

Bounds on sombor index and inverse sum indeg (*ISI*) index of graph operations

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Abstract: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $d_G(u)$ the degree of a vertex $u \in V(G)$. The Sombor index of G is defined as $SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$, whereas, the inverse sum indeg (*ISI*) index is defined as $ISI(G) = \sum_{uv \in E(G)} \frac{d_u d_v}{d_u + d_v}$. In this paper, we compute the bounds in terms of maximum degree, minimum degree, order and size of the original graphs G and H for Sombor and *ISI* indices of several graph operations like corona product, cartesian product, strong product, composition and join of graphs.

Keywords: Sombor index; inverse sum indeg index; graph operations; corona product; cartesian product

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1. Introduction

We consider only simple, finite, undirected and connected graphs. Let G be such a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. The cardinality $|V(G)| = n_G$ is called order of G and the cardinality $|E(G)| = m_G$ is the size of G . The degree $d_G(u)$ of a vertex u in G is the number of edges incident with u . The minimum degree (resp. maximum degree) of vertex in G is denoted by δ_G (resp. Δ_G). If the vertex u_i is adjacent to vertex u_j , then the edge connecting them is denoted by $u_i u_j$. A regular graph is a graph in which degree of each vertex is same. A complete graph of order n is denoted by K_n .

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In the chemical and mathematical literature, a topological index is a numerical quantity that is derived from a (chemical) graph such that it remains the same under graph isomorphism. They have several applications in chemistry, pharmacology, materials science, among other, [7, 21]. In 2010, Vukičević and Gašperov [22] introduced the inverse sum indeg index (shortly *ISI* index) as

$$ISI(G) = \sum_{uv \in E(G)} \frac{d_u d_v}{d_u + d_v}. \quad (1)$$

The *ISI* index is a well-studied topological index and has several applications in quantitative structure-activity or structure-property relationships (QSAR/QSPR) [10, 11, 16]. Zangi *et al.* [24] found basic properties of the *ISI* matrix. They also gave some bounds for the *ISI* energy of graphs. The *ISI* index and *ISI* energy of the molecular graphs of hyaluronic acid-paclitaxel conjugates was obtained by Havare [10]. For more results on *ISI* index and *ISI* energy, see [3, 13].

In [5] Gutman defined new vertex-degree-based topological index called Sombor index, denoted by $SO(G)$ and defined as:

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}. \quad (2)$$

The study of the Sombor index of graphs has attracted a significant amount of attention within a very short span of time. Redžepović [12] found that the Sombor index has good predictive potential for statistical modeling of enthalpy of vaporization and entropy for alkanes. Cruz *et al.* [4] investigated the Sombor index of chemical graphs and characterized the extremal graphs with respect to the Sombor index over the sets of (connected) chemical graphs, chemical trees, and hexagonal systems. For more related results, one may refer to [6, 8, 17, 19, 23] and the references therein.

In chemical graph theory, some chemical graphs obtained by the use of different graph operations (graph products) are very interesting to investigate. In 2015, Shetty *et al.* [20] gave formulae for harmonic index of some graph operations. The exact expressions for first and second Zagreb indices and hyper Wiener index was found by Khalifeh *et al.* [14, 15]. Akhter *et al.* [1, 2] computed the exact formulae and the bounds for general sum-connectivity index of some graph operations.

In this paper, we compute some bounds for the Sombor and inverse sum indeg (*ISI*) index of several graph operations. These graph operations include corona product, cartesian product, strong product, join and composition of graphs.

2. Bounds for the Sombor and inverse sum indeg (*ISI*) indices

In this section, we derive some bounds for Sombor and inverse sum indeg (*ISI*) indices of several graph operations. Let G and H be two simple connected graphs whose vertex sets are disjoint. For each $u \in V(G)$ and $v \in V(H)$, we have

$$\Delta_G \geq d_G(u), \quad \delta_G \leq d_G(u), \quad (3)$$

$$\Delta_H \geq d_H(v), \quad \delta_H \leq d_H(v). \quad (4)$$

The equality holds if and only if G and H are regular graphs.

