Communications in Combinatorics and Optimization
Vol. 9, No. 3 (2024), pp. 567-578
https://doi.org/10.22049/cco.2023.28011.1503

CCO
Commun. Comb. Optim.

Research Article

A study on structure of codes over $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$

G. Karthick

Department of Mathematics, Presidency University, Bangalore, Karnataka, India
karthygowtham@gmail.com

Received: 19 September 2022; Accepted: 18 April 2023
Published Online: 25 April 2023

Abstract: We study $(1+2u+2v)$-constacyclic code over a semi-local ring $S = \mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$ with the condition $u^2 = 3u, v^2 = 3v$, and $uv = vu = 0$, we show that $(1+2u+2v)$-constacyclic code over S is equivalent to quasi-cyclic code over $\mathbb{Z}_4$ by using two new Gray maps from S to $\mathbb{Z}_4$. Also, for odd length n we have defined a generating set for constacyclic codes over S. Finally, we obtained some examples which are new to the data base [Database of $\mathbb{Z}_4$ codes [online], http://$\mathbb{Z}_4$ Codes.info(Accessed March 2, 2020)].

Keywords: Non-chain ring. Linear code. Non-chain ring. Gray map. Linear code

AMS Subject classification: 94B05, 94B15, 94B35, 94B60

1. Introduction

Cyclic codes have been well studied due to their algebraic structures. It has been playing a crucial role in its preferable applications. Pless et al. [13] discussed $\mathbb{Z}_4$ cyclic codes and proved the existence of idempotent generators for certain cyclic codes. In 2014, Yildiz et al [17] determined algebraic structures of codes over the ring $\mathbb{Z}_4 + u\mathbb{Z}_4$ and they obtained the basic facts about their generators with this they conducted a computer search and obtained many new linear codes over $\mathbb{Z}_4$. Later, Ashraf et al. [2] studied $(1+u)$-constacyclic codes over $\mathbb{Z}_4+u\mathbb{Z}_4$. In 2015 and 2018 Martinez-Moro et al. and Yilditz et al. studied linear codes and self-dual codes over $\mathbb{Z}_4[x]/\langle x^2 + 2x\rangle$ which is isomorphic to $\mathbb{Z}_4[x]/\langle x^2 - 1\rangle$ in [10, 18], respectively. Also, Yu et al. [19] defined new Gray maps over $\mathbb{Z}_4[u]/\langle u^2\rangle$ and obtained good binary codes are constructed using $(1+u)$ and Cengellenmis et al. [5] also studied constacyclic code over this ring. On the other hand, Shi et al. [14] studied $(1+2u)$-constacyclic codes over $\mathbb{Z}_4[u]/\langle u^2 - 1\rangle$ and they obtained new $\mathbb{Z}_4$ codes with better parameter. Ozen et al. [12] studied $(2+u)$ constacyclic code over $\mathbb{Z}_4[u]/\langle u^2 - 1\rangle$ and they obatined new $\mathbb{Z}_4$ codes with better parameter. These studies produced many significant linear codes to improve the

online database [Database of $\mathbb{Z}_4$ codes [online], http://$\mathbb{Z}_4$ Codes.info(Accessed March 2, 2020)]. In 2017, Ozen et al. [11] studied the cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$, where $u^3 = 0$ and determined their minimal spanning sets they have also obtained many new quarternary linear codes from the $\mathbb{Z}_4$-images of these codes. Recently, Islam et al. [8] and Islam and Prakash [9] discussed the $\mathbb{Z}_4$-images of constacyclic codes over $\mathbb{Z}_4[u]/\langle u^k\rangle$, and $\mathbb{Z}_4[u,v]/\langle u^2, v^2, uv - vu\rangle$, respectively.
On the other side, the codes over non-commutative rings was studied by, Boucher et al. [3] he introduced the skew cyclic (or θ-cyclic) code which is a generalized class of cyclic codes. Skew cyclic codes over arbitrary length was studied by Irfan et al. [15]. Later, skew cyclic and skew constacyclic codes over finite rings gained much attention of many mathematician [4, 6, 7, 16].
Inspired by the above results, this paper considers constacyclic codes over the non-chain finite commutative ring $S = \mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4,\ u^2 = 3u, v^2 = 3v$, and $uv = vu = 0$. The rest of this paper is organized as follows. Section 2 gives some preliminary results. Gray maps for $(1 + 2u + 2v)$-constacyclic codes are studied in Section 3. The structure of $(1 + 2u + 2v)$-constacyclic code and their generating polynomials are discussed in Section 4 with some examples in Section 5.

2. Preliminaries

Let $S = \mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4,\ u^2 = 3u, v^2 = 3v,$ and $uv = vu = 0$ be a commuative ring of order 64 with a unique maiximal ideal $\langle u, v, 2\rangle$, then the quotient ring $\frac{S}{\langle u,v,2\rangle}$ is isomorphic to $\mathbb{Z}_2$. Any element in the ring S can be uniquely written as $a + ub + vc$ where a, b and c are elements of $\mathbb{Z}_4$. A non-empty subset C of R^n is said to be a linear code of length n if C is an R-submodule of S^n. The elements of C are called codewords.
An element $a + ub + vc$ is said to be unit in S only if a is a unit element in S. Let α be a unit in S then we define α-constacyclic shift as folows

$$\phi_\alpha(c_0, c_1, \ldots, c_{n-1}) = (\alpha c_{n-1}, c_0, \ldots, c_{n-2}).$$

A code whose codewords satisfy this shift is called an α-constacyclic code. When $\alpha = 1$ then α-constacyclic is a cyclic code and when $\alpha = -1$ then α-constacyclic is a negacyclic code.
It is convineant to identify each code word of α-constacyclic code as a polynomial in $\frac{S[x]}{(x^n-\alpha)}$ through a linear map ϕ as given below

$$\phi : C \mapsto \frac{S[x]}{(x^n - \alpha)}, \quad \phi(c_0, c_1, \ldots, c_{n-1}) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}.$$

Then set of α-constacyclic code words in R^n can be seen as a polynomial collection over $\frac{S[x]}{(x^n-\alpha)}$. And it can be seen that each α-cyclic shift in C represent $xc(x)$ in codoamin and thus we have the following theorem.

