\sqrt{4\delta^2 + \{1 + \delta\}^2}. \tag{15}$$

Similarly,

$$\sum_{\{v_i^{t-1}v_j^t : v_iv_j \in E(G)\}} \sqrt{4d_G(v_i)^2 + \{1 + d_G(v_j)\}^2} \leq \sum_{\{v_i^{t-1}v_j^t : v_iv_j \in E(G)\}} \sqrt{4\Delta^2 + \{1 + \Delta\}^2}$$

and,

$$\sum_{\{v_i^t v_j^{t-1} : v_iv_j \in E(G)\}} \sqrt{\{1 + d_G(v_i)\}^2 + 4d_G(v_j)^2} \leq \sum_{\{v_i^t v_j^{t-1} : v_iv_j \in E(G)\}} \sqrt{\{1 + \Delta\}^2 + 4\Delta^2}.$$

So,

$$\sum_{\{v_i^{t-1}v_j^t : v_iv_j \in E(G)\}} \sqrt{4d_G(v_i)^2 + \{1 + d_G(v_j)\}^2} + \sum_{\{v_i^t v_j^{t-1} : v_iv_j \in E(G)\}} \sqrt{\{1 + d_G(v_i)\}^2 + 4d_G(v_j)^2}$$
$$\leq 2m\sqrt{4\Delta^2 + \{1 + \Delta\}^2}. \tag{16}$$

Applying equations (15) and (16) in equation (14), we get the result

$$2tSO(G) + SO_{V,n}(G) + 2m\sqrt{4\delta^2 + (1+\delta)^2} \leq SO(\mu_t(G))$$
$$\leq 2tSO(G) + SO_{V,n}(G) + 2m\sqrt{4\Delta^2 + (1+\Delta)^2}.$$

When $\Delta = \delta = k$,

$$SO(\mu_t(G)) = 2tSO(G) + SO_{V,n}(G) + 2m\sqrt{4k^2 + (1+k)^2}.$$

□

Now, we determine the upper bound for the Sombor Index of the t-Mycielskian of k-regular graphs like the generalized Peterson graphs, Hypercube graphs, etc.

Table 2. Edge partition of $\mu_t(G)$ Mycielskian graph based on degrees of end vertices of each edge.

$(d_{\mu_t(G)}(v_i), d_{\mu_t(G)}(v_j))$ where $v_i, v_j \in V(\mu_t(G))$	Number of edges
$(2k, 2k)$	$m + 2m(t-1)$
$(2k, k+1)$	$2m$
$(k+1, n)$	n

G is a k-regular graph.

Theorem 7. *For any k-regular graph G of order n and any integer $t \geq 1$,*

$$SO(\mu_t(G)) \leq (4t+2)\,SO(G) + 2nk\sqrt{2}. \tag{17}$$

Proof. Given G is a k-regular graph with order n and size m,

$$SO(G) = m\sqrt{k^2 + k^2} = mk\sqrt{2}. \tag{18}$$

By using the edge partition based on the degrees of end vertices of each edge of graph of $\mu_t(G)$ given in Table 2, we can compute the Sombor index of the t-Mycielskian graph of k-regular graph G, using equation (1) as follows:

$$SO(\mu_t(G)) = (m + (t-1)(2m))\sqrt{(2k)^2 + (2k)^2} + (2m)\sqrt{(2k)^2 + (k+1)^2}$$
$$+ n\sqrt{(k+1)^2 + n^2}. \tag{19}$$

Now, it is obvious that

$$\sqrt{(2k)^2+(k+1)^2} \leq \sqrt{(2k)^2+(2k)^2} \tag{20}$$
$$\sqrt{n^2+(k+1)^2} \leq \sqrt{(2k)^2+(2k)^2}. \tag{21}$$

Applying equations (20) and (21) in equation (19), we get

$$\begin{aligned} SO\left(\mu_t\left(G\right)\right) &\leq \left(m+\left(t-1\right)\left(2m\right)\right)\sqrt{(2k)^2+(2k)^2}+\left(2m\right)\sqrt{(2k)^2+(2k)^2} \\ &+n\sqrt{(2k)^2+(2k)^2} \\ &= \left(m+\left(t-1\right)\left(2m\right)+2m+n\right)\sqrt{(2k)^2+(2k)^2} \\ &= \left(m\left(2t+1\right)+n\right)\left(2k\right)\sqrt{2} \\ &= 2mk\sqrt{2}\left(2t+1\right)+2nk\sqrt{2} \\ &= 2SO\left(G\right)\left(2t+1\right)+2nk\sqrt{2} \quad (\textit{using equation } (18)). \end{aligned}$$

Therefore, we have the result

$$SO\left(\mu_t\left(G\right)\right) \leq \left(4t+2\right)SO\left(G\right)+2nk\sqrt{2}.$$

□

Remark 3. The equality in Theorem 7 is attained for $G \cong P_2$ (or $G \cong K_2$). In this case, $k=1$ and $n=2$, so $m=\frac{nk}{2}=\frac{(2)(1)}{2}=1$. So,

$$\begin{aligned} SO\left(\mu_t\left(G\right)\right) &= \left(1+\left(t-1\right)\left(2\right)\right)\sqrt{(2)^2+(2)^2}+\left(2\right)\sqrt{(2)^2+(1+1)^2}+n\sqrt{(1+1)^2+2^2} \\ &= 2\sqrt{2}\left(3+2t\right) \\ &= \left(4t+2\right)SO\left(G\right)+2nk\sqrt{2}. \end{aligned}$$

Note that $\mu_t\left(P_2\right)=C_{2t+3}$, since $\left|V\left(\mu_t\left(G\right)\right)\right|=\left(t+1\right)n+1=nt+n+1$ (Figure 4).

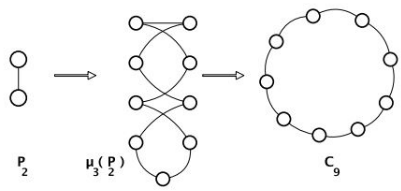

Figure 4. $\mu_3\left(P_2\right)$ [26]

Table 3. Edge partition of $\mu_m(K_n)$ Mycielskian graph based on degrees of end vertices of each edge, in each level.

Level in $\mu_m(K_n)$	$(d_{\mu_m(K_n)}(v_i), d_{\mu_m(K_n)}(v_j))$ $v_i, v_j \in V(\mu_m(K_n))$	Number of edges
Level 0	$(2(n-1), 2(n-1))$	$\frac{n(n-1)}{2}$
between levels $p-1$ to p; $1 \leq p \leq m-1$	$(2(n-1), 2(n-1))$	$(m-1)\,2\left[\frac{n(n-1)}{2}\right]$ $= (m-1)\,n(n-1)$
between levels $m-1$ and m	$(2(n-1), n)$	$2\left[\frac{n(n-1)}{2}\right]$ $= n(n-1)$
between level m and universal vertex z	(n, n)	n

Here, we use Proposition 3 to determine $d_{\mu_m(K_n)}(v)$ for $v \in V(\mu_m(K_n))$

3.2. Sombor Index of the generalized Mycielskian of the complete graph, K_m

Theorem 8. *The Sombor Index of $\mu_m(K_n)$ is given by*

$$SO(\mu_m(K_n)) = n(n-1)^2(2m-1)\sqrt{2} + n^2\sqrt{2} + n(n-1)\sqrt{4(n-1)^2 + n^2}. \tag{22}$$

Proof. ***Level*** 0

In the 0^{th}-level, $\mu_m(K_n)$ contains K_n itself as an isomorphic subgraph, with $d_{\mu_m(K_n)}(v_i^0) = 2d_{K_n}(v_i) = 2(n-1)$ (Proposition 3), $1 \leq i \leq n$.