2.1. The corona product

The corona product of G and H , denoted by $G \odot H$, is a graph obtained by taking one copy of G and n_G copies of H and joining the vertex u that is on i -th position in G to every vertex in i -th copy of H . The order and size of $G \odot H$ are $n_G(1 + n_H)$ and $m_G + n_G m_H + n_G n_H$, respectively. The degree of a vertex $u \in V(G \odot H)$ is given by

$$d_{G \odot H}(u) = \begin{cases} d_G(u) + n_H & \text{if } u \in V(G), \\ d_H(u) + 1 & \text{if } u \in V(H). \end{cases} \tag{5}$$

In the following theorem, the bounds on the Sombor index of corona product of two graphs are computed.

Theorem 1. *Let G and H be graphs. Then $\alpha_1 \leq SO(G \odot H) \leq \alpha_2$, where*

$$\begin{aligned} \alpha_1 &= \sqrt{2}m_G(\delta_G + n_H) + \sqrt{2}n_G m_H(\delta_H + 1) + n_G n_H \sqrt{(\delta_G + n_H)^2 + (\delta_H + 1)^2}, \\ \alpha_2 &= \sqrt{2}m_G(\Delta_G + n_H) + \sqrt{2}n_G m_H(\Delta_H + 1) + n_G n_H \sqrt{(\Delta_G + n_H)^2 + (\Delta_H + 1)^2}. \end{aligned}$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (5) in equation (2), we obtain

$$\begin{aligned} SO(G \odot H) &= \sum_{uv \in E(G)} \sqrt{(d_G(u) + n_H)^2 + (d_G(v) + n_H)^2} \\ &\quad + n_G \sum_{uv \in E(H)} \sqrt{(d_H(u) + 1)^2 + (d_H(v) + 1)^2} \\ &\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} \sqrt{(d_G(u) + n_H)^2 + (d_H(v) + 1)^2} \\ &\leq \sum_{uv \in E(G)} \sqrt{2(\Delta_G + n_H)^2} + n_G \sum_{uv \in E(H)} \sqrt{2(\Delta_H + 1)^2} \\ &\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} \sqrt{(\Delta_G + n_H)^2 + (\Delta_H + 1)^2} \\ &= \sqrt{2}m_G(\Delta_G + n_H) + \sqrt{2}n_G m_H(\Delta_H + 1) \\ &\quad + n_G n_H \sqrt{(\Delta_G + n_H)^2 + (\Delta_H + 1)^2}. \end{aligned} \tag{6}$$

Similarly, we can compute

$$SO(G \odot H) \geq \sqrt{2}m_G(\delta_G + n_H) + \sqrt{2}n_G m_H(\delta_H + 1) + n_G n_H \sqrt{(\delta_G + n_H)^2 + (\delta_H + 1)^2}. \tag{7}$$

The equality in (6) and (7) holds if and only if G and H are regular graphs. □

Let $t \geq 1$ and \overline{K}_t be the complement of K_t . Then t -thorny graph of G is the corona product of G and \overline{K}_t . The following corollary is an easy consequence of Theorem 1.

Corollary 1. *For a graph G , the following holds:*

$$\begin{aligned} \sqrt{2}m_G(\delta_G + t) + n_Gt\sqrt{(\delta_G + t)^2 + 1} &\leq SO(G \odot \overline{K}_t) \\ &\leq \sqrt{2}m_G(\Delta_G + t) + n_Gt\sqrt{(\Delta_G + t)^2 + 1}. \end{aligned}$$

The next theorem gives the bounds on the *ISI* index of corona product of two graphs.

Theorem 2. *Let G and H be graphs. Then $\alpha_1 \leq ISI(G \odot H) \leq \alpha_2$, where*

$$\begin{aligned} \alpha_1 &= \frac{m_G(\delta_G + n_H)^2}{2(\Delta_G + n_H)} + \frac{n_Gm_H(\delta_H + 1)^2}{2(\Delta_H + 1)} + \frac{n_Gn_H(\delta_G + n_H)(\delta_H + 1)}{\Delta_G + \Delta_H + n_H + 1}, \\ \alpha_2 &= \frac{m_G(\Delta_G + n_H)^2}{2(\delta_G + n_H)} + \frac{n_Gm_H(\Delta_H + 1)^2}{2(\delta_H + 1)} + \frac{n_Gn_H(\Delta_G + n_H)(\Delta_H + 1)}{\delta_G + \delta_H + n_H + 1}. \end{aligned}$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (5) in equation (1), we obtain