Theorem 1. *Let C be a linear code of length n over S. Then C is a α-constacyclic over S if and only if C is an ideal of $\frac{S[x]}{(x^n-\alpha)}$.*

Let $r = (r_1, r_2, \ldots, r_m) \in \mathbb{Z}_4^{mn}$ where $r_i \in \mathbb{Z}_4^n$ for $i = \{1, 2, \cdots, m\}$ then we define a map $\upsilon : \mathbb{Z}_4^{mn} \to \mathbb{Z}_4^{mn}$, $\upsilon(r_1, r_2, \ldots, r_m) = (\sigma(r_1), \sigma(r_2), \ldots, \sigma(r_m))$ where σ is cyclic shift operator defined above if a code C is closed under this shift operator then we call it as quasi cyclic code of index m.

Definition 1. Let C be a linear code of length n over $\mathbb{Z}_4$. Then C is said to be r-cyclic code if $\sigma^r(C) = C$, where σ is the cyclic shift operator. Note that for $r \geq 2$, every cyclic code is r-cyclic but not conversely.

Note: From now α represent the unit element $1 + 2u + 2v$.

3. Gray Maps over S and their Properties

In this section we define two different Gray maps and shown that the Gray images α-constacyclic code is cyclic and quasi cyclic code over $\mathbb{Z}_4$ where $\alpha = 1 + 2u + 2v$.

Definition 2. Let γ_1 be linear map defined from S to $\mathbb{Z}_4^2$,

$$\gamma_1(a + ub + vc) = (2a + 3b + 3c, 2a + b + c).$$

The Gray map γ_1 can be extended for length n. The Lee weight of $a \in \mathbb{Z}_4$ is defined as $\min(a, 4 - a)$ and is denoted as $w_L(a)$. For any element $r = (a + ub + vc) \in S$ we define the Lee weight of a code as $w_L(r) = w_L(\gamma_1(r))$. Then Lee distance of code C is $d_L(C) = min(w_L(c_i - c_j))$ where $c_i, c_j \in C$.

Lemma 1. *Let γ_1 be the gray map defined then it satisfies $\sigma\gamma_1(s) = \gamma_1\phi_\alpha(s)$ where σ represents the cyclic shift operator and s is an element in S^n.*

Proof. Let $s = s_0, s_1, \ldots, s_{n-1}$ where $s_i = a_i + ub_i + vc_i$. We have

$$\begin{aligned}
\sigma\gamma_1(s) &= \sigma\gamma_1(s_0, s_1, \ldots, s_{n-1}) \\
&= \sigma(2a_0 + 3b_0 + 3c, 2a_1 + 3b_1 + 3c_1, \ldots, 2a_{n-1} + 3b_{n-1} + 3c_{n-1}, 2a_0 \\
&\quad + b_0 + c_0, 2a_1 + b_1 + c_1, \ldots, 2a_{n-1} + b_{n-1} + c_{n-1}) \\
&= (2a_{n-1} + b_{n-1} + c_{n-1}, 2a_0 + 3b_0 + 3c, \ldots, 2a_{n-1} + 3b_{n-1} + 3c_{n-1}, 2a_0 \\
&\quad + b_0 + c_0, 2a_1 + b_1 + c_1, \ldots, 2a_{n-1} + b_{n-1} + c_{n-1}).
\end{aligned}$$

On the other hand

$$\begin{aligned}\gamma_1\phi_\alpha(s) &= \gamma_1\phi_\alpha(s_0, s_1, \ldots, s_{n-1})\\ &= \gamma_1(\alpha s_{n-1}, s_0, \ldots, s_{n-2})\\ &= \gamma_1(a_{n-1} + u(3b_{n-1} + 2a_{n-1}) + v(3c_{n-1} + 2a_{n-1}), a_0 + ub_0\\ &\quad +vc_0, \ldots, a_{n-2} + ub_{n-2} + vc_{n-2})\\ &= (2a_{n-1} + b_{n-1} + c_{n-1}, 2a_0 + 3b_0 + 3c, \ldots, 2a_{n-1} + 3b_{n-1} + 3c_{n-1}, 2a_0\\ &\quad +b_0 + c_0, 2a_1 + b_1 + c_1, \ldots, 2a_{n-1} + b_{n-1} + c_{n-1}).\end{aligned}$$

□

Theorem 2. *Let C be a α-constacyclic code then $\gamma_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Let C be a α-constacyclic code then it for each $a \in C$ we have $\phi_\alpha(a) \in C$. Thus by usingn Lemma 1 we have $\sigma\gamma_1(C) = \gamma_1\phi_\alpha(C) = \gamma_1(C)$, implies $\gamma_1(C)$ is a cyclic code of length $2n$ over S. □