Between levels $p-1$ ***to*** p***;*** $1 \leq p \leq m-1$

For $1 \leq p \leq m-1$, between the $(p-1)^{th}$-level and the p^{th}-level, $\mu_m(K_n)$ contains $\frac{n(n-1)}{2}$ $\{v_i^{p-1}, v_j^p\}$-edges and $\frac{n(n-1)}{2}$ $\{v_i^p, v_j^{p-1}\}$-edges, where $\{v_i, v_j\} \in E(K_n)$, with $d_{\mu_m(K_n)}\left(v_i^{p-1}\right) = 2d_{K_n}(v_i) = 2(n-1) = d_{\mu_m(K_n)}(v_i^p)$ (Proposition 3), $1 \leq i \leq n$.

Between levels $m-1$ ***and*** m

Between the $(m-1)^{th}$-level and the m^{th}-level, $\mu_m(K_n)$ contains $\frac{n(n-1)}{2}$ $\{v_i^{m-1}, v_j^m\}$-edges and $\frac{n(n-1)}{2}$ $\{v_i^m, v_j^{m-1}\}$-edges, where $\{v_i, v_j\} \in E(K_n)$, with $d_{\mu_m(K_n)}\left(v_i^{m-1}\right) = 2d_{K_n}(v_i) = 2(n-1)$, $d_{\mu_m(K_n)}(v_i^m) = 1 + d_{K_n}(v_i) = 1 + (n-1) = n$ (Proposition 3), $1 \leq i \leq n$.

Between level m ***and universal vertex*** z

Between the m^{th}-level and vertex z, $\mu_m(K_n)$ contains n $\{v_j^m, z\}$-edges, where $v_i \in V(K_n)$, with $d_{\mu_m(K_n)}(v_i^m) = 1 + d_{K_n}(v_i) = 1 + (n-1) = n$ (Proposition 3), $1 \leq i \leq n$ and $d_{\mu_m(K_n)}(z) = n$.

Table 4. Edge partition of $\mu_m\left(C_n^2\right)$ Mycielskian graph based on degrees of end vertices of each edge, in each level.

Level in $\mu_m\left(C_n^2\right)$	$\left(d_{\mu_m(C_n^2)}(v_i), d_{\mu_m(C_n^2)}(v_j)\right)$ where $v_i, v_j \in V\left(\mu_m\left(C_n^2\right)\right)$	Number of edges
Level 0	$(8,8)$	$2n$
between levels $p-1$ to p; $1 \leq p \leq m-1$	$(8,8)$	$(m-1)\,2\,(2n)$ $=(m-1)\,(4n)$
between levels $m-1$ and m	$(8,5)$	$2\,(2n)=4n$
between level m and universal vertex z	$(5,n)$	n

Here, we use Proposition 3 to determine $d_{\mu_m(C_n^2)}(v)$ for $v \in V\left(\mu_m\left(C_n^2\right)\right)$

By using Table 3, equation (22) can be easily computed. Therefore, applying equation (1), we get

$$\begin{aligned} SO\left(\mu_m\left(K_n\right)\right) = {} & \frac{n\,(n-1)}{2}\sqrt{(2\,(n-1))^2+(2\,(n-1))^2} \\ & + (m-1)\,n\,(n-1)\sqrt{(2\,(n-1))^2+(2\,(n-1))^2} \\ & + n\,(n-1)\sqrt{(2\,(n-1))^2+n^2} + n\sqrt{n^2+n^2}. \end{aligned}$$

Hence,

$$SO\left(\mu_m\left(K_n\right)\right) = n\,(n-1)^2\,(2m-1)\sqrt{2} + n^2\sqrt{2} + n\,(n-1)\sqrt{4\,(n-1)^2+n^2}.$$

□

The Python code to verify the above-derived result is given in section 6.

3.3. Sombor Index of the generalized Mycielskian of the circulant graph, C_n^2

Theorem 9. *The Sombor Index of $\mu_m\left(C_n^2\right)$ is given by*

$$SO\left(\mu_m\left(C_n^2\right)\right) = 16n\,(m-1)\sqrt{2} + 4n\sqrt{89} + n\sqrt{n^2+5^2}. \tag{23}$$

Proof. We note that in C_n^2, $d_{C_n^2}(v) = 4\;\forall\, v \in V\left(C_n^2\right)$ and, $\left|E\left(C_n^2\right)\right| = 2n$.

Level 0

In the 0^{th}-level, $\mu_m\left(C_n^2\right)$ contains C_n^2 itself as an isomorphic subgraph, with $d_{\mu_m(C_n^2)}\left(v_i^0\right) = 2d_{C_n^2}\left(v_i\right) = 8$ (Proposition 3), $1 \leq i \leq n$.

Between levels $p-1$ ***to*** p***;*** $1 \leq p \leq m-1$

For $1 \leq p \leq m-1$, between the $(p-1)^{th}$-level and the p^{th}-level, $\mu_m\left(C_n^2\right)$ contains $2n$ $\{v_i^{p-1}, v_j^p\}$-edges and $2n$ $\{v_i^p, v_j^{p-1}\}$-edges, where $\{v_i, v_j\} \in E\left(C_n^2\right)$, with $d_{\mu_m(C_n^2)}\left(v_i^{p-1}\right) = 2d_{C_n^2}(v_i) = 8 = d_{\mu_m(C_n^2)}(v_i^p)$ (Proposition 3), $1 \leq i \leq n$.

Between levels $m-1$ ***and*** m

Between the $(m-1)^{th}$-level and the m^{th}-level, $\mu_m\left(C_n^2\right)$ contains $2n$ $\{v_i^{m-1}, v_j^m\}$-edges and $2n$ $\{v_i^m, v_j^{m-1}\}$-edges, where $\{v_i, v_j\} \in E\left(C_n^2\right)$, with $d_{\mu_m(C_n^2)}\left(v_i^{m-1}\right) = 2d_{C_n^2}(v_i) = 8d_{\mu_m(C_n^2)}(v_i^m) = 1 + d_{C_n^2}(v_i) = 5$ (Proposition 3), $1 \leq i \leq n$.