$$\begin{aligned} ISI(G \odot H) &= \sum_{uv \in E(G)} \frac{(d_G(u) + n_H)(d_G(v) + n_H)}{d_G(u) + d_G(v) + 2n_H} \\ &\quad + n_G \sum_{uv \in E(H)} \frac{(d_H(u) + 1)(d_H(v) + 1)}{d_H(u) + d_H(v) + 2} \\ &\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} \frac{(d_G(u) + n_H)(d_H(v) + 1)}{d_G(u) + d_H(v) + n_H + 1} \\ &\leq \sum_{uv \in E(G)} \frac{(\Delta_G + n_H)^2}{2(\delta_G + n_H)} + n_G \sum_{uv \in E(H)} \frac{(\Delta_H + 1)^2}{2(\delta_H + 1)} \\ &\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} \frac{(\Delta_G + n_H)(\Delta_H + 1)}{\delta_G + \delta_H + n_H + 1} \\ &= \frac{m_G(\Delta_G + n_H)^2}{2(\delta_G + n_H)} + \frac{n_Gm_H(\Delta_H + 1)^2}{2(\delta_H + 1)} + \frac{n_Gn_H(\Delta_G + n_H)(\Delta_H + 1)}{\delta_G + \delta_H + n_H + 1}. \end{aligned} \tag{8}$$

Similarly, we can compute

$$ISI(G \odot H) \geq \frac{m_G(\delta_G + n_H)^2}{2(\Delta_G + n_H)} + \frac{n_Gm_H(\delta_H + 1)^2}{2(\Delta_H + 1)} + \frac{n_Gn_H(\delta_G + n_H)(\delta_H + 1)}{\Delta_G + \Delta_H + n_H + 1}. \tag{9}$$

The equality in (8) and (9) holds if and only if G and H are regular graphs. □

The following corollary is an easy consequence of Theorem 2.

Corollary 2. For a graph G , the following holds:

$$\begin{aligned} \frac{m_G(\delta_G + t)^2}{2(\Delta_G + t)} + \frac{n_G t(\delta_G + t)(\delta_H + 1)}{\Delta_G + t + 1} &\leq ISI(G \odot \overline{K_t}) \\ &\leq \frac{m_G(\Delta_G + t)^2}{2(\delta_G + t)} + \frac{n_G t(\Delta_G + t)}{\delta_G + t + 1}. \end{aligned}$$

2.2. The cartesian product

The cartesian product of G and H , denoted by $G \times H$, is a graph whose vertex set is $V(G \times H) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ whenever $[v_1$ and v_2 are adjacent in H and $u_1 = u_2]$ or $[u_1$ and u_2 are adjacent in G and $v_1 = v_2]$. The order of the cartesian product of two graphs is the product of number of vertices of G and H , and the size is $m_G n_H + m_H n_G$. If G and H are regular graphs then $G \times H$ is also regular graph. The degree of a vertex $(u, v) \in V(G \times H)$ is

$$d_{G \times H}((u, v)) = d_G(u) + d_H(v). \tag{10}$$

In the following theorem, we compute bounds on the Sombor index of $G \times H$.

Theorem 3. Let G and H be graphs. Then

$$\sqrt{2}m_{G \times H}(\delta_G + \delta_H) \leq SO(G \times H) \leq \sqrt{2}m_{G \times H}(\Delta_G + \Delta_H).$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (10) in equation (2), we obtain

$$\begin{aligned} SO(G \times H) &= \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} \sqrt{(d_G(u_1) + d_H(v_1))^2 + (d_G(u_1) + d_H(v_2))^2} \\ &+ \sum_{v_1 \in V(H)} \sum_{u_1 u_2 \in E(G)} \sqrt{(d_G(u_1) + d_H(v_1))^2 + (d_G(u_2) + d_H(v_1))^2} \\ &\leq \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} \sqrt{2(\Delta_G + \Delta_H)^2} \\ &+ \sum_{v_1 \in V(H)} \sum_{u_1 u_2 \in E(G)} \sqrt{2(\Delta_G + \Delta_H)^2} \\ &= \sqrt{2}(n_G m_H + n_H m_G)(\Delta_G + \Delta_H) \\ &= \sqrt{2}m_{G \times H}(\Delta_G + \Delta_H). \end{aligned} \tag{11}$$

One can analogously compute the following:

$$SO(G \times H) \geq \sqrt{2}m_{G \times H}(\delta_G + \delta_H). \tag{12}$$

The equality in (11) and (12) obviously holds if and only if G and H are regular graphs. □

Now, we compute the bounds on the *ISI* index of $G \times H$.