Definition 3. Let $s = (s_0, s_1, \ldots, s_{n-1}) \in S^n$ where $s_i = a_i + ub_i + vc_i$ then define the permutation of Gray image γ_1 from S^n to $\mathbb{Z}_4^{2n}$ as γ_1^* given by

$$\begin{aligned}\gamma_1^*(s_0, s_1, \ldots, s_{n-1}) &= (2a_0 + 3b_0 + c_0, 2a_0 + b_0 + c_0, 2a_1 + 3b_1 + c_1, 2a_1 + b_1 + c_1, \ldots, 2a_{n-1}\\ &\quad +3b_{n-1} + 3c_{n-1}, 2a_{n-1} + b_{n-1} + c_{n-1}).\end{aligned}$$

Lemma 2. *Let γ_1^* be permutation Gray map then it satisfies $\gamma_1^*(\sigma)(s) = \sigma^2(\gamma_1^*)(s)$ where s is an element in S.*

Proof. Let $s = s_0, s_1, \ldots, s_{n-1}$ where $s_i = a_i + wb_i$. We have

$$\begin{aligned}\gamma_1^*(\sigma)(s) &= \gamma_1^*(\sigma)(s_0, s_1, \ldots, s_{n-1})\\ &= \gamma_1^*(s_{n-1}, s_0, \ldots, s_{n-2})\\ &= (2a_{n-1} + 3b_{n-1} + 3c_{n-1}, 2a_{n-1} + b_{n-1} + c_{n-1}, 2a_0 + 3b_0 + 3c_0, 2a_0 +\\ &\quad b_0 + c_0, \cdots, 2a_{n-2} + 3b_{n-2} + 3c_{n-1}, 2a_{n-2} + b_{n-2} + c_{n-2}).\end{aligned}$$

On the other side we have,

$$\begin{aligned}\sigma^2(\gamma_1^*)(s) &= \sigma^2(\gamma_1^*)(s_0, s_1, \ldots, s_{n-1})\\ &= \sigma^2(2a_0 + 3b_0 + c_0, 2a_0 + b_0 + c_0, 2a_1 + 3b_1 + c_1, 2a_1 + b_1 + c_1, \ldots, 2a_{n-1}\\ &\quad +3b_{n-1} + 3c_{n-1}, 2a_{n-1} + b_{n-1} + c_{n-1})\\ &= (2a_{n-1} + 3b_{n-1} + 3c_{n-1}, 2a_{n-1} + b_{n-1} + c_{n-1}, 2a_0 + 3b_0 + 3c_0, 2a_0\\ &\quad +b_0 + c_0, \ldots, 2a_{n-2} + 3b_{n-2} + 3c_{n-1}, 2a_{n-2} + b_{n-2} + c_{n-2}).\end{aligned}$$

□

Theorem 3. *If C be a cyclic code of length n then $\gamma_1^*(C)$ is a two cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Let C be a cyclic code of length n then it satisfies $\sigma(c) \in C$ for all $c \in C$. Using Lemma 2 we have $\gamma_1^*\sigma(C) = \gamma_1^*(C) = \sigma^2\gamma_1^*(C)$. Hence $\gamma_1^*(C)$ is a two cyclic code of length $2n$ over $\mathbb{Z}_4$. □

Definition 4. Let γ_2 be a linear map defined from S to $\mathbb{Z}_4^2$ by

$$\gamma_2(a + ub + vc) = (a + 2b + 2c, 2b + 2c, a).$$

The map γ_2 can be extended to length n. For any element $r = (a + ub + vc) \in S$ we define the Lee weight of a code as $w_L(r) = w_L(\gamma_2(r))$. Then Lee distance of code C is $d_L(C) = \min(w_L(c_i - c_j))$ where $c_i, c_j \in C$.

Lemma 3. *Let γ_2 be a gray map defined in Definition 4 then it satisfies $\upsilon_3\gamma_2(s) = \gamma_2\phi_\alpha(s)$ for any $s \in S^n$.*

Proof. Let $s = (s_0, s_1, \ldots, s_{n-1})$ where $s_i = a_i + ub_i + vc_i$. Then we have

$$\begin{aligned}
\upsilon_3\gamma_2(s) &= \upsilon_3\gamma_2(s_0, s_1, \ldots, s_{n-1}) \\
&= \upsilon_3(a_0 + 2b_0 + 2c_0, a_1 + 2b_1 + 2c_1, \ldots, a_{n-1} + 2b_{n-1} + 2c_{n-1}, 2b_0 + 2c_0, 2b_1 \\
&\quad +2c_1, \cdots, 2b_{n-1} + 2c_{n-1}, a_0, a_1, \ldots, a_{n-1}) \\
&= (a_{n-1} + 2b_{n-1} + 2c_{n-1}, \ldots, a_{n-2} + 2b_{n-2} + 2c_{n-2}, 2b_{n-1} + 2c_{n-1}, 2b_0 \\
&\quad +2c_0, \ldots, 2b_{n-2} + 2c_{n-2}, a_{n-1}, a_0, \ldots, a_{n-2}).
\end{aligned}$$

Thus, on the other hand

$$\begin{aligned}
\gamma_2\phi_\alpha(s) &= (s_0, s_1, \ldots, s_{n-1}) \\
&= \gamma_2(\alpha s_{n-1}, s_0, \ldots, s_{n-2}) \\
&= \gamma_2(a_{n-1} + u(3b_{n-1} + 2a_{n-1}) + v(3c_{n-1} + 2a_{n-1}), a_0 + ub_0 + vc_0, \ldots, a_{n-2} \\
&\quad +ub_{n-2} + vc_{n-2}) \\
&= (a_{n-1} + 2b_{n-1} + 2c_{n-1}, a_0 + 2b_0 + 2c_0 \ldots, a_{n-2} + 2b_{n-2} \\
&\quad +2c_{n-2}, 2b_{n-1} + 2c_{n-1}, 2b_0 + 2c_0, \ldots, 2b_{n-2} + 2c_{n-2}, a_{n-1}, a_0, \ldots, a_{n-2}).
\end{aligned}$$

□

Hence, we have the following theorem.

