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Research Article

Algorithmic complexity of triple Roman dominating functions on graphs

Abolfazl Poureidi* **and Jafar Fathali**†

Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran
*a.poureidi@shahroodut.ac.ir
†fathali@shahroodut.ac.ir

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Abstract: Given a graph $G = (V, E)$, a function $f : V \to \{0, 1, 2, 3, 4\}$ is a triple Roman dominating function (TRDF) of G, for each vertex $v \in V$, (i) if $f(v) = 0$, then v must have either one neighbour in V_4, or either two neighbours in $V_2 \cup V_3$ (one neighbour in V_3) or either three neighbours in V_2, (ii) if $f(v) = 1$, then v must have either one neighbour in $V_3 \cup V_4$ or either two neighbours in V_2, and if $f(v) = 2$, then v must have one neighbour in $V_2 \cup V_3 \cup V_4$. The triple Roman domination number of G is the minimum weight of an TRDF f of G, where the weight of f is $\sum_{v \in V} f(v)$. The triple Roman domination problem is to compute the triple Roman domination number of a given graph. In this paper, we study the triple Roman domination problem. We show that the problem is NP-complete for the star convex bipartite and the comb convex bipartite graphs and is APX-complete for graphs of degree at most 4. We propose a linear-time algorithm for computing the triple Roman domination number of proper interval graphs. We also give an $(2H(\Delta(G) + 1) - 1)$-approximation algorithm for solving the problem for any graph G, where $\Delta(G)$ is the maximum degree of G and $H(d)$ denotes the first d terms of the harmonic series. In addition, we prove that for any $\varepsilon > 0$ there is no $(1/4 - \varepsilon) \ln |V|$-approximation polynomial-time algorithm for solving the problem on bipartite and split graphs, unless NP $\subseteq$ DTIME $(|V|^{O(\log \log |V|)})$.

Keywords: Triple Roman domination, Approximation algorithm, NP-complete, Proper interval graph, APX-complete

AMS Subject classification: 05C69

1. Introduction

Let $G = (V, E)$ be a graph such that V denotes the vertex set of G and E denotes the edge set of G. Let $N_G(v) = \{u \in V : uv \in E\}$, $N_G[v] = N_G(v) \cup \{v\}$ and $\deg_G(v) = |N_G(v)|$. A *pendant vertex* is a vertex with degree one. The *maximum degree* of a graph G, denoted by $\Delta(G)$, is $\Delta(G) = \max\{\deg_G(v) : v \in V\}$. A graph

* *Corresponding Author*

is *complete* if there is an edge between any pair of its vertices. The *induced subgraph* $G[S]$ for any $S \subseteq V$ is the graph whose vertex set is S and edge set consists of all edges in E that have both endpoints in S. A *clique* of G is a subset $S \subseteq V$ such that $G[S]$ is a complete graph. A graph $G = (V, E)$ is an *intersection graph* for a family of nonempty sets $\mathcal{F}$ if each vertex in V is corresponding to a set in $\mathcal{F}$ and two vertices are adjacent in G if and only if the intersection of their corresponding sets in $\mathcal{F}$ is nonempty. An *interval graph* $G = (V, E)$ is an intersection graph for a family of intervals on the real line. A *proper interval graph* is an interval graph in which no interval properly contains another. A *tree* is a connected graph with no cycles. A tree $T = (V, E)$ is called a *star* if $|V| = 2$ or $|V| \geq 3$ and T contains exactly one vertex that is not pendant and is called the *central vertex* of the star. A *path* is a tree with exactly two pendant vertex and a *comb graph* is a tree that is obtained by attaching a pendant vertex to each vertex of a path. A graph $G = (V, E)$ is called a *bipartite graph* if V can be partitioned into two subsets X and Y such that each edge in E has one end in X and one end in Y, denoted by $G = (X, Y, E)$. Let $G = (X, Y, E)$ be a bipartite graph. The graph G is a *tree convex bipartite* graph [12] if there is a tree $T = (X, F)$ such that the induced subgraph $T[N_G(v)]$ is connected for each vertex $v \in Y$. When T is a star (resp., comb), then G is a *star (resp., comb) convex bipartite* graph. Let $H(d)$ denote the first d terms of the harmonic series, that is, $H(d) = \sum_{i=1}^{d} 1/i$. Note that $H(d) \leq \ln(d) + 1$.
Given a graph $G = (V, E)$ and a function f from V to $\{0, 1, \ldots, t\}$, where $t > 0$ is an integer, the *weight* of f, denoted by $w(f)$, is equal to $\sum_{v \in V} f(v)$. We denote f by $(V_0, V_1, \ldots, V_k)$, where $V_i = \{v \in V : f(v) = i\}$ for all $0 \leq i \leq t$. A function $f : V \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) of G, if each vertex $v \in V$ with $f(v) = 0$ is adjacent to a vertex $u \in D$ with $f(u) = 2$. The *Roman domination number* of G is the minimum weight of an RDF f of G. Beeler et al. [6] initiated the study of double Roman dominating functions, a stronger version of Roman domination functions. A *double Roman dominating function* (DRDF) of G is a function $f : V \to \{0, 1, 2, 3\}$ such that for each $v \in V$:

1. with $f(v) = 0$, there is a vertex $u \in N_G(v)$ with $f(u) = 3$ or there are vertices $x, y \in N_G(v)$ with $f(x) = f(y) = 2$, and
2. with $f(v) = 1$, there is a vertex $u \in N_G(v)$ with $f(u) > 1$.