Between level m ***and universal vertex*** z

Between the m^{th}-level and vertex z, $\mu_m\left(C_n^2\right)$ contains n $\{v_j^m, z\}$-edges, where $v_i \in V\left(C_n^2\right)$, with $d_{\mu_m(C_n^2)}(v_i^m) = 1 + d_{C_n^2}(v_i) = 5$ (Proposition 3), $1 \leq i \leq n$ and $d_{\mu_m(C_n^2)}(z) = n$.
By using Table 4, and applying equation (1), we get

$$\begin{aligned} SO\left(\mu_m\left(C_n^2\right)\right) &= 2n\sqrt{8^2+8^2} + (m-1)(4n)\sqrt{8^2+8^2} + (4n)\sqrt{8^2+5^2} + n\sqrt{n^2+5^2} \\ &= (2n)\left(8\sqrt{2}\right) + 4n(m-1)\left(8\sqrt{2}\right) + 4n\sqrt{89} + n\sqrt{n^2+5^2}. \end{aligned}$$

Hence,

$$SO\left(\mu_m\left(C_n^2\right)\right) = 16n(m-1)\sqrt{2} + 4n\sqrt{89} + n\sqrt{n^2+5^2}.$$

□

3.4. Sombor Index of the generalized Mycielskian of the cycle graph, C_n

Theorem 10. *The Sombor Index of* $\mu_m(C_n)$ *is given by*

$$SO\left(\mu_m(C_n)\right) = 16n(m-1)\sqrt{2} + 4n\sqrt{89} + n\sqrt{n^2+5^2}. \tag{24}$$

Proof. ***Level*** 0

In the 0^{th}-level, $\mu_m(C_n)$ contains C_n itself as an isomorphic subgraph, with $d_{\mu_m(C_n)}\left(v_i^0\right) = 2d_{C_n}(v_i) = 4$ (Proposition 3), $1 \leq i \leq n$.

Between levels $p-1$ ***to*** p***;*** $1 \leq p \leq m-1$

For $1 \leq p \leq m-1$, between the $(p-1)^{th}$-level and the p^{th}-level, $\mu_m(C_n)$ contains n $\{v_i^{p-1}, v_j^p\}$-edges and n $\{v_i^p, v_j^{p-1}\}$-edges, where $\{v_i, v_j\} \in E(C_n)$, with $d_{\mu_m(C_n)}\left(v_i^{p-1}\right) = 2d_{C_n}(v_i) = 4 = d_{\mu_m(C_n)}(v_i^p)$ (Proposition 3), $1 \leq i \leq n$.

Between levels $m-1$ ***and*** m

Between the $(m-1)^{th}$-level and the m^{th}-level, $\mu_m(C_n)$ contains n $\{v_i^{m-1}, v_j^m\}$-edges and n $\{v_i^m, v_j^{m-1}\}$-edges, where $\{v_i, v_j\} \in E(C_n)$, with $d_{\mu_m(C_n)}\left(v_i^{m-1}\right) = 2d_{C_n}(v_i) = 4$, $d_{\mu_m(C_n)}(v_i^m) = 1 + d_{C_n}(v_i) = 3$, (Proposition 3).

Table 5. Edge partition of $\mu_m(C_n)$ Mycielskian graph based on degrees of end vertices of each edge, in each level.

Level in $\mu_m(C_n)$	$(d_{\mu_m(C_n)}(v_i), d_{\mu_m(C_n)}(v_j))$ where $v_i, v_j \in V(\mu_m(C_n))$	Number of edges
Level 0	$(4,4)$	n
between levels $p-1$ to p; $1 \leq p \leq m-1$	$(4,4)$	$(m-1)(2n)$
between levels $m-1$ and m	$(4,3)$	$2n$
between level m and universal vertex z	$(3,n)$	n

Here, we use Proposition 3 to determine $d_{\mu_m(C_n)}(v)$ for $v \in V(\mu_m(C_n))$

Between level m ***and universal vertex*** z

Between the m^{th}-level and vertex z, $\mu_m(C_n)$ contains n $\{v_j^m, z\}$-edges, where $v_i \in V(C_n)$, with $d_{\mu_m(C_n)}(v_i^m) = 1 + d_{C_n}(v_i) = 3$ (Proposition 3), $1 \leq i \leq n$ and $d_{\mu_m(C_n)}(z) = n$.
By using Table 5 and equation (1), we can compute

$$\begin{aligned} SO(\mu_m(C_n)) &= n\sqrt{4^2+4^2} + (m-1)(2n)\sqrt{4^2+4^2} + (2n)\sqrt{4^2+3^2} + n\sqrt{n^2+3^2} \\ &= 4n\sqrt{2} + 8n(m-1)\sqrt{2} + 10n + n\sqrt{n^2+3^2}. \end{aligned}$$

Therefore, we get

$$SO(\mu_m(C_n)) = 4n(2m-1)\sqrt{2} + 10n + n\sqrt{n^2+3^2}.$$

□

3.5. Sombor Index of the generalized Mycielskian of the path graph, P_n

Theorem 11. *The Sombor Index of $\mu_m(P_n)$ is given by*

$$\begin{aligned} SO(\mu_m(P_n)) &= 8m\sqrt{5} + (8m-4)(n-3)\sqrt{2} + 2\sqrt{13} + 10(n-3) \\ &\quad + 2\sqrt{n^2+2^2} + (n-2)\sqrt{n^2+3^2}. \end{aligned} \tag{25}$$

Proof. ***Level*** 0

In the 0^{th}-level, $\mu_m(P_n)$ contains P_n itself as an isomorphic subgraph, with $d_{\mu_m(P_n)}(v_1^0) = d_{\mu_m(P_n)}(v_n^0) = 2d_{C_n}(v_1) = 2d_{C_n}(v_n) = 2$ (Proposition 3), and $d_{\mu_m(P_n)}(v_i^0) = 2d_{C_n}(v_i) = 2(2) = 4$ (Proposition 3), $1 < i < n$.

Table 6. Edge partition of $\mu_m(P_n)$ Mycielskian graph based on degrees of end vertices of each edge, in each level.

Level in $\mu_m(P_n)$	$(d_{\mu_m(P_n)}(v_i), d_{\mu_m(P_n)}(v_j))$ where $v_i, v_j \in V(\mu_m(P_n))$	Number of edges
Level 0	$(2,4)$ $(4,4)$	2 $(n-3)$
between levels $p-1$ to p; $1 \le p \le m-1$	$(2,4)$ $(4,4)$	$(m-1)(4)$ $(m-1)(2(n-3))$
between levels $m-1$ and m	$(2,3)$ $(2,4)$ $(4,3)$	2 2 $2(n-3)$
between level m and universal vertex z	$(2,n)$ $(3,n)$	2 $(n-2)$

Here, we use Proposition 3 to determine $d_{\mu_m(P_n)}(v)$ for $v \in V(\mu_m(P_n))$.

