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Research Article

On γ-free, γ-totally-free and γ-fixed sets in graphs

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Abstract: Let $G = (V, E)$ be a connected graph. A subset S of V is called a γ-free set if there exists a γ-set D of G such that $S \cap D = \emptyset$. If further the induced subgraph $H = G[V - S]$ is connected, then S is called a cc-γ-free set of G. We use this concept to identify connected induced subgraphs H of a given graph G such that $\gamma(H) \leq \gamma(G)$. We also introduce the concept of γ-totally-free and γ-fixed sets and present several basic results on the corresponding parameters.

Keywords: Domination, domination number, γ-set, γ-free set, γ-totally-free set, γ-fixed set

AMS Subject classification: 05C69

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected and connected graph with neither loops nor multiple edges. For graph theoretic terminologies we refer to [1]. For domination related concepts we refer to [2].
A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to a vertex in S. The domination number γ of G is the minimum cardinality of a dominating set of G. A dominating set S of G with $|S| = \gamma$ is called a γ-set of G. A

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dominating set S is called a connected dominating set if the induced subgraph $G[S]$ is connected. The connected domination number γ_c of G is the minimum cardinality of a connected dominating set of G. A vertex v of G is called a support vertex if v is adjacent to a pendent vertex. If the number of pendent vertices adjacent to v is at least two, then v is a strong support vertex. The corona of two graphs G and H, denoted by $G \circ H$, is the graph obtained from one copy of G and $|V(G)|$ copies of H and i^{th} vertex of G is joined to every vertex in the i^{th} copy of H.
Let G be a connected graph and let H be a connected induced subgraph of G. Then $\gamma(H)$ may be equal to or less than or greater than $\gamma(G)$.

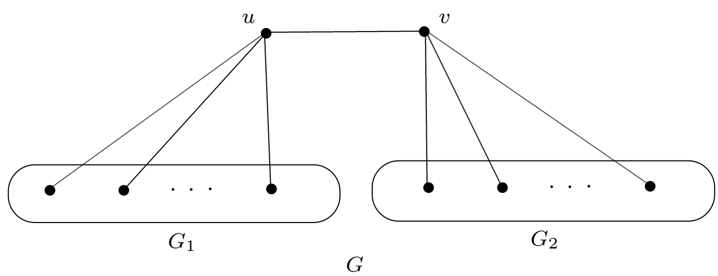

Figure 1. **A graph G with all three types of induced subgraphs**

Example 1. Consider the graph G given in Figure 1, where G_1 is any graph with $\gamma(G_1) \geq 3$. Clearly $\gamma(G) = 2$. Let $H_1 = G_1$, $H_2 = G_1 + u$, $H_3 = (v_1, u, v, w_1)$, where $v_1 \in V(G_1)$ and $w_1 \in V(G_2)$. Clearly H_1, H_2, H_3 are connected induced subgraphs of G. Also $\gamma(H_1) > \gamma(G)$, $\gamma(H_2) < \gamma(G)$ and $\gamma(H_3) = \gamma(G)$. Thus G contains all three types of induced subgraphs.

The following is a fundamental problem in domination:

Problem 1. Let H be a connected induced subgraph of a connected graph G. Under what conditions $\gamma(H) \leq \gamma(G)$?

In this paper we introduce the concepts of γ-free set, γ-totally-free set, γ-fixed set and *cc*-γ-free set in graphs. We use the concept of *cc*-γ-free set to identify connected induced subgraphs H of a given connected graph G such that $\gamma(H) \leq \gamma(G)$. The significance and use of this concept is given in the concluding section of the paper.

2. γ-free sets

Sampathkumar and Neeralagi introduced the concept of γ-free vertex, γ-totally free vertex and γ-fixed vertex in [4]. We extend this notion to subsets of V and define new parameters based on these concepts.

Definition 1. Let $G = (V, E)$ be a connected graph. A subset S of V is called a γ-free set if there exists a γ-set D of G such that $D \cap S = \emptyset$.

A subset of a γ-free set is γ-free and hence γ-freeness is a hereditary property. A γ-free set S is a maximal γ-free set if and only if $S \cup \{v\}$ is not a γ-free set for all $v \in V - S$.

Definition 2. The minimum cardinality of a maximal γ-free set of G is called the γ-free number of G and is denoted by $\gamma_{fr}(G)$.