Theorem 4. *Let G and H be graphs. Then*

$$\frac{1}{2}m_{G \times H} \frac{(\delta_G + \delta_H)^2}{\Delta_G + \Delta_H} \leq ISI(G \times H) \leq \frac{1}{2}m_{G \times H} \frac{(\Delta_G + \Delta_H)^2}{\delta_G + \delta_H}.$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (10) in equation (1), we obtain

$$\begin{aligned} ISI(G \times H) &= \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} \frac{(d_G(u_1) + d_H(v_1))(d_G(u_1) + d_H(v_2))}{2d_G(u_1) + d_H(v_1) + d_H(v_2)} \\ &+ \sum_{v_1 \in V(H)} \sum_{u_1 u_2 \in E(G)} \frac{(d_G(u_1) + d_H(v_1))(d_G(u_2) + d_H(v_1))}{d_G(u_1) + d_G(u_2) + 2d_H(v_1)} \\ &\leq \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} \frac{(\Delta_G + \Delta_H)^2}{2(\delta_G + \delta_H)} + \sum_{v_1 \in V(H)} \sum_{u_1 u_2 \in E(G)} \frac{(\Delta_G + \Delta_H)^2}{2(\delta_G + \delta_H)} \quad (13) \\ &= \frac{1}{2}(n_G m_H + n_H m_G) \frac{(\Delta_G + \Delta_H)^2}{\delta_G + \delta_H} \\ &= \frac{1}{2}m_{G \times H} \frac{(\Delta_G + \Delta_H)^2}{\delta_G + \delta_H}. \end{aligned}$$

One can analogously compute the following:

$$ISI(G \times H) \geq \frac{1}{2}m_{G \times H} \frac{(\delta_G + \delta_H)^2}{\Delta_G + \Delta_H}. \quad (14)$$

The equality in (13) and (14) obviously holds if and only if G and H are regular graphs. \square

2.3. The strong product

The strong product of G and H , denoted by $G \boxtimes H$, is a graph whose vertex set is $V(G \boxtimes H) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \boxtimes H$ whenever $[v_1$ and v_2 are adjacent in H and $u_1 = u_2]$ or $[u_1$ and u_2 are adjacent in G and $v_1 = v_2]$ or $[u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)]$. The order of $G \boxtimes H$ is the product of number of vertices of G and H , and the size is $n_G m_H + n_H m_G + 2m_G m_H$. The degree of a vertex $(u, v) \in V(G \boxtimes H)$ is

$$d_{G \boxtimes H}((u, v)) = d_G(u) + d_H(v) + d_G(u)d_H(v). \quad (15)$$

We compute bounds on the Sombor index of $G \boxtimes H$ in the following theorem.

Theorem 5. *Let G and H be graphs. Then*

$$\sqrt{2}m_{G\boxtimes H}(\delta_G + \delta_H + \delta_G\delta_H) \leq SO(G \boxtimes H) \leq \sqrt{2}m_{G\boxtimes H}(\Delta_G + \Delta_H + \Delta_G\Delta_H).$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (15) in equation (2), we obtain