***Between levels** $p-1$ **to** p; $1 \le p \le m-1$*

For $1 \le p \le m-1$, between the $(p-1)^{th}$-level and the p^{th}-level, $\mu_m(P_n)$ contains $(n-3)$ $\{v_i^{p-1}, v_j^p\}$-edges and $(n-3)$ $\{v_i^p, v_j^{p-1}\}$-edges, where $\{v_i, v_j\} \in E(P_n)$, with $d_{\mu_m(P_n)}\left(v_i^{p-1}\right) = 2d_{P_n}(v_i) = d_{\mu_m(P_n)}(v_i^p) = 4$ (Proposition 3), $1 < i < n$; and contains 4 edges, viz., $\{v_1^{p-1}, v_2^p\}$, $\{v_1^p, v_2^{p-1}\}$, $\{v_n^{p-1}, v_{n-1}^p\}$, $\{v_n^p, v_{n-1}^{p-1}\}$, with $d_{\mu_m(P_n)}\left(v_1^{p-1}\right) = d_{\mu_m(P_n)}\left(v_n^{p-1}\right) = d_{\mu_m(P_n)}(v_1^p) = d_{\mu_m(P_n)}(v_n^p) = 2$.

***Between levels** $m-1$ **and** m*

Between the $(m-1)^{th}$-level and the m^{th}-level, $\mu_m(P_n)$ contains $(n-3)$ $\{v_i^{m-1}, v_j^m\}$-edges and $(n-3)$ $\{v_i^m, v_j^{m-1}\}$-edges, where $\{v_i, v_j\} \in E(P_n)$, with $d_{\mu_m(P_n)}\left(v_i^{m-1}\right) = 2d_{P_n}(v_i) = 4$ (Proposition 3), $1 < i < n$, and $d_{\mu_m(P_n)}(v_i^m) = 1 + d_{P_n}(v_i) = 1 + 2 = 3$ (Proposition 3), $1 < i < n$; 2 edges, viz., $\{v_1^{m-1}, v_2^m\}, \{v_n^{m-1}, v_{n-1}^m\}$, with $d_{\mu_m(P_n)}\left(v_1^{m-1}\right) = d_{\mu_m(P_n)}\left(v_n^{m-1}\right) = 2d_{P_n}(v_1) = 2d_{P_n}(v_n) = 2$ (Proposition 3); and 2 edges, viz., $\{v_1^m, v_2^{m-1}\}, \{v_n^m, v_{n-1}^{m-1}\}$, with $d_{\mu_m(P_n)}(v_1^m) = d_{\mu_m(P_n)}(v_n^m) = 1 + d_{P_n}(v_1) = 1 + d_{P_n}(v_n) = 1 + 1 = 2$ (Proposition 3).

***Between level** m **and universal vertex** z*

Between the m^{th}-level and vertex z, $\mu_m(P_n)$ contains n $\{v_j^m, z\}$-edges, where $v_i \in V(C_n)$, with $d_{\mu_m(P_n)}(v_i^m) = 1 + d_{P_n}(v_i) = 1 + 2 = 3$ (Proposition 3), $1 \le i \le n$; $d_{\mu_m(P_n)}(v_1^m) = 1 + d_{P_n}(v_1) = d_{\mu_m(P_n)}(v_n^m) = 1 + d_{P_n}(v_n) = 1 + 1 = 2$ and $d_{\mu_m(P_n)}(z) = n$.

By using Table 6 and equation (1), we can compute

$$SO(\mu_m(P_n)) = 2\sqrt{2^2+4^2} + (n-3)\sqrt{4^2+4^2} + (m-1)(4)\sqrt{2^2+4^2} + (m-1)(2(n-3))\sqrt{4^2+4^2} + 2\sqrt{2^2+3^2} + 2\sqrt{2^2+4^2} + 2(n-3)\sqrt{4^2+3^2} + 2\sqrt{2^2+n^2} + (n-2)\sqrt{n^2+3^2}$$

which is,

$$SO(\mu_m(P_n)) = 4\sqrt{5} + 8(m-1)\sqrt{5} + 4\sqrt{5} + (n-3)(4)\sqrt{2} + (m-1)(2n-6)(4)\sqrt{2} + 2\sqrt{13} + 10(n-3) + 2\sqrt{n^2+2^2} + (n-2)\sqrt{n^2+3^2}.$$

Therefore, we get

$$SO(\mu_m(P_n)) = 8m\sqrt{5} + (8m-4)(n-3)\sqrt{2} + 2\sqrt{13} + 10(n-3) + 2\sqrt{n^2+2^2} + (n-2)\sqrt{n^2+3^2}.$$

□

4. Some more results on generalized Sierpiński and Mycielskian graphs

In [33], the author has checked if the Sombor Index derived for the simplicial network models is in line with some of the important results on the Sombor Index. The following remarks show some results that are analogously derived from some important bounds established on Sombor Index:

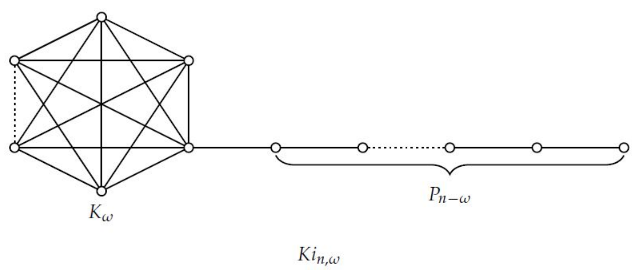

Figure 5. **The long kite graph** $Ki_{n,\omega}$ [11]

Remark 4. Let $\Omega(n,\omega)$ denote the set of connected graphs of order n with clique number ω. The long kite graph $Ki_{n,\omega}$ is a graph of order n obtained from a clique K_ω and a path $P_{n-\omega}$ by adding an edge between a vertex from the clique and an endpoint from the path (see Figure 5). It is shown in [11], that any graph $G \in \Omega(n,\omega)$ has Sombor index lower bounded by

$$SO(G) \geq SO(Ki_{n,\omega}) \sim \binom{\omega-1}{2}\sqrt{2}(\omega-1) + 2\sqrt{2}n + \sqrt{2}\omega^2.$$

Also, $SO(G) \geq SO(P_n)$. Now, for any positive integer t, $S(t,G)$ is of order n^t and $\omega(S(t,G)) = \omega(G)$ [31]; $\mu_t(G)$ is of order $(t+1)n+1 = nt+n+1$ and $\omega(\mu_t(G)) = \omega(G)$. Therefore,

$$SO(S(t,G)) > SO\left(Ki_{n^t,\omega}\right) \sim \binom{\omega-1}{2}\sqrt{2}(\omega-1) + 2\sqrt{2}n^t + \sqrt{2}\omega^2 \tag{26}$$

and,

$$SO(\mu_t(G)) > SO(Ki_{nt+n+1,\omega}) \sim \binom{\omega-1}{2}\sqrt{2}(\omega-1) + 2\sqrt{2}(nt+n+1) + \sqrt{2}\omega^2. \tag{27}$$

Also,

$$SO(S(t,G)) > SO(P_{n^t}) \tag{28}$$

and,

$$SO(\mu_t(G)) > SO(P_{nt+n+1}). \tag{29}$$

Remark 5. It is shown in [9], that for any triangle-free graph G of order n with m edges, maximum degree Δ and minimum degree δ,

$$SO(G) \leq \begin{cases} m\sqrt{\delta^2+(n-\delta)^2} & if \quad \Delta+\delta \leq n \\ m\sqrt{\Delta^2+(n-\Delta)^2} & if \quad \Delta+\delta \geq n. \end{cases}$$