Theorem 4. *Let C be a α-constacyclic code then $\delta_2(C)$ is a quasi cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Since C is a α-constacyclic code then $\phi_\alpha(s) \in C$ for all $s \in C$. Then by using 3 we have $\gamma_2\phi_\alpha(C) = \gamma_2(C) = \upsilon_3\gamma_2(C)$. Implies $\gamma_2(C)$ is a quasi cyclic code of length $2n$ with index 3. □

Definition 5. Let $s = (s_0, s_1, \ldots, s_{n-1}) \in S^n$ where $s_i = a_i + ub_i + vc_i$ then define permutation of the Gray image γ_2 from S^n to $\mathbb{Z}_4^{2n}$ as γ_2^* given by

$$\begin{aligned}\gamma_2^*(s_0, s_1, \ldots, s_{n-1}) &= (a_0 + 2b_0 + 2c_0, 2b_0 + 2c_0, a_0, a_1 + 2b_1 + 2c_1, 2b_1 + 2c_1, a_1, \cdots, a_{n-1} \\ &\quad +2b_{n-1} + 2c_{n-1}, 2b_{n-1} + 2c_{n-1}, a_{n-1}).\end{aligned}$$

Lemma 4. *Let γ_2^* be permutation Gray map then it satisfies $\gamma_2^*(\sigma)(s) = \sigma^3(\gamma_2^*)(s)$ where s is an element in S.*

Proof. Let $s = s_0, s_1, \ldots, s_{n-1}$ where $s_i = a_i + wb_i$. We have

$$\begin{aligned}\gamma_2^*(\sigma)(s) &= \gamma_2^*(\sigma)(s_0, s_1, \ldots, s_{n-1}) \\ &= \gamma_2^*(s_{n-1}, s_0, \ldots, s_{n-2}) \\ &= (a_{n-1} + 2b_{n-1} + 2c_{n-1}, 2b_{n-1} + 2c_{n-1}, a_{n-1}a_0 + 2b_0 + 2c_0, 2b_0 \\ &\quad +2c_0, a_0, \ldots, a_{n-1} + 2b_{n-1} + 2c_{n-1}, 2b_{n-1} + 2c_{n-1}, a_{n-1}).\end{aligned}$$

On the other side we have

$$\begin{aligned}\sigma^3(\gamma_2^*)(s) &= \sigma^2(\gamma_2^*)(s_0, s_1, \ldots, s_{n-1}) \\ &= \sigma^3(a_0 + 2b_0 + 2c_0, 2b_0 + 2c_0, a_0, a_1 + 2b_1 + 2c_1, 2b_1 + 2c_1, a_1, \ldots, a_{n-1} \\ &\quad +2b_{n-1} + 2c_{n-1}, 2b_{n-1} + 2c_{n-1}, a_{n-1}) \\ &= (a_{n-1} + 2b_{n-1} + 2c_{n-1}, 2b_{n-1} + 2c_{n-1}, a_{n-1}a_0 + 2b_0 + 2c_0, 2b_0 \\ &\quad +2c_0, a_0, \ldots, a_{n-1} + 2b_{n-1} + 2c_{n-1}, 2b_{n-1} + 2c_{n-1}, a_{n-1}).\end{aligned}$$

□

Theorem 5. *If C be a cyclic code of length n then $\gamma_2^*(C)$ is a three cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Proof is similar to the proof of Theorem 3. □

Corollary 1. *Let C be a linear code of odd length n over S. Then C is a cyclic code if and only if $\varphi(C)$ is an α-constacyclic code where $\varphi : S^n \longrightarrow S^n$ defined by $\varphi(c_0, c_1, \ldots, c_{n-1}) = (c_0, \alpha c_1, \ldots, \alpha^{n-2}c_{n-2}, \alpha^{n-1}c_{n-1})$.*

Definition 6. [12] Let n be an odd positive integer and $\xi = (1, n+1)(3, n+3)\cdots(2i+1, n+2i+1)\cdots(n-2, 2n-2)$ a permutation of $\{0, 1, \ldots, 2n-1\}$. Then Nechaev's permutation π is defined by $\pi(c_0, c_1, \ldots, c_{2n-1}) = (c_{\xi(0)}, c_{\xi(1)}, \ldots, c_{\xi(2n-1)})$.