$$\begin{aligned} SO(G \boxtimes H) &= \sum_{u_1 \in V(G)} \sum_{v_1, v_2 \in E(H)} \sqrt{(d_G(u_1) + d_H(v_1) + d_G(u_1)d_H(v_1))^2 + (d_G(u_1) + d_H(v_2) + d_G(u_1)d_H(v_2))^2} \\ &+ \sum_{v_1 \in V(H)} \sum_{u_1, u_2 \in E(G)} \sqrt{(d_G(u_1) + d_H(v_1) + d_G(u_1)d_H(v_1))^2 + (d_G(u_2) + d_H(v_1) + d_G(u_2)d_H(v_1))^2} \\ &+ 2 \sum_{u_1, u_2 \in E(G)} \sum_{v_1, v_2 \in E(H)} \sqrt{(d_G(u_1) + d_H(v_1) + d_G(u_1)d_H(v_1))^2 + (d_G(u_2) + d_H(v_2) + d_G(u_2)d_H(v_2))^2} \\ &\leq \sum_{u_1 \in V(G)} \sum_{v_1, v_2 \in E(H)} \sqrt{2(\Delta_G + \Delta_H + \Delta_G\Delta_H)^2} \\ &+ \sum_{v_1 \in V(H)} \sum_{u_1, u_2 \in E(G)} \sqrt{2(\Delta_G + \Delta_H + \Delta_G\Delta_H)^2} \\ &+ 2 \sum_{u_1, u_2 \in E(G)} \sum_{v_1, v_2 \in E(H)} \sqrt{2(\Delta_G + \Delta_H + \Delta_G\Delta_H)^2} \\ &= \sqrt{2}(n_G m_H + n_H m_G + 2m_G m_H)(\Delta_G + \Delta_H + \Delta_G\Delta_H) \\ &= \sqrt{2}m_{G\boxtimes H}(\Delta_G + \Delta_H + \Delta_G\Delta_H). \end{aligned} \tag{16}$$

Analogously, one can compute the following:

$$SO(G \boxtimes H) \geq \sqrt{2}m_{G\boxtimes H}(\delta_G + \delta_H + \delta_G\delta_H). \tag{17}$$

If G and H are regular graphs then the equality in (16) and (17) holds. □

Next, we compute bounds on the *ISI* index of $G \boxtimes H$.

Theorem 6. *Let G and H be graphs. Then*

$$\frac{1}{2}m_{G\boxtimes H} \frac{(\delta_G + \delta_H + \delta_G\delta_H)^2}{\Delta_G + \Delta_H + \Delta_G\Delta_H} \leq ISI(G \boxtimes H) \leq \frac{1}{2}m_{G\boxtimes H} \frac{(\Delta_G + \Delta_H + \Delta_G\Delta_H)^2}{\delta_G + \delta_H + \delta_G\delta_H}.$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (15) in equation (1), we obtain

$$\begin{aligned} ISI(G \boxtimes H) &= \sum_{u_1 \in V(G)} \sum_{v_1, v_2 \in E(H)} \frac{(d_G(u_1) + d_H(v_1) + d_G(u_1)d_H(v_1))(d_G(u_1) + d_H(v_2) + d_G(u_1)d_H(v_2))}{2d_G(u_1) + d_H(v_1) + d_H(v_2) + d_G(u_1)(d_H(v_1) + d_H(v_2))} \\ &+ \sum_{v_1 \in V(H)} \sum_{u_1, u_2 \in E(G)} \frac{(d_G(u_1) + d_H(v_1) + d_G(u_1)d_H(v_1))(d_G(u_2) + d_H(v_1) + d_G(u_2)d_H(v_1))}{d_G(u_1) + d_G(u_2) + 2d_H(v_1) + d_H(v_1)(d_G(u_1) + d_G(u_2))} \\ &+ 2 \sum_{u_1, u_2 \in E(G)} \sum_{v_1, v_2 \in E(H)} \frac{(d_G(u_1) + d_H(v_1) + d_G(u_1)d_H(v_1))(d_G(u_2) + d_H(v_2) + d_G(u_2)d_H(v_2))}{d_G(u_1) + d_G(u_2) + d_H(v_1) + d_H(v_2) + d_G(u_1)d_H(v_1) + d_G(u_2)d_H(v_2)} \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} \frac{(\Delta_G + \Delta_H + \Delta_G \Delta_H)^2}{2(\delta_G + \delta_H + \delta_G \delta_H)} + \sum_{v_1 \in V(H)} \sum_{u_1 u_2 \in E(G)} \frac{(\Delta_G + \Delta_H + \Delta_G \Delta_H)^2}{2(\delta_G + \delta_H + \delta_G \delta_H)} \\
 &+ 2 \sum_{u_1 u_2 \in E(G)} \sum_{v_1 v_2 \in E(H)} \frac{(\Delta_G + \Delta_H + \Delta_G \Delta_H)^2}{2(\delta_G + \delta_H + \delta_G \delta_H)} \tag{18} \\
 &= \frac{1}{2} (n_G m_H + n_H m_G + 2m_G m_H) \frac{(\Delta_G + \Delta_H + \Delta_G \Delta_H)^2}{\delta_G + \delta_H + \delta_G \delta_H} \\
 &= \frac{1}{2} m_{G \boxtimes H} \frac{(\Delta_G + \Delta_H + \Delta_G \Delta_H)^2}{\delta_G + \delta_H + \delta_G \delta_H}.
 \end{aligned}$$