Lemma 1. *Let G be any connected graph of order n. Then $\gamma_{fr}(G) = n - \gamma(G)$.*

Proof. Let D be a γ-set of G and let $S = V - D$. Since $D \cap S = \emptyset$, S is a γ-free set of G. Also for any $v \in D$, $S_1 = S \cup \{v\}$ is not a γ-free set of G since $|V - S_1| = |D - \{v\}| = \gamma(G) - 1$. Thus S is a maximal γ-free set and $|S| = n - \gamma(G)$. Hence $\gamma_{fr}(G) \leq n - \gamma(G)$. Now, let X be any maximal γ-free set of G. Let D be a γ-set of G such that $D \cap X = \emptyset$. Hence $D \subseteq V - X$. If there exists a vertex $v \in (V - X) - D$, then $X \cup \{v\}$ is a γ-free set which is a contradiction. Thus $\gamma_{fr}(G) \geq n - \gamma(G)$ and hence $\gamma_{fr}(G) = n - \gamma(G)$. □

Definition 3. A γ-free set S of G is called a *cc*-γ-free set if the induced subgraph $H = G[V - S]$ is connected.

Lemma 2. *Every connected graph G admits a cc-γ-free set.*

Proof. Let D be a γ-set of G. It follows from Lemma 1 that $S = V - D$ is a maximal γ-free set of G. Let $\mathcal{F} = \{H : H$ is a connected induced subgraph of G and $D \subseteq V(H)\}$. Since $G \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. Choose $H \in \mathcal{F}$, such that $|V(H)|$ is minimum. Since $D \subseteq V(H)$, we have $S_1 = V(G) - V(H) \subseteq S$. Since S is γ-free, S_1 is γ-free and $H = G[V - S_1]$ is connected. Hence S_1 is a *cc*-γ-free set of G. □

Definition 4. Let G be a connected graph. Then $\max\{|S| : S$ is a *cc*-γ-free set of G $\}$ is called the *cc*-γ-free number of G and is denoted by $cc\gamma_{fr}(G)$.

Example 2. For the path $P_6 = (v_1, v_2, v_3, v_4, v_5, v_6)$, $D = \{v_2, v_5\}$ is the unique γ-set and hence any γ-free set is a subset of $V - D = \{v_1, v_3, v_4, v_6\}$. Clearly $S_1 = \{v_1, v_6\}$ is a γ-free set of P_6 and $H = P_6[V - S_1] = P_4$. Thus $cc\gamma_{fr}(P_6) = 2$ and $\gamma_{fr}(P_6) = 4$.

Theorem 1. *Let G be a connected graph. Then $cc\gamma_{fr}(G) \leq \gamma_{fr}(G)$ and equality holds if and only if $\gamma(G) = \gamma_c(G)$, where $\gamma_c(G)$ is the connected domination number of G.*

Proof. Let S be a cc-γ-free set of G. Since S is a γ-free set, there exists a γ-set D of G such that $D \cap S = \emptyset$. Hence $cc\gamma_{fr}(G) = |S| \leq |V - D| = n - \gamma(G) = \gamma_{fr}(G)$. Thus $cc\gamma_{fr}(G) \leq \gamma_{fr}(G)$. Now, suppose $cc\gamma_{fr}(G) = \gamma_{fr}(G) = n - \gamma(G)$. Let S be a cc-γ-free set of G. Since $|S| = n - \gamma(G)$, it follows that $S = V - D$, where D is a γ-set of G. Also $G[V - S] = G[D]$ is connected and hence D is a connected dominating set of G. Hence $\gamma(G) = \gamma_c(G) = |D|$. The converse is obvious. □

Example 3. Let G be a connected graph of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let H_i be any graph of order k_i, $1 \leq i \leq n$. Then the graph G^* obtained from G by joining v_i to all the vertices of H_i is called a generalized corona of G and is denoted by $G \circ (H_1, H_2, \ldots, H_n)$. Clearly $\gamma(G^*) = \gamma_c(G^*) = |V(G)| = n$. Hence $cc\gamma_{fr}(G^*) = \gamma_{fr}(G^*)$.