Lemma 5. *Let γ_1 be the Gray map defined in Definition 2. Then $\gamma_1\varphi = \pi\gamma_1$ where π is Nechaev's permutation and φ is the map defined in Corollary 1.*

Proof. Let $s_i = a_i + ub_i + vc_i \in S$ for $0 \leq i \leq n-1$. Then $s = (s_0, s_1, \ldots, s_{n-1}) \in S^n$ and

$$\begin{aligned}
\gamma_1\varphi(z) &= \gamma_1\varphi(s_0, s_1, \ldots, s_{n-1}) \\
&= \gamma_1(s_0, \alpha s_1, \ldots, \alpha^{n-1}s_{n-1}) \\
&= (2a_0 + 3b_0 + 3c_0, 2a_1 + b_1 + c_1, \ldots, 2a_{n-1} + 3b_{n-1} + 3c_{n-1}, 3b_0 + c_0, \\
&\quad 2a_1 + b_1 + c_1, \ldots, 3b_{n-1} + c_{n-1}).
\end{aligned}$$

Further,

$$\begin{aligned}
\pi\gamma_1(z) &= \pi\gamma_1(z_0, z_1, \ldots, z_{n-1}) \\
&= \pi(2a_0 + 3b_0 + 3c, 2a_1 + 3b_1 + 3c_1, \cdots, 2a_{n-1} + 3b_{n-1} + 3c_{n-1}, 2a_0 \\
&\quad + b_0 + c_0, 2a_1 + b_1 + c_1, \cdots, 2a_{n-1} + b_{n-1} + c_{n-1}) \\
&= (2a_0 + 3b_0 + 3c_0, 2a_1 + b_1 + c_1, \ldots, 2a_{n-1} + 3b_{n-1} + 3c_{n-1}, 3b_0 + c_0, \\
&\quad 2a_1 + b_1 + c_1, \ldots, 3b_{n-1} + c_{n-1}).
\end{aligned}$$

and therefore $\gamma_1\varphi = \pi\gamma_1$. □

Theorem 6. *For a cyclic code C of odd length n over R, let $T = \gamma_1(C)$. Then $\pi(T)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Let C be a cyclic code and $T = \delta_1(C)$. Then by Lemma 5, $\pi\gamma_1(C) = \pi(T) = \psi_1\varphi(C)$. From Corollary 1, $\varphi(C)$ is an α-constacyclic code. Hence, by Theorem 2, $\delta_1\varphi(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$, and thus $\pi(T)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. □

Lemma 6. *Let γ_2 be the Gray map defined in Definition 4. Then $\gamma_2\varphi = \pi\gamma_2$ where π is Nechaev's permutation and φ is the map defined in Corollary 1.*

Proof. The proof is similar to that of Lemma 5 and so is omitted. □

Theorem 7. *For a cyclic code C of odd length n over R, let $T = \gamma_2(C)$. Then $\pi(T)$ is a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_4$.*

Proof. The proof is similar to that of Theorem 6 and so is omitted. □

4. Structure of $(1 + 2u + 2v)$-constacyclic code

In this section we study the structure of cyclic code and α-constacyclic code over S. Let $e_1 = (1+u+v), e_2 = -u$ and $e_3 = -v$, it satisfies $e_ie_j = 0(i \neq j),\ e_i^2 = e_i$ and $e_1 + e_2 + e_3 = 1$. Thus, any element in S can be uniquely expressed as $re_1 + se_2 + te_3$ where $r = a, s = (3b + a)$ and $t = (3c + a)$ are elements in $\mathbb{Z}_4$.
Let A, B a non empty set then define $A \oplus B = \{a + b \mid a \in A, b \in B\}$ and $A \otimes B = \{a, b \mid a \in A, b \in B\}$. Let C be a linear code over S, $C_1 = \{a \mid ae_1 + be_2 + ce_3 \in C\}, C_2 = \{b \mid ae_1 + be_2 + ce_3 \in C\}$ and $C_3 = \{c \mid ae_1 + be_2 + ce_3 \in C\}$. Thus, $C = \bigoplus_{i=1}^{3} C_i$. Note that whenever C is linear in S then $C_i's$ are linear over $\mathbb{Z}_4$.

Theorem 8. *Let C be a linear code then C is cyclic code of length n over S if and only if C_1, C_2 and C_3 are cyclic code over $\mathbb{Z}_4$.*

Proof. Let $c = c_0, c_1, \ldots, c_{n-1} \in C$ where $c_i = e_1a_i + e_2b_i + e_3d_i$. Let C be cyclic over S then $\sigma(c) = \sigma(a)e_1 + \sigma(b)e_2 + \sigma(d)e_3 \in C$. Implies C_1, C_2 and C_3 are cyclic code over $\mathbb{Z}_4$.
Let C_1 be a cyclic codes. Then $\sigma(a) \in C_1$ implies $\sigma(a)e_1 + be_2 + de_3 \in C$ and so $e_1(\sigma(a)e_1 + be_2 + de_3) = \sigma(a)e_1 \in C$ for some $b \in C_2, d \in C_3$. In a similar way $\sigma(b)e_2 \in C, \sigma(d)e_3 \in C$, using linearity we have $\sigma(a)e_1 + \sigma(b)e_2 + \sigma(d)e_3 = \sigma(c) \in C$. Hence, C is cyclic over S. □

Lemma 7. *[1] Let C be a cyclic code of length n over $\mathbb{Z}_4$.*

1. *If n is odd then $\mathbb{Z}_4[x]/(x^n - 1)$ is a principal ideal ring and $C = (f(x), 2g(x)) = (f(x) + 2g(x))$ where $f(x)$ and $g(x)$ generate cyclic codes with $g(x)|f(x)|(x^n - 1) \mod 4$.*

Theorem 9. *Let C be a cyclic code of odd length n. Then there exist $g(x)$ such that $C = \langle g(x) \rangle$.*