In particular, the Mycielskian construction is used to produce families of larger triangle-free graphs with increasing chromatic numbers. Now, for a triangle-free graph G of order n and size m, $\mu_t(G)$ is a triangle-free graph of order $nt+n+1$ and size $m(2t+1)+n$. $\Delta_{\mu_t(G)} = \max\{2\Delta, \Delta+1, n\}$ and $\delta_{\mu_t(G)} = \min\{2\delta, \delta+1, n\}$. Therefore,

$$SO(\mu_t(G)) \leq \begin{cases} (m(2t+1)+n)\sqrt{\delta^2_{\mu_t(G)} + \left(nt+n+1-\delta_{\mu_t(G)}\right)^2} & if \quad \Delta_{\mu_t(G)}+\delta_{\mu_t(G)} \leq nt+n+1 \\ (m(2t+1)+n)\sqrt{\Delta^2_{\mu_t(G)} + \left(nt+n+1-\Delta_{\mu_t(G)}\right)^2} & if \quad \Delta_{\mu_t(G)}+\delta_{\mu_t(G)} \geq nt+n+1. \end{cases} \tag{30}$$

Remark 6. It is shown in [9], for a graph G with n vertices, maximum degree Δ and minimum degree δ,

$$\frac{n\delta^2}{\sqrt{2}} \leq SO(G) \leq \frac{n\Delta^2}{\sqrt{2}}$$

with equality (left and right) holding if and only if G is a regular graph. Now, for $t \geq 2$ and $E(G) \neq \phi$, $\Delta_{S(t,G)} = 1+\Delta$ [3] and $\delta_{S(t,G)} = \delta$,

$$\frac{n^t\delta^2}{\sqrt{2}} < SO(S(t,G)) < \frac{n^t(\Delta+1)^2}{\sqrt{2}}. \tag{31}$$

Also $\Delta_{\mu_t(G)} = \max\{2\Delta, \Delta+1, n\}$ and $\delta_{\mu_t(G)} = \min\{2\delta, \delta+1, n\}$. So,

$$\frac{(nt+n+1)\,\delta^2_{\mu_t(G)}}{\sqrt{2}} \leq SO\left(S\left(t,G\right)\right) \leq \frac{(nt+n+1)\,\Delta^2_{\mu_t(G)}}{\sqrt{2}} \tag{32}$$

with equality (left and right) holding if and only if G is a regular graph.

Remark 7. It is given in [18] that for any connected graph G of order n,

$$SO\left(P_n\right) \leq SO\left(G\right) \leq SO\left(K_n\right)$$

with equality holding if and only if $G \cong P_n$ or $G \cong K_n$. In [9], Lemma 1 (i) states that for a graph G we have $SO\left(G\right) > SO\left(G-e\right)$, where $e = v_iv_j$ is any edge in G. Thus, we note that by deleting an edge from the graph G, its Sombor Index necessarily decreases. Also, every simple graph with n vertices is a spanning subgraph of the corresponding complete graph with n vertices. Here, $|E\left(P_n\right)| \leq |E\left(G\right)| \leq |E\left(K_m\right)|$ and so $|E\left(S\left(t,P_n\right)\right)| \leq |E\left(S\left(t,G\right)\right)| \leq |E\left(S\left(t,K_m\right)\right)|$. Hence, we can deduce

$$SO\left(S\left(t,P_n\right)\right) \leq SO\left(S\left(t,G\right)\right) \leq SO\left(S\left(t,K_n\right)\right) \tag{33}$$

and,

$$SO\left(\mu_t\left(P_n\right)\right) \leq SO\left(\mu_t\left(G\right)\right) \leq SO\left(\mu_t\left(K_n\right)\right) \tag{34}$$

with equality holding (in both equation (38) and (39) if and only if $G \cong P_n$ or $G \cong K_n$.

Remark 8. It is shown in [9], that for any triangle-free graph G of order n with m edges, maximum degree Δ and minimum degree δ,

$$SO\left(G\right) \leq \begin{cases} m\sqrt{\delta^2+(n-\delta)^2} & if \quad \Delta+\delta \leq n \\ m\sqrt{\Delta^2+(n-\Delta)^2} & if \quad \Delta+\delta \geq n. \end{cases}$$

In particular, the Mycielskian construction is used to produce families of larger triangle-free graphs with increasing chromatic numbers. Now, for a triangle-free graph G of order n and size m, $\mu_t\left(G\right)$ is a triangle-free graph of order $nt+n+1$ and size $m\left(2t+1\right)+n$. $\Delta_{\mu_t(G)} = \max\{2\Delta, \Delta+1, n\}$ and $\delta_{\mu_t(G)} = \min\{2\delta, \delta+1, n\}$. Therefore,

$$SO\left(\mu_t\left(G\right)\right) \leq \begin{cases} \left(m\left(2t+1\right)+n\right)\sqrt{\delta^2_{\mu_t(G)}+\left(nt+n+1-\delta_{\mu_t(G)}\right)^2} & if \quad \Delta_{\mu_t(G)}+\delta_{\mu_t(G)} \leq nt+n+1 \\ \left(m\left(2t+1\right)+n\right)\sqrt{\Delta^2_{\mu_t(G)}+\left(nt+n+1-\Delta_{\mu_t(G)}\right)^2} & if \quad \Delta_{\mu_t(G)}+\delta_{\mu_t(G)} \geq nt+n+1. \end{cases} \tag{35}$$

Remark 9. It is shown in [9], for a graph G with n vertices, maximum degree Δ and minimum degree δ,

$$\frac{n\delta^2}{\sqrt{2}} \leq SO\left(G\right) \leq \frac{n\Delta^2}{\sqrt{2}}.$$

with equality (left and right) holding if and only if G is a regular graph. Now, for $t \geq 2$ and $E(G) \neq \phi$, $\Delta_{S(t,G)} = 1 + \Delta$ [3] and $\delta_{S(t,G)} = \delta$,

$$\frac{n^t \delta^2}{\sqrt{2}} < SO(S(t,G)) < \frac{n^t (\Delta+1)^2}{\sqrt{2}}. \tag{36}$$

Also $\Delta_{\mu_t(G)} = \max\{2\Delta, \Delta+1, n\}$ and $\delta_{\mu_t(G)} = \min\{2\delta, \delta+1, n\}$. So,

$$\frac{(nt+n+1)\,\delta^2_{\mu_t(G)}}{\sqrt{2}} \leq SO(S(t,G)) \leq \frac{(nt+n+1)\,\Delta^2_{\mu_t(G)}}{\sqrt{2}} \tag{37}$$

with equality (left and right) holding if and only if G is a regular graph.