Also for the path P_n with $n \leq 4$, the cycle C_n with $n \leq 4$, the complete graph K_n and the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ with $n_i \geq 2$ for all i, we have $\gamma = \gamma_c$. Hence for all these graphs, $cc\gamma_{fr}(G) = \gamma_{fr}(G) = n - \gamma(G)$. We now proceed to determine $cc\gamma_{fr}(G)$ for graphs with $\gamma \neq \gamma_c$.

Theorem 2. *Let T be any tree of order n. Then $cc\gamma_{fr}(T) = k$, where k is the number of pendant vertices of T.*

Proof. Let L and A denote respectively the set of all pendant vertices and the set of all support vertices of T. Clearly L is cc-γ-free set of T. Now let $v \in A$ and let $u \in L$ be the leaf adjacent to v. Since any γ-set D of T contains one of the vetices u, v, it follows that $L \cup \{v\}$ is not a γ-free set. Also if $w \in V - (L \cup A)$, then $T[V - (L \cup \{w\})]$ is not connected and hence $L \cup \{w\}$ is not a cc-γ-free set of T. Thus L is a maximal cc-γ-free set of T and hence $cc\gamma_{fr}(T) \geq |L| = k$. Now, let S be any maximal cc-γ-free set of T. If $u \in L$, $v \in A$ and $uv \in E(T)$, then S contains exactly one of the vertices u and v. Hence we may assume without loss of generality that $L \subseteq S$ and $S \cap A = \emptyset$. Since S is a cc-γ-free set; it follows that $S = L$. Hence $cc\gamma_{fr}(T) = k$. □

Corollary 1. *Let T be a tree of order n. Then $2 \leq cc\gamma_{fr}(T) \leq n-1$. Also $cc\gamma_{fr}(T) = 2$ if and only if T is the path P_n and $cc\gamma_{fr}(T) = n - 1$ if and only if T is the star $K_{1,n-1}$.*

It follows from Theorem 1 that $cc\gamma_{fr}(K_n) = n - 1$. Thus $K_{1,n-1}$ and K_n are two graphs of order n with $cc\gamma_{fr}(G) = n-1$. The following result gives a characterization of all graphs of order n with $cc\gamma_{fr}(G) = n - 1$.

Theorem 3. *Let G be a connected graph of order n. Then $cc\gamma_{fr}(G) = n - 1$ if and only if $\gamma(G) = 1$.*

Proof. Suppose $cc\gamma_{fr}(G) = n - 1$. Let S be a cc-γ-free set of G with $|S| = n - 1$ and $V - S = \{v\}$. Since S is a γ-free set of G, it follows that $\{v\}$ is a dominating set of G and hence $\gamma(G) = 1$. Conversely suppose $\gamma(G) = 1$ and let $\{v\}$ be a dominating set of G. Then $V - \{v\}$ is a maximal cc-γ-free set of G. Hence $cc\gamma_{fr}(G) = n - 1$. □

3. On γ-totally-free sets

Definition 5. A subset S of V is called a γ-totally-free set if $D \cap S = \emptyset$ for all γ-sets D of G. The maximum cardinality of a γ-totally-free set of G is called as the γ-totally-free number of G and is denoted by $\gamma_{tf}(G)$.

Observation 4.

(i) Let A denote the union of all γ-sets of a graph G. Then G admits a γ-totally-free set if and only if $A \neq V$. In this case $V - A$ is a totally γ-free set and $\gamma_{tf}(G) = |V - A|$.

(ii) Since a γ-totally free set is a γ-free set, it follows that $\gamma_{tf}(G) \leq \gamma_{fr}(G) = n - \gamma(G)$. Furthermore, $\gamma_{tf}(G) = \gamma_{fr}(G)$ if and only if G has a unique γ-set.

(iii) If G is a vertex transitive graph, then every vertex of G lies in a γ-set of G and hence G does not admit a γ-totally free set. In particular the complete graph K_m and the cycle C_n do not admit a γ-totally-free set. Also $K_{r,s}$ where $r, s \geq 2$ does not admit a γ-totally-free set.

Theorem 5. *A path $P_n = (v_1, v_2, \cdots, v_n)$ admits a γ-totally-free set if and only if $n \not\equiv 1 (mod\ 3)$. Also*

$$\gamma_{tf}(P_n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 (mod\ 3); \\ \lfloor \frac{n}{3} \rfloor & \text{if } n \equiv 2 (mod\ 3). \end{cases}$$

Proof. We consider the following cases.