Analogously, one can compute the following:

$$ISI(G \boxtimes H) \geq \frac{1}{2} m_{G \boxtimes H} \frac{(\delta_G + \delta_H + \delta_G \delta_H)^2}{\Delta_G + \Delta_H + \Delta_G \Delta_H}. \tag{19}$$

If G and H are regular graphs then the equality in (18) and (19) holds. □

2.4. The join of graphs

The join of G and H , denoted by $G + H$, is a union of graphs G and H together with all the edges joining the sets of vertices of G and H . The order and size of $G + H$ are $n_G n_H$ and $m_G + m_H + n_G n_H$, respectively. The degree of a vertex u in $G + H$ is given by

$$d_{G+H}(u) = \begin{cases} d_G(u) + n_H & \text{if } u \in V(G), \\ d_H(u) + n_G & \text{if } u \in V(H). \end{cases} \tag{20}$$

We compute bounds on the Sombor index for join of two graphs in the following theorem.

Theorem 7. *Let G and H be graphs. Then $\alpha_1 \leq SO(G + H) \leq \alpha_2$, where*

$$\begin{aligned}
 \alpha_1 &= \sqrt{2}m_G(\delta_G + n_H) + \sqrt{2}m_H(\delta_H + n_G) + n_G n_H \sqrt{(\delta_G + n_H)^2 + (\delta_H + n_G)^2}, \\
 \alpha_2 &= \sqrt{2}m_G(\Delta_G + n_H) + \sqrt{2}m_H(\Delta_H + n_G) + n_G n_H \sqrt{(\Delta_G + n_H)^2 + (\Delta_H + n_G)^2}.
 \end{aligned}$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (20) in equation (2), we obtain

$$\begin{aligned}
 SO(G + H) &= \sum_{uv \in E(G)} \sqrt{(d_G(u) + n_H)^2 + (d_G(v) + n_H)^2} + \\
 &\sum_{uv \in E(H)} \sqrt{(d_H(u) + n_G)^2 + (d_H(v) + n_G)^2} \\
 &+ \sum_{u \in V(G)} \sum_{v \in V(H)} \sqrt{(d_G(u) + n_H)^2 + (d_H(v) + n_G)^2} \\
 &\leq \sum_{uv \in E(G)} \sqrt{2(\Delta_G + n_H)^2} + \sum_{uv \in E(H)} \sqrt{2(\Delta_H + n_G)^2} \\
 &+ \sum_{u \in V(G)} \sum_{v \in V(H)} \sqrt{(\Delta_G + n_H)^2 + (\Delta_H + n_G)^2} \\
 &= \sqrt{2}m_G(\Delta_G + n_H) + \sqrt{2}m_H(\Delta_H + n_G) \\
 &+ n_G n_H \sqrt{(\Delta_G + n_H)^2 + (\Delta_H + n_G)^2}.
 \end{aligned} \tag{21}$$

Similarly, we can show that

$$SO(G + H) \geq \sqrt{2}m_G(\delta_G + n_H) + \sqrt{2}m_H(\delta_H + n_G) + n_G n_H \sqrt{(\delta_G + n_H)^2 + (\delta_H + n_G)^2}. \tag{22}$$

If G and H are regular graphs then we obtain the equality in (21) and (22). \square

Now, the bounds on the ISI index for join of two graphs is given in the following theorem.