Remark 10. It is given in [18] that for any connected graph G of order n,

$$SO(P_n) \leq SO(G) \leq SO(K_n)$$

with equality holding if and only if $G \cong P_n$ or $G \cong K_n$. In [9], Lemma 1 (i) states that for a graph G we have $SO(G) > SO(G-e)$, where $e = v_i v_j$ is any edge in G. Thus, we note that by deleting an edge from the graph G, its Sombor Index necessarily decreases. Also, every simple graph with n vertices is a spanning subgraph of the corresponding complete graph with n vertices. Here, $|E(P_n)| \leq |E(G)| \leq |E(K_m)|$ and so $|E(S(t,P_n))| \leq |E(S(t,G))| \leq |E(S(t,K_m))|$. Hence, we can deduce

$$SO(S(t,P_n)) \leq SO(S(t,G)) \leq SO(S(t,K_n)) \tag{38}$$

and,

$$SO(\mu_t(P_n)) \leq SO(\mu_t(G)) \leq SO(\mu_t(K_n)) \tag{39}$$

with equality holding (in both equation (38) and (39) if and only if $G \cong P_n$ or $G \cong K_n$.

5. Annexure 1: Python code to calculate $SO(S(n,K_m))$

```
#Sombor Index of Generalised Sierpinski Graph of Complete Graph, i.e., S(n,K_m)
import math

def search(list,n) :    # to search in a list
    for i in range(len(list)) :
        if list[i] == n :
            return True
    return False

def loop(v) :    # to generate the vertex set of S(n,K_m)
    temp = []
    for j in range(1,m+1,1) :
        for k in v :
            temp.append(str(j)+k)
    return temp

def edge_loop1(edge_set) :  # to generate the edge set of S(n,K_m)
    temp = []
    for j in range(1,m+1,1) :
        for k in edge_set :
```

```
            temp.append((str(j)+str(k[0]),str(j)+str(k[1])))
    return temp

def edge_loop2(extra_edge,i) :  # to generate the edge set of S(n,K_m)
    add_edge = []
    temp = ()
    if i == 2 :
        for e in extra_edge :
            for j in range(int(e[0])+1,m+1,1) :
                temp = (str(e[0])+str(j),str(j)+str(e[0]))
                if search(add_edge,temp) == False :
                    add_edge.append(temp)
        return add_edge
    else :
        for e in extra_edge :
            temp = (str(e[0])+str(e[0][-1]),str(e[0][-1])+str(e[0]))
            add_edge.append(temp)
        return add_edge

def degree_radius(i,j) :    # to calculate degree radius of an edge
    return math.sqrt(pow(i,2) + pow(j,2))



def degree(degree_vertex,vertex) :  # to search the degree of a vertex in S(n,K_m)
    for v in degree_vertex :
        if v[0] == vertex :
            #print(v[1])
            return v[1]


if __name__=="__main__" :
    m = int(input("Enter the order of the complete graph: "))
    n = int(input("Enter the dimension 'n' of the generalized Sierpinski Graph of the complete graph: "))

# vertex set of the complete graph K_m
    v = []
    for i in range(1,m+1,1) :
        v.append(str(i))
    print("The vertex set of K_",m," is:",v)

# vertex set of S(n,K_m)
    vertex = []
    if n > 1 :
        for i in range(2,n+1,1) :
            vertex = loop(v)
            v = vertex
    else :
        vertex = v
    print("The vertex set of S(",n,",K_",m,") is:",vertex)
    print("Number of vertices in S(",n,",K_",m,") is:",len(vertex))

# edge set of the complete graph K_m
    edge_set = []
    for i in range(1,m+1,1) :
        for j in range(i,m+1,1) :
            if (i != j) :
                edge = (i,j)
                edge_set.append(edge)
    print("The edge set of K_",m," is:",edge_set)

# to generate the additional edges of S(2,K_m)
    add_edge = []
    temp = ()
    for e in edge_set :
        for j in range(int(e[0])+1,m+1,1) :
            temp = (str(e[0])+str(j),str(j)+str(e[0]))
            if search(add_edge,temp) == False :
                add_edge.append(temp)
```

```
    #print("The additional edges of S(2,K_",m,"):",add_edge)

# edge set of S(n,K_m)
    sp_edge = []
    spedge1 = []
    spedge2 = []
    extra_edge = edge_set
    if n > 1 :
        for i in range(2,n+1,1) :
            spedge1 = edge_loop1(edge_set)
            print(spedge1)
            #print(len(spedge1))
            spedge2 = edge_loop2(extra_edge,i)
            extra_edge = spedge2
            print(spedge2)
            #print(len(spedge2))
            sp_edge = spedge1 + spedge2
            edge_set = sp_edge

    else :
        sp_edge = edge_set
    print("The edge set of S(",n,",K_",m,") is:",sp_edge)
    print("Number of edges in S(",n,",K_",m,") is:",len(sp_edge))

# to calculate the degree of the end vertices of each edge in S(n,K_m), i.e. (d_{S(n,K_m)}(v_i),d_{S(n,K_m)}(v_j)) where (v_i,v_j) is an edge of S(n,K_m)
    degree_vertex = []
    for u in vertex :
        for i in range(1,n,1) :
            if u[0] == u[i] :
                flag = 1
                continue
            else :
                flag = 0
                break
        if flag == 1 :
            degree_vertex.append((u,m-1))
        else :
            degree_vertex.append((u,m))
    #print(degree_vertex)
    #print(len(degree_vertex))

# Calculation of the Sombor Index of S(n,K_m)
    so = []
    for an edge in sp_edge :
        i = degree(degree_vertex,edge[0])
        j = degree(degree_vertex,edge[1])
        #print(i,j)
        so.append(degree_radius(i,j))
    #print(len(so))

    SO = 0
    for ele in range(0, len(so)) :
        SO = SO + so[ele]

    print("The Sombor Index for",n,"-Sierpinski of K_",m,"is:",SO)
```