Proof. Let C be a cyclic code. By Theorem 8 we have C_1, C_2 and C_3 are cyclic. Since C_1, C_2 and C_3 are cyclic by Lemma 7, $C_i = \langle g_i(x) \rangle$. Thus, given any element in e_iC_i we have $e_ia_i(x)g_i(x) \in e_iC_i$ for some $a_i(x) \in \mathbb{Z}_4[x]$. Then using the representation of C in S we have $\sum_{i=1}^{3} e_ia_i(x)g_i(x) \in C$. Multiply by e_i we get $\langle e_ig_i(x) \rangle \subseteq C$. Hence, $g(x) = e_1g_1(x) + e_2g_2(x) + e_3g_3(x)$ generates C. □

Theorem 10. *Let C be linear code then C is α-constacyclic code iff C_1 is cyclic, C_2 and C_3 are Negacyclic code of length n over $\mathbb{Z}_4$.*

Proof. First let C be α-constacyclic code over R. Let $a = (a_0, a_1, \ldots, a_{n-1}) \in C_1, b = (b_0, b_1, \ldots, b_{n-1}) \in C_2$ and $d = (d_0, d_1, \ldots, d_{n-1}) \in C_3$ then $ae_1 + be_2 + de_3 \in S$. Since C is α-constacyclic code,

$$\phi_\alpha(c_0, c_1, \ldots, c_{n-1}) = (\alpha c_{n-1}, c_0, \ldots, c_{n-1}).$$

Since, $(e_1 + e_2 + e_3)(1 + 2u + 2v) = e_1 - e_2 - e_3$. We have $\phi_{-1}(b) \in C_2, \phi_{-1}(b) \in C_3, \sigma(a) \in C_1$. Hence, C_1 is cyclic and C_2, C_3 are negacyclic codes.
Conversely, we assume that C_1 is cyclic code and C_2, C_3 are negacyclic code. Let $(c_0, c_1, \cdots, c_{n-1}) \in C$ where $c_i = e_1 a_i + e_2 b_i + e_3 c_i$. Since C_1 is cyclic and C_2, C_3 are negacyclic $(\phi_{-1}(b), \phi_{-1}(d)) \in (C_2, C_3)$ and $\sigma(a) \in C_1$, we have $\sigma(a)e_1 + \phi_{-1}(b) + \phi_{-1}(d) \in C$. That is, $(\alpha c_{n-1}, c_0, \ldots, c_{n-2}) \in C$. Hence, C is α-constacyclic code. □

Theorem 11. *Let n be an odd integer. Then the map $\tau : S[x]/\langle x^n - 1\rangle \longrightarrow S[x]/\langle x^n - \alpha\rangle$ defined by $\tau(f(x)) = f(\alpha x)$ is a ring isomorphism.*

Proof. Let $f(x) = g(x)$ in $S[x]/\langle x^n - 1\rangle$. Then $f(x) \equiv g(x) \bmod (x^n - 1)$. Replacing x by αx on both sides gives $f(\alpha x) - g(\alpha x) \equiv 0 \bmod (x^n \alpha^n - 1)$ which implies that $f(\alpha x) - g(\alpha x) \equiv 0 \bmod \alpha^n(x^n - \alpha)$ since $\alpha^n = \alpha$ for an odd integer n. Thus, $f(\alpha x) = g(\alpha x)$ in $R[x]/\langle x^n - \alpha\rangle$, so τ is an injective and well-defined map. Moreover, since $S[x]/\langle x^n - 1\rangle$ and $S[x]/\langle x^n - \alpha\rangle$ are finite rings with the same number of elements and τ is injective, then τ is surjective. Further, one can check that τ is a ring homomorphism. Hence, τ is a ring isomorphism. □

Corollary 2. *Let C be a linear code of odd length n over S. Then C is a cyclic code if and only if $\tau(C)$ is an α-constacyclic code over S.*

Theorem 12. *Let C be a α-constacyclic code over S then there exist a polynomial $g(x)$ such that $C = \langle g(x)\rangle$.*

Proof. The proof is similar to the proof of Theorem 9. □

Note: Let $a(x) + ub(x) + vc(x) = g_1(x)e_1 + g_2(x)e_2 + g_3(x)e_1(x)$. Then

$$a(x) = g_1(x), b(x) = g_1(x) + 3g_2(x), c(x) = g_1(x) + 3g_3(x).$$

Theorem 13. *Let γ_1 be the gray map defined and if $C = \langle g_1(x) + (g_1(x) + 3g_2(x))u + (g_1(x) + 3g_3(x))v\rangle$ be α-constacyclic code then $\gamma_1(C)$ is a cyclic code over $\mathbb{Z}_4$ and is generated by $(g_2(x) + x^n 3g_2(x)), (g_3(x) + x^n 3g_3(x))$.*

Proof. Let $r(x) \in C$ then there exist $h_i(x) \in \mathbb{Z}_4[x]$ such that

$$\begin{aligned} r(x) &= (h_1(x)g_1(x) + (h_1(x)g_1(x) + 3h_2(x)g_2(x))u \\ &\quad + (h_1(x)g_1(x) + 3h_3(x)g_3(x))v) \\ \gamma_1(r(x)) &= (h_2(x)g_2(x) + h_3(x)g_3(x), 3h_2(x)g_2(x) + 3h_3(x)g_3(x)) \\ &= h_2(x)(g_2(x), 3g_2(x)) + h_3(g_3(x), 3g_3(x)). \end{aligned}$$

Hence, $\gamma_1(r(x)) \in \frac{\mathbb{Z}_4}{(x^n-1)} \times \frac{\mathbb{Z}_4}{(x^n-1)}$, Using the fact $a, b \in \frac{\mathbb{Z}_4}{(x^n-1)} \times \frac{\mathbb{Z}_4}{(x^n-1)}$ implies $a + x^n b \in \frac{\mathbb{Z}_4}{(x^{2n}-1)}$, we have that $\gamma_1(C) = \langle (g_2(x) + x^n 3g_2(x)), (g_3(x) + x^n 3g_3(x))\rangle$ is a cyclic code over $\frac{\mathbb{Z}_4}{(x^{2n}-1)}$. □

The proof of the following theorem is similar to the proof of Theorem 13.