Case 1. $n \equiv 1 \pmod 3$.

Let $n = 3k+1$. Then the γ-sets of P_n are given by, $D_1 = \{v_i : i \equiv 2 \pmod 3\} \cup \{v_n\}$, $D_2 = \{v_i : i \equiv 1 \pmod 3\} \cup \{v_{n-2}\}$, $D_3 = \{v_i : i \equiv \ \pmod 3\} \cup \{v_1\}$ and $D_4 = \{v_i : i \equiv 1 \pmod 3\} \cup \{v_3\}$. Clearly $\cup_{i=1}^{4} D_i = V$ and hence P_n does not admit γ-totally-free set.

Case 2. $n \equiv 0 \pmod 3$.

In this case $D = \{v_i : i \equiv 2 \pmod 3\}$ is the unique γ-set of P_n and $|D| = \frac{n}{3}$. Hence $\gamma_{tf}(P_n) = n - \frac{n}{3} = \frac{2n}{3}$.

Case 3. $n \equiv 2 \pmod 3$.

Let $n = 3k + 2$. Any γ-set containing v_n is of the form $\{v_n\} \cup D$, where D is a γ-set of $P_{n-2} = \{v_1, v_2, \ldots, v_{3k}\}$. This path has unique γ-set $\{v_i : i \equiv 2 \pmod 3\}$. Hence only γ-set containing v_n is $D_1 = \{v_n\} \cup \{v_i : i \equiv 2 \pmod 3\}$.

Similarly only γ-set containing v_1 is of the form $D_2 = \{v_1\} \cup \{v_i : i \equiv 1 \pmod 3\}$.

Any γ-set containing v_{n-1} is of the form $\{v_{n-1}\} \cup D$, where D is a γ-set of $P_n - \{v_{3k}, v_{3k+1}, v_{3k+2}\}$ or $P_n - \{v_{3k+1}, v_{3k+2}\}$. The only γ-sets containing v_{n-1} are given by, $D_3 = \{v_{n-1}\} \cup \{v_i : i \equiv 1 \pmod 3\}$ and $D_4 = \{v_{n-1}\} \cup \{v_i : i \equiv 2 \pmod 3\}$. Similarly the only γ-sets containing v_2 are given by, $D_5 = \{v_2\} \cup \{v_i : i \equiv 1 \pmod 3\}$

and $D_6 = \{v_2\} \cup \{v_i : i \equiv 2 \pmod 3\}$. Hence $\cup_{i=1}^{6} D_i = \{v_i : i \not\equiv 0 \pmod 3\}$. Therefore $\{v_i : i \equiv 0 \pmod 3\}$ is a γ-totally-free set of P_n of maximum cardinality. Hence $\gamma_{tf}(P_n) = \lfloor \frac{n}{3} \rfloor$. □

Definition 6. A γ-totally-free set S of G is called a *cc*-γ-totally-free set if $G[V - S]$ is connected.

Example 4. Let H be any connected graph with $\gamma(H) \geq 2$ and let $G = H + K_k$, $k \geq 2$. Then $V(H)$ is a *cc*-γ-totally-free set of G and $\gamma(H) > \gamma(G)$.

4. On γ-fixed sets

Definition 7. Let $G = (V, E)$ be a connected graph. A subset S of V is called a γ-fixed set if $D \cap S \neq \emptyset$ for all γ-sets D in G.

A superset of a γ-fixed set is γ-fixed and hence γ-fixedness is a super-hereditary property. A γ-fixed set S is a minimal γ-fixed set if and only if $S - \{v\}$ is not a γ-fixed set for all $v \in S$.

Definition 8. The minimum cardinality of a minimal γ-fixed set of G is called the γ-fixed number of G and is denoted by $\gamma_{fi}(G)$. The maximum cardinality of a minimal γ-fixed set of G is called the Γ-fixed number of G and is denoted by $\Gamma_{fi}(G)$.