6. Annexure 2: Python code to calculate $SO\left(\mu_m\left(K_n\right)\right)$

```
#Sombor Index for m-Mycielski of complete graphs
import math

def reverse(x) :    # function to return the reverse of a string x
    return x[::-1]

def search(list,n) :    # to search in a list
    for i in range(len(list)) :
        if list[i] == n :
            return True
```

```
    return False

def degree_radius(i,j) : # to calculate degree radius of an edge
    #print(math.sqrt((i**2)+(j**2)))
    return math.sqrt((i**2)+(j**2))

def degree(degree_vertex,vertex) :   # to search the degree of a vertex in mu_{K_m
    }
    for v in degree_vertex :
        if v[0] == vertex :
            return v[1]

if __name__=="__main__" :
    n = int(input("Enter the number of vertices in the complete graph:"))
    m = int(input("Enter m(level):"))

# vertex set of mu_{K_m}
    vertex = []
    for i in range(m+1) :
        for j in range(1,n+1,1) :
            t = (i,j)
            vertex.append(t)
    vertex.append((m+1,n+1))
    print("The vertex set of mu_{K_",m,"}:",vertex)

# degree of each vertex in mu_{K_m}
    degree_vertex = []
    for t in vertex :
        if (t[0] >= 0) and (t[0] <= m-1) :
            temp = (t,2*(n-1))
            degree_vertex.append(temp)
        elif t[0] == m :
            temp = (t,1+(n-1))
            degree_vertex.append(temp)
        else :
            temp = (t,n)
            degree_vertex.append(temp)
    #for t in degree_vertex :
        #print(t)

# edge set of K_m
    edge_set = []
    for i in range(1,n+1,1) :
        for j in range(i,n+1,1) :
            if i != j :
                edge = (i,j)
                edge_set.append(edge)
    #print(edge_set)

#edge set of mu_{K_m}
    Mycielski_edge_set = []

    for e in edge_set :
        edge = ((0,e[0]),(0,e[1]))
        Mycielski_edge_set.append(edge)
    for p in range(1,m+1,1) :
        for i in range(1,n+1,1) :
            for j in range(1,n+1,1) :
                temp = (i,j)
                if (search(edge_set,temp) == True) or (search(edge_set,reverse(
    temp)) == True) :
                    edge = ((p-1,i),(p,j))
                    Mycielski_edge_set.append(edge)
    for i in range(1,n+1,1) :
        edge = ((m,i),(m+1,n+1))
        Mycielski_edge_set.append(edge)

    print("The edge set of mu_{K_",m,"}:")
    for x in Mycielski_edge_set :
        print(x)
```

```
    print("Number of edges of mu_{K_",m,"}:",len(Mycielski_edge_set))

# Calculation of the Sombor Index of mu_{K_m}
    so = []
    for edge in Mycielski_edge_set :
        i = degree(degree_vertex,edge[0])
        j = degree(degree_vertex,edge[1])
        #print(i,j)
        so.append(degree_radius(i,j))
        #print(len(so))

    SO = 0
    for ele in range(0, len(so)) :
        SO = SO + so[ele]

    print("The Sombor Index for",m,"-Mycielski of K_",n,"is:",SO)
```

7. Conclusion

In this paper, two graph networks, viz. Sierpiński and Mycielskian graphs have been investigated. They find widespread applications in various fields of research:

i. *Sierpiński Graphs*: Sierpiński graphs are fractal graphs known for their self-similarity and recursive construction. They find applications in various fields, including:

- *Image Processing*: Sierpiński graphs can be utilized in image compression and image analysis. Due to their self-similar nature, they can be used to generate fractal patterns that closely approximate complex images. This technique is known as fractal image compression, and it follows for efficient storage and transmission of images with reduced data size.
- *Computer Graphics*: In computer graphics, Sierpiński graphs are used to generate visually appealing and intricate patterns, often used for background textures or a basis for procedural modeling. The recursive construction of Sierpiński graphs enables the generation of complex shapes with simple rules, making them useful for procedural content generation in games and simulations.
- *Wireless Sensor Networks*: In the context of wireless sensor networks, Sierpiński graphs have been studied for their potential use in network topology design. The self-similar and scalable nature of Sierpiński graphs can be leveraged to create efficient network structures, where nodes are placed at different hierarchical levels to optimize energy consumption and communication efficiency.

ii. *Mycielskian Graphs*: Mycielskian graphs are a family of graphs constructed from an original graph . They are known for their properties related to graph coloring and structural characteristics. Their applications include:

- *Graph Coloring*: Mycielskian graphs are widely used in graph theory to study graph coloring problems. The Mycielskian operation increases the chromatic number of a graph while preserving some other properties. The study of Mycielskian graphs contributes to the understanding of chromatic numbers and the development of graph coloring algorithms.

- *Network Design*: Mycielskian graphs have been explored for their potential in network design and optimization. In particular, they have been used in the design of interconnection networks for their parallel computing systems. The properties of Mycielskian graphs can be leveraged to create efficient interconnection networks with desirable properties.

Sierpiński and Sierpiński-type graphs are studied in fractal theory and appear naturally in diverse areas of mathematics and in several scientific fields. And as for the Mycielskian constructions, their major strength is their ability to build large families of graphs with a given parameter fixed and other parameters strictly growing. As with many mathematical concepts, the application of Sierpiński graphs and Mycielskian graphs continues to evolve as researchers explore new possibilities and uncover additional uses in various fields. These graphs can find potential use in mathematical chemistry in the future.

In parallel to the development of the organization of complex networks [33], there has been a lot of research on degree-based topological indices, which are capable of characterizing network structure and dynamics with interdisciplinary applications across mathematics, chemistry, informatics and physics [8, 12, 17]. The Sombor index (SO), the reduced Sombor index (SO_{red}), and the average Sombor index (SO_{avg}) were used to model entropy and enthalpy of vaporization of alkanes. Simple linear models that use one of these indices as an only predictor, showed satisfactory predictive potential, indicating that this topological index may be used successfully in modelling the thermodynamic properties of compounds. Also, it is established that compared with the predictive capability of other degree-based indices, Sombor indices demonstrate better predictive potential [30].

In subsections 2.2, 3.2, 3.3, 3.4 and 3.5 we provide some simple computed formulas to give the Sombor Index of the generalized Sierpiński graph $S(n, K_m)$ and the m-Mycielskian of some specific graphs like K_n, C_n^2, C_n and P_n. It is easy to recognize the mechanical pattern used to derive the formulas and hence we put forth the open problem to develop a general algorithm to calculate the Sombor Index of the generalized Sierpiński and m-Mycielskian of any graph G. The Python codes provided in Annexures 5 and 6 serve to develop more intuition for the same. Some novel upper and lower bounds in subsections 2.1 and 3.1, have been deduced in an attempt to sharpen the focus on the more generalized result. In Remark 2, the Sombor Index for the subdivided-line graph of a graph G, considered to be a further generalization of Sierpiński-like graphs, has been derived. However, the problem to formulate the Sombor Index of the n-iterated subdivided-line graph of G remains open.

Acknowledgements. The first author would like to thank her project advisor, Dr. Radha R. Iyer, Department of Mathematics, Amrita School of Physical Sciences, to whom she is deeply grateful for her undying encouragement, guidance, criticism, and faith. She is thankful to her wonderful parents, Mr. Sachinandan Chanda and Mrs. Sumita Chanda, her friends Megha and Priyamvada, and her family for their undeterred support, constant motivation, and inspiration while carrying out her research. She is also thankful to the anonymous referees and reviewers of this paper for their criticism and guidance which helped her polish her work for the better.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] V. Anandkumar and R.R. Iyer, *On the hyper-Zagreb index of some operations on graphs*, Int. J. Pure Appl. Math. **112** (2017), 239–252 http://dx.doi.org/10.12732/ijpam.v112i2.2.

[2] A. Behtoei and M. Anbarloei, *Gutman index of the Mycielskian and its complement*, arXiv:1502.01580 (2015).

[3] A. Behtoei, M. Khatibi, and F. Attarzadeh, *Degree sequence of the generalized sierpiński graph*, 2020, pp. 88–97 https://doi.org/10.11575/cdm.v15i3.68174.

[4] D. Boutin, S. Cockburn, L. Keough, S. Loeb, K.E. Perry, and P. Rombach, *Symmetry parameters for Mycielskian graphs*, pp. 99–117 https://doi.org/10.1007/978–3–030–77983–2–5, Springer International Publishing, Cham, 2021.