Theorem 14. *Let γ_2 be the gray map defined and if $C = \langle g_1(x) + (g1(x) + 3g_2(x))u + (g_1(x) + 3g_3(x))v\rangle$ be α-constacyclic code then $\gamma_2(C)$ is a quasicyclic code of length $3n$ over $\mathbb{Z}_4$ and is generated by $(g_1(x) + x^{2n}g_1(x)), (2g_2(x) + x^n g_2(x))$ and $(2g_3(x) + x^n g_3(x))$*

5. Examples

In this Section we have computed some codes using Magma Computational Algebra System. Some codes presented here is new to the Database [Database of $\mathbb{Z}_4$ codes [online], http://$\mathbb{Z}_4$ Codes.info(Accessed March 2, 2020)].

Example 1. Let C be a α-constacyclic code of length 7. Then by Theorem 10 C_1 is cyclic and C_2, C_3 are negacyclic codes over $\mathbb{Z}_4$. C is generated by $g(x) = e_1g_1(x) + e_2g_2(-x) + e_3g_3(-x)$ where, $g_1(x) = x^4 + x^3 + 3x^2 + 2x + 1$, $g_2(x) = x^4 + x^3 + 3x^2 + 2x + 1$, and $g_3(x) = x^3 + 3x^2 + 2x + 3$. So $\gamma_2(C)$ is a linear code of parameter $((21, 4^8 2^3, 3))$ and hence by Theorem 7, $\pi(\gamma_2(C))$ is quasi cyclic code.

Example 2. Let C be a α-constacyclic code of length 7 then by Theorem 10 C_1 is cyclic and C_2, C_3 are negacyclic codes over $\mathbb{Z}_4$. C is generated by $g(x) = e_1g_1(x) + e_2g_2(-x) + e_3g_3(-x)$ where $g_1(x) = x^4 + x^3 + 3x^2 + 2x + 1$, $g_2(x) = x^4 + x^3 + 3x^2 + 2x + 1$ and $g_3(x) = x^4 + x^3 + 3x^2 + 2x + 1$. So $\gamma_2(C)$ is a linear code of parameter $((21, 4^6 2^3, 4))$ and by Theorem 7, $\pi(\gamma_2(C))$ is quasi cyclic code.

Example 3. Let C be a cyclic code of length 15 then by Theorem 8 C_1, C_2, C_3 are cyclic codes over $\mathbb{Z}_4$. C is generated by $g(x) = e_1g_1(x) + e_2g_2(x) + e_3g_3(x)$ where $g_1(x) = x^6 + 2x^4 + x^3 + 3x^2 + x + 1$, $g_2(x) = x^4 + 3x^3 + 2x^2 + 1$ and $g_3(x) = x + 3$. So $\gamma_2(C)$ is a linear code of parameter $((45, 4^{18} 2^{13}, 3))$.

Example 4. Let C be a cyclic code of length 15 then by Theorem 8 C_1, C_2, C_3 are cyclic codes over $\mathbb{Z}_4$. C is generated by $g(x) = e_1g_1(x) + e_2g_2(x) + e_3g_3(x)$ where $g_1(x) = x^7 + 3x^6 + 2x^5 + 3x^4 + 2x^3 + 2x^2 + 3$, $g_2(x) = 2x^{10} + 2x^5 + 2$ and $g_3(x) = 2x^6 + 2x^3 + 2x^2 + 2x + 2$. Thus $\gamma_2(C)$ is a linear code of parameter $((45, 4^{16} 2^{10}, 3))$.

In the below table we have computed some codes using Magma Computational Algebra System. (* represents the code is new in the Database [Database of $\mathbb{Z}_4$ codes [online], http://$\mathbb{Z}_4$ Codes.info(Accessed March 2, 2020)])

n	$g_1(x)$	$g_2(x)$	$g_3(x)$	$\gamma_1(C)$	$\gamma_2(C)$
9	$x^3 + 2x + 1$	$g_1(x)$	$g_1(x)$	$((18, 4^{12}2^4, 2))$	$((27, 4^{18}, 2^4, 2))^*$
7	$x^4 + x^3 + 3x^2 + 3$	$x^3 + 2x^2 + x + 3$	$g_2(x)$	$((14, 4^8 2^0, 3))$	$((21, 4^{11}2^6, 2))^*$
7	$x^4 + 3x^3 + 3x^2 + 3$	$x^4 + x^3 + 3x^2 + 2x + 1$	$g_2(x)$	$((14, 4^6 2^0, 4))$	$((21, 4^9 2^2, 3))^*$
7	$x^4 + 3x^3 + 3x^2 + 3$	$x^4 + x^3 + 3x^2 + 2x + 1$	$g_2(x)$	$((14, 4^6 2^0, 4))$	$((21, 4^9 2^2, 3))^*$
9	$x^8 + x^7 + 3x^6 + x^5 + x^4 + 3x^3 + x^2 + x + 3$	$x^7 + 3x^6 + x^4 + 3x^3 + 3x + 1$	$g_2(x)$	–	$((27, 4^6 2^2, 3))^*$
9	$x^8 + x^7 + 3x^6 + x^5 + x^4 + 3x^3 + x^2 + x + 3$	$x^3 + 2x + 1$	$g_2(x)$	–	$((27, 4^{14}2^4, 2))^*$

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] T. Abualrub and I. Siap, *Reversible cyclic codes over* $\mathbb{Z}_4$, Australas. J. Comb. **38** (2007), 195–206.