Theorem 8. *Let G and H be graphs. Then $\alpha_1 \leq ISI(G + H) \leq \alpha_2$, where*

$$\begin{aligned} \alpha_1 &= \frac{1}{2}m_G \frac{(\delta_G + n_H)^2}{\Delta_G + n_H} + \frac{1}{2}m_H \frac{(\delta_H + n_G)^2}{\Delta_H + n_G} + n_G n_H \frac{(\delta_G + n_H)(\delta_H + n_G)}{\Delta_G + \Delta_H + n_H + n_G}, \\ \alpha_2 &= \frac{1}{2}m_G \frac{(\Delta_G + n_H)^2}{\delta_G + n_H} + \frac{1}{2}m_H \frac{(\Delta_H + n_G)^2}{\delta_H + n_G} + n_G n_H \frac{(\Delta_G + n_H)(\Delta_H + n_G)}{\delta_G + \delta_H + n_H + n_G}. \end{aligned}$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (20) in equation (1), we obtain

$$\begin{aligned} ISI(G + H) &= \sum_{uv \in E(G)} \frac{(d_G(u) + n_H)(d_G(v) + n_H)}{d_G(u) + d_G(v) + 2n_H} \\ &+ \sum_{uv \in E(H)} \frac{(d_H(u) + n_G)(d_H(v) + n_G)}{d_H(u) + d_H(v) + 2n_G} \\ &+ \sum_{u \in V(G)} \sum_{v \in V(H)} \frac{(d_G(u) + n_H)(d_H(v) + n_G)}{d_G(u) + d_H(v) + n_H + n_G} \\ &\leq \sum_{uv \in E(G)} \frac{(\Delta_G + n_H)^2}{2(\delta_G + n_H)} + \sum_{uv \in E(H)} \frac{(\Delta_H + n_G)^2}{2(\delta_H + n_G)} \\ &+ \sum_{u \in V(G)} \sum_{v \in V(H)} \frac{(\Delta_G + n_H)(\Delta_H + n_G)}{\delta_G + \delta_H + n_H + n_G} \\ &= \frac{1}{2}m_G \frac{(\Delta_G + n_H)^2}{\delta_G + n_H} + \frac{1}{2}m_H \frac{(\Delta_H + n_G)^2}{\delta_H + n_G} + n_G n_H \frac{(\Delta_G + n_H)(\Delta_H + n_G)}{\delta_G + \delta_H + n_H + n_G}. \end{aligned} \tag{23}$$

Similarly, we can show that

$$ISI(G + H) \geq \frac{1}{2}m_G \frac{(\delta_G + n_H)^2}{\Delta_G + n_H} + \frac{1}{2}m_H \frac{(\delta_H + n_G)^2}{\Delta_H + n_G} + n_G n_H \frac{(\delta_G + n_H)(\delta_H + n_G)}{\Delta_G + \Delta_H + n_H + n_G}. \tag{24}$$

If G and H are regular graphs then we obtain the equality in (23) and (24). \square

2.5. The composition

The composition or lexicographic product of G and H , denoted by $G[H]$, is the graph whose vertex set is $V(G[H]) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G[H]$ whenever $[u_1 u_2 \in E(G)]$ or $[v_1$ and v_2 are adjacent in H and $u_1 = u_2]$. The order of $G[H]$ is the product of number of vertices of G and H , and size is $m_G n_H^2 + n_G m_H$. The degree of a vertex $(u, v) \in V(G[H])$ is

$$d_{G[H]}((u, v)) = n_H d_G(u) + d_H(v). \quad (25)$$

In the following theorem, we calculate bounds on the Sombor index for composition of two graphs.