[5] V. Chvátal, *The minimality of the Mycielski graph*, Graphs and Combinatorics (Berlin, Heidelberg) (R.A. Bari and F. Harary, eds.), Springer Berlin Heidelberg, 1974, pp. 243–246.

[6] M.T. Cronin, J. Leszczynski, and T. Puzyn, *Recent Advances in QSAR Studies Methods and Applications*, Challenges and Advances in Computational Chemistry and Physics, vol. 8, Springer, Dordrecht, Heidelberg, London, New York, 2010.

[7] R. Cruz, I. Gutman, and J. Rada, *Sombor index of chemical graphs*, Appl. Math. Comput. **399** (2021), Article ID: 126018 https://doi.org/10.1016/j.amc.2021.126018.

[8] R. Cruz, A. Santamaría-Galvis, and J. Rada, *Extremal values of vertex-degree-based topological indices of coronoid systems*, Int. J. Quantum Chem. **121** (2020), no. 6, https://doi.org/10.1002/qua.26536.

[9] K.C. Das, A.S. Çevik, I.N. Cangul, and Y. Shang, *On Sombor index*, Symmetry **13** (2021), no. 1, Article number: 140 https://doi.org/10.3390/sym13010140.

[10] K.C. Das, S. Das, and B. Zhou, *Sum-connectivity index of a graph*, Frontiers of

Mathematics in China **11** (2016), 47–54
https://doi.org/10.1007/s11464–015–0470–2.

[11] K.C. Das and Y. Shang, *Some extremal graphs with respect to Sombor index*, Mathematics **9** (2021), no. 11, Article ID: 1202
https://doi.org/10.3390/math9111202.

[12] Kinkar Chandra Das, Suresh Elumalai, and Selvaraj Balachandran, *Open problems on the exponential vertex-degree-based topological indices of graphs*, Discrete Appl. Math. **293** (2021), 38–49
https://doi.org/10.1016/j.dam.2021.01.018.

[13] J. Devillers and A.T. Balaban (eds.), *Topological Indices and Related Descriptors in QSAR and QSPR*, CRC Press, London, 1999.

[14] A. Estrada-Moreno and J.A. Rodríguez-Velázquez, *On the general Randić index of polymeric networks modelled by generalized Sierpiński graphs*, Discrete Appl. Math. **263** (2019), 140–151
https://doi.org/10.1016/j.dam.2018.03.032.

[15] C. Gopika, J. Geetha, and K. Somasundaram, *Weighted PI index of tensor product and strong product of graphs*, Discrete Math. Algorithms Appl. **13** (2021), Atricle ID: 2150019
https://doi.org/10.1142/S1793830921500191.

[16] S. Gravier, M. Kovse, and A. Parreau, *Generalized sierpiński graphs 1*, Posters at EuroComb'11, Rényi Institute, Budapest, 2011.

[17] I. Gutman, *Degree-based topological indices*, Croatica Chemica Acta **86** (2013), no. 4, 351–361
https://doi.org/10.5562/cca2294.

[18] ______, *Geometric approach to degree–based topological indices: Sombor indices*, MATCH Commun. Math. Comput. Chem. **86** (2021), no. 1, 11–16.

[19] I. Gutman, N. Gürsoy, A. Gürsoy, and A. Ülker, *New bounds on Sombor index*, Commun. Comb. Optim. **8** (2023), no. 2, 305–311
https://dx.doi.org/10.22049/cco.2022.27600.1296.

[20] Toru Hasunuma, *Structural properties of subdivided-line graphs*, J. Discrete Algorithms **31** (2015), 69–86, 24th International Workshop on Combinatorial Algorithms (IWOCA 2013).

[21] A.M. Hinz, S. Klavžar, and S.S. Zemljič, *A survey and classification of sierpiński-type graphs*, Discrete Appl. Math. **217** (2017), 565–600
https://doi.org/10.1016/j.dam.2016.09.024.

[22] M. Imran, S. Hafi, W. Gao, and M.R. Farahani, *On topological properties of sierpiński networks*, Chaos, Solitons & Fractals **98** (2017), 199–204
https://doi.org/10.1016/j.chaos.2017.03.036.

[23] I. Javaid, H. Benish, M. Imran, A. Khan, and Z. Ullah, *On some bounds of the topological indices of generalized Sierpiński and extended Sierpiński graphs*, J. Ine. Appl. **2019** (2019), Article number: 37
https://doi.org/10.1186/s13660–019–1990–1.

[24] I. Milovanović, E. Milovanović, A. Ali, and M. Matejić, *Some results on the Sombor indices of graphs*, Contrib. Math. **3** (2021), 59–67

https://doi.org/10.47443/cm.2021.0024.

[25] J. Mycielski, *Sur le coloriage des graphs*, Colloquium Math. **3** (1955), no. 2, 161–162 (Fre).

[26] P.G. Nayana and R.R. Iyer, *On secure domination number of generalized Mycielskian of some graphs*, J. Intelligent & Fuzzy Sys **44** (2023), no. 3, 4831–4841 https://doi.org/10.3233/JIFS–223326.

[27] C. Phanjoubam and S. Mawiong, *On Sombor index and some topological indices*, Iranian J. Math. Chem. **12** (2021), no. 4, 209–215 https://doi.org/10.22052/ijmc.2021.243137.1588.

[28] S. Ramakrishnan, J. Senbagamalar, and J. Baskar Babujee, *Topological indices of molecular graphs under specific chemical reactions*, Int. J. Comput. Algorithm **2** (2013), 68–74 https://doi.org/10.20894/IJCOA.101.002.001.019.

[29] H. Ramane, I. Gutman, K. Bhajantri, and D. Kitturmath, *Sombor index of some graph transformations*, Commun. Comb. Optim. **8** (2023), no. 1, 193–205 https://dx.doi.org/10.22049/cco.2021.27484.1272.

[30] I. Redžepović, *Chemical applicability of Sombor indices: Survey*, J. Serbian Chem. Soc. **86** (2021), no. 5, 445–457 https://doi.org/10.2298/JSC201215006R.

[31] J.A. Rodríguez-Velázquez, E.D. Rodríguez-Bazan, and A. Estrada-Moreno, *On generalized Sierpiński graphs*, Discuss. Math. Graph Theory **37** (2017), no. 3, 547–560 https://doi.org/10.7151/dmgt.1945.

[32] Y. Shang, *On the number of spanning trees, the laplacian eigenvalues, and the laplacian estrada index of subdivided-line graphs*, Open Math. **14** (2016), no. 1, 641–648 https://doi.org/10.1515/math–2016–0055.

[33] ______, *Sombor index and degree-related properties of simplicial networks*, Appl. Math. Comput. **419** (2022), Article ID: 126881 https://doi.org/10.1016/j.amc.2021.126881.

[34] M. Stiebitz, *Beiträge zur theorie der färbungskritischen graphen*, Ph.D. thesis, Technical University Ilmenau, 1985.