[2] M. Ashraf and G. Mohammad, $(1+u)$*-constacyclic codes over* $\mathbb{Z}_4 + u\mathbb{Z}_4$, arXiv:1504.03445v1. (2015).

[3] D. Boucher, W. Geiselmann, and F. Ulmer, *Skew-cyclic codes*, Appl. Algebra Eng. Comm. Compute. **18** (2007), no. 4, 379–389. https://doi.org/10.1007/s00200-007-0043-z.

[4] D. Boucher, P. Solé, and F. Ulmer, *Skew constacyclic codes over Galois rings*, Adv. Math. Commun. **2** (2008), no. 3, 273–292. http://doi.org/10.3934/amc.2008.2.273.

[5] Y. Cengellenmis, A. Dertli, and N. Aydın, *Some constacyclic codes over* $\mathbb{Z}_4[u]/\langle u^2\rangle$*, new Gray maps, and new quaternary codes*, Algebra Colloq. **25** (2018), no. 3, 369–376. https://doi.org/10.1142/S1005386718000263.

[6] J. Gao, F. Ma, and F. Fu, *Skew constacyclic codes over the ring* $\mathbb{F}_q + v\mathbb{F}_q$, Appl. Comput. Math. **6** (2017), no. 3, 286–295.

[7] F. Gursoy, I. Siap, and B. Yildiz, *Construction of skew cyclic codes over* $\mathbb{F}_q + v\mathbb{F}_q$, Adv. Math. Commun. **8** (2014), no. 3, 313–322. https://doi.org/10.3934/amc.2014.8.313.

[8] H. Islam, T. Bag, and O. Prakash, *A class of constacyclic codes over* $\mathbb{Z}_4[u]/\langle u^k\rangle$, J. Appl. Math. Comput. **60** (2019), no. 1,2, 237–251. https://doi.org/10.1007/s12190-018-1211-y.

[9] H. Islam and O. Prakash, *A class of constacyclic codes over the ring* $\mathbb{Z}_4[u,v]/\langle u^2, v^2, uv-vu\rangle$ *and their Gray images*, Filomat **33** (2019), no. 8, 2237–2248.

[10] E. Martinez-Moro, S. Szabo, and B. Yildiz, *Linear codes over* $\mathbb{Z}_4[x]/\langle x^2+2x\rangle$, Int. J. Inf. Coding Theory **3** (2015), no. 1, 78–96.

[11] M. Özen, N.T. Özzaim, and N. Aydin, *Cyclic codes over* $\mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$, Turkish J. Math. **41** (2017), no. 5, 1235–1247. http://doi.org/10.3906/mat-1602-35.

[12] M. Özen, F.Z. Uzekmek, N. Aydin, and N. Özzaim, *Cyclic and some constacyclic codes over the ring* $\mathbb{Z}_4[u]/\langle u^2-1\rangle$, Finite Fields Appl. **38** (2016), 27–39.

[13] V.S. Pless and Z. Qian, *Cyclic codes and quadratic residue codes over* $\mathbb{Z}_4$, IEEE Trans. Inform. Theory **42** (1996), no. 5, 1594–1600.

https://doi.org/10.1109/18.532906.

[14] M. Shi, L. Qian, L. Sok, N. Aydin, and P. Solé, *On constacyclic codes over* $\mathbb{Z}_4[u]/\langle u^2 - 1\rangle$ *and their Gray images*, Finite Fields Appl. **45** (2017), 86–95.

[15] I. Siap, T. Abualrub, N. Aydin, and P. Seneviratne, *Skew cyclic codes of arbitrary length*, Int. J. Inf. Coding Theory **2** (2011), no. 1, 10–20. http://doi.org/10.1504/IJICOT.2011.044674.

[16] T. Yao, M. Shi, and P. Solé, *Skew cyclic codes over* $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, J. Algebra Comb. Discrete Struct. Appl. **2** (2015), no. 3, 163–168.

[17] B. Yildiz and N. Aydin, *On cyclic codes over* $\mathbb{Z}_4 + u\mathbb{Z}_4$ *and their* $\mathbb{Z}_4$*–images*, Int. J. Inf. Coding Theory **2** (2014), no. 4, 226–237. https://doi.org/10.1504/IJICOT.2014.066107.

[18] B. Yildiz and A. Kaya, *Self-dual codes over* $\mathbb{Z}_4[x]/\langle x^2 + 2x\rangle$ *and the* $\mathbb{Z}_4$*-images*, Int. J. Inf. Coding Theory **5** (2018), no. 2, 142–154.

[19] H. Yu, Y. Wang, and M. Shi, $(1+u)$*-constacyclic codes over* $\mathbb{Z}_4 + u\mathbb{Z}_4$, SpringerPlus **5** (2016), no. 1, Artice number 1325.