Theorem 9. *Let G and H be graphs. Then*

$$\sqrt{2}m_{G[H]}(n_H \delta_G + \delta_H) \leq SO(G[H]) \leq \sqrt{2}m_{G[H]}(n_H \Delta_G + \Delta_H).$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (25) in equation (2), we obtain

$$\begin{aligned} SO(G[H]) &= \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} \sqrt{(n_H d_G(u_1) + d_H(v_1))^2 + (n_H d_G(u_1) + d_H(v_2))^2} \\ &+ \sum_{v_1 \in V(H)} \sum_{v_2 \in V(H)} \sum_{u_1 u_2 \in E(G)} \sqrt{(n_H d_G(u_1) + d_H(v_1))^2 + (n_H d_G(u_2) + d_H(v_2))^2} \\ &\leq \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} \sqrt{2(n_H \Delta_G + \Delta_H)^2} \\ &+ \sum_{v_1 \in V(H)} \sum_{v_2 \in V(H)} \sum_{u_1 u_2 \in E(G)} \sqrt{2(n_H \Delta_G + \Delta_H)^2} \\ &= \sqrt{2}(n_G m_H + n_H^2 m_G)(n_H \Delta_G + \Delta_H) \\ &= \sqrt{2}m_{G[H]}(n_H \Delta_G + \Delta_H). \end{aligned} \quad (26)$$

Analogously, one can compute the upper bound

$$SO(G[H]) \geq \sqrt{2}m_{G[H]}(n_H \delta_G + \delta_H). \quad (27)$$

The equality in (26) and (27) obviously holds if and only if G and H are regular graphs. \square

In the next theorem, we compute bounds on the *ISI* index for composition of two graphs.

Theorem 10. *Let G and H be graphs. Then*

$$\frac{1}{2}m_{G[H]} \frac{(n_H\delta_G + \delta_H)^2}{n_H\Delta_G + \Delta_H} \leq ISI(G[H]) \leq \frac{1}{2}m_{G[H]} \frac{(n_H\Delta_G + \Delta_H)^2}{n_H\delta_G + \delta_H}.$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (3), (4) and (25) in equation (1), we obtain

$$\begin{aligned} ISI(G[H]) &= \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} \frac{(n_H d_G(u_1) + d_H(v_1))(n_H d_G(u_1) + d_H(v_2))}{2n_H d_G(u_1) + d_H(v_1) + d_H(v_2)} \\ &+ \sum_{v_1 \in V(H)} \sum_{v_2 \in V(H)} \sum_{u_1 u_2 \in E(G)} \frac{(n_H d_G(u_1) + d_H(v_1))(n_H d_G(u_2) + d_H(v_2))}{n_H(d_G(u_1) + d_G(u_2)) + d_H(v_1) + d_H(v_2)} \\ &\leq \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} \frac{(n_H \Delta_G + \Delta_H)^2}{2(n_H \delta_G + \delta_H)} \\ &+ \sum_{v_1 \in V(H)} \sum_{v_2 \in V(H)} \sum_{u_1 u_2 \in E(G)} \frac{(n_H \Delta_G + \Delta_H)^2}{2(n_H \delta_G + \delta_H)} \\ &= \frac{1}{2}(n_G m_H + n_H^2 m_G) \frac{(n_H \Delta_G + \Delta_H)^2}{n_H \delta_G + \delta_H} \\ &= \frac{1}{2}m_{G[H]} \frac{(n_H \Delta_G + \Delta_H)^2}{n_H \delta_G + \delta_H}. \end{aligned} \tag{28}$$

Analogously, one can compute the upper bound

$$ISI(G[H]) \geq \frac{1}{2}m_{G[H]} \frac{(n_H \delta_G + \delta_H)^2}{n_H \Delta_G + \Delta_H}. \tag{29}$$

The equality in (28) and (29) obviously holds if and only if G and H are regular graphs. □

Remark 1. Recently, Milovanović et al. [18] proved that for a connected graph G

$$SO(G) \geq \frac{\sqrt{2}}{2}M_1(G).$$

The equality holds if and only if G is regular. Here, $M_1(G) = \sum_{uv \in E(G)}(d_u + d_v) = \sum_{u \in V(G)} d_u^2$ is the first Zagreb index [9]. As a consequence of this result, upper bounds on the first Zagreb index follow from the upper bounds on the Sombor index reported in this paper.

3. Conclusion

In this paper, the bounds for Sombor index of several graph operations in terms of maximum degree, minimum degree, order and size of the original graphs G and H are computed. These graph operations include corona product, cartesian product, strong product, composition and join of graphs. Analogously, we calculated the bounds for inverse sum indeg (*ISI*) index.

It still remains an open problem to compute the bounds for Sombor and *ISI* indices of graph operations like disjunction, symmetric index of graphs and several others.

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