Theorem 6. *A γ-fixed set S of G is a minimal γ-fixed set if and only if for every $v \in S$, there exists a γ-set D such that $D \cap S = \{v\}$.*

Proof. Suppose S is a minimal γ-fixed set of G. Hence for all $v \in S$, $S - \{v\}$ is not a γ-fixed set. Therefore there exists a γ-set D of G such that $D \cap (S - \{v\}) = \emptyset$. Hence $D \cap S = \{v\}$. The Converse is obvious. □

Example 5. If G is a graph with $\gamma(G) = 1$, then $S = \{v : deg(v) = n - 1\}$ is the only minimal γ-fixed set in G. Hence $\gamma_{fi}(G) = \Gamma_{fi}(G) = |S|$. In particular $\gamma_{fi}(K_n) = \Gamma_{fi}(K_n) = n$.

Example 6. Let $G = K_{r,s}$ be a complete bipartite graph with $r \leq s$. Let V_1 and V_2 be the partite sets of G with $|V_1| = r$ and $|V_2| = s$. Then V_1 and V_2 are minimal γ-fixed sets in G. Hence $\gamma_{fi}(G) = r$ and $\Gamma_{fi}(G) = s$.

Observation 7. If G has a unique γ-set D, then $S = \{v\}$ is a minimal γ-fixed set for any $v \in D$. Hence $\gamma_{fi}(G) = \Gamma_{fi}(G) = 1$.

Lemma 3. *Let G be a graph of order $n \geq 3$ with $\delta = 1$. Then*

$$\gamma_{fi}(G) = \begin{cases} 1 & \text{if } G \text{ has a strong support vertex;} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. If G has a strong vertex v, let $S = \{v\}$. Otherwise let $S = \{v, u\}$, where v is a support vertex and u is the pendent neighbor of v. Clearly S is a minimal γ-fixed set of G and hence the result follows. □

Corollary 2. $\gamma_{fi}(P_n) = \begin{cases} 1 & \text{if } n \leq 3; \\ 2 & \text{otherwise.} \end{cases}$

Corollary 3. *Let G be a graph of order $n \geq 2$. Then $\gamma_{fi}(G \circ K_1) = 2$.*

Theorem 8. *For any graph G, $\gamma_{fi}(G) \leq \delta + 1$. Equality holds if and only if $N[v]$ is a minimal γ-fixed set of G for every vertex v with $deg(v) = \delta$.*

Proof. Let $deg(v) = \delta$. Then $D \cap N[v] \neq \emptyset$ for all γ-sets D of G. Hence $N[v]$ is a γ-fixed set. Let $S \subseteq V$ be a minimal γ-fixed set of G. Then $\gamma_{fi}(G) \leq |S| \leq \delta + 1$ and equality holds if and only if $N[v]$ is a minimal γ-fixed set of G. □

Definition 9. A subset S of V is said to be a *cc*-γ-fixed set of G if S is γ-fixed and $G[V - S]$ is connected.

Question 1. Which graphs admit *cc*-γ-fixed set?

5. Conclusion and scope

Let G be a connected graph and let S be a *cc*-γ-free set of G. Then the induced subgraph $H = G[V - S]$ is connected and there exists a γ-set D of G such that $D \subseteq V(H)$. Hence it follows that $\gamma(H) \leq \gamma(G)$. Thus *cc*-γ-free set serves as an useful tool in identifying connected induced subgraphs H of G with $\gamma(H) \leq \gamma(G)$. The queens graph Q_n has vertex set V of order n^2(in $1 - 1$ correspondence with the n^2 cells of an $n \times n$ chessboard) where two vertices u and v are adjacent if and only if a queen at u can reach v in a single move. Hence $\gamma(Q_n)$ is the minimum number of queens to be placed on an $n \times n$ chessboard such that every cell is either occupied by a queen or attached by a queen. One of the most interesting open problems on $\gamma(Q_n)$ is the following:

Problem 2. Is $\gamma(Q_n) \leq \gamma(Q_{n+1})$?

Though the problem appears to be obviously true, no proof has yet been formed and for a solution to this problem a $ 100 price is offered by S.T. Hedetniemi ([3], pp 157). This problem is a special case of Problem 1.2.
We observe that if it can be proved that $V(Q_{n+1}) - V(Q_n)$ is a *cc*-γ-free set of Q_{n+1}, then we get an affirmative answer to the above problem. Also similar problems for other domination related parameters such as connected domination number, total domination number, independent domination number, super domination number and 2-domination number can be investigated.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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