The *double Roman domination number* of G is the minimum weight of an DRDF f of G.
The double Roman domination of graphs has been studied in the literature, for example [16, 17]. In 2019, Abdollahzadeh Ahangar et al. [1] introduced a generalization of the DRDFs in which any undefended place could be defended from a sudden attack with, at least, k legions without leaving any neighboring strong-city without military forces.
We use the notation used in [1]. Let $G = (V, E)$ be a graph and let $f : V \to \{0, 1, \ldots, k + 1\}$ for a given positive integer k. Given a vertex $v \in V$, the *active neighbourhood* of v, denoted by $AN(v)$, is the set of vertices $w \in N_G(v)$ such that

$f(w) \geq 1$ and let $AN[v] = AN(v) \cup \{v\}$. The function f is a $[k]$-RDF, if for each vertex $v \in V$ with $f(v) < k$,

$$f(AN[v]) \geq |AN(v)| + k.$$

Denote the minimum weight of an $[k]$-RDF of G by $\gamma_{[kR]}(G)$. Note that for $k \in \{1, 2\}$ the $[k]$-RDF definition matches that of the RDF and DRDF. Authors [1] focused their attention to the *triple Roman domination number* ($k = 3$) case, so that for any vertex $v \in V$ with $f(v) < 3$, it must happen that $f(AN[v]) \geq |AN(v)| + 3$. More precisely, for each vertex $v \in V$, the following conditions hold.

1. If $h(v) = 0$, then v must have either one neighbour in V_4, or either two neighbours in $V_2 \cup V_3$ (one neighbour in V_3) or either three neighbours in V_2.
2. If $h(v) = 1$, then v must have either one neighbour in $V_3 \cup V_4$ or either two neighbours in V_2.
3. If $h(v) = 2$, then v must have one neighbour in $V_2 \cup V_3 \cup V_4$.

The *triple Roman domination problem* is to compute the *triple Roman domination number*, the minimum weight of a *triple Roman dominating function* (TRDF), of a given graph. Authors [1] proved that the triple Roman domination problem is NP-complete for chordal graphs and bipartite graphs. Moreover, they showed that it is possible to compute the triple Roman domination number of bounded clique-width graphs in linear-time. Triple Roman domination has been studied by several authors [2, 4, 9, 10, 14].
The organization of the paper as follows. In Section 2, we prove that the triple Roman domination problem is NP-complete even for the star convex bipartite graphs and the comb convex bipartite graphs. In Section 3, we propose a linear-time algorithm for computing the triple Roman domination number of proper interval graphs. In Section 4, we prove that for any $\varepsilon > 0$ there is no $(1/4 - \varepsilon) \ln |V|$-approximation polynomial-time algorithm for solving the triple Roman domination problem on bipartite and split graphs, unless NP $\subseteq$ DTIME $(|V|^{O(\log \log |V|)})$. In Section 5, we first give an $(2H(\Delta(G) + 1) - 1)$–approximation algorithm for computing the triple Roman domination number of graphs. Finally, APX-completeness of the triple Roman domination problem for graphs of degree at most 4 is proven.

2. NP-complete results

In this section, we show that the decision version of triple Roman domination problem is NP-complete even when restricted to the star convex bipartite graphs and the comb convex bipartite graphs. For this purpose, we present polynomial-time reductions from a well-known NP-complete problem, 3-SAT [11], to the triple Roman domination problem. The 3-SAT problem and the decision version of triple Roman domination problem are defined as follows:

3-SAT
Instance: A boolean formula Φ in 3-conjunctive normal form.
Question: Is Φ satisfiable?

Let $C = \{c_1, \cdots, c_l\}$ be a set of $l \geq 1$ clauses, let $X = \{x_1, \cdots, x_k\}$ be a set of $k \geq 3$ variables and let $[1, t] = \{1, 2, \ldots, t\}$, where t is a positive integer. $\Phi = \{C, X\}$ is called an instance of 3-SAT if the clause c_j, $j \in [1, l]$, is of the form $c_j = \{y_{1j}, y_{2j}, y_{3j}\}$ such that each of y_{1j}, y_{2j} and y_{3j} is either a variable or the negation of a variable in X.

Triple Roman Domination (TRD)
Instance: A graph G and a positive integer t.
Question: Is there an TRDF f of G with $w(f) \leq t$?

Theorem 1. *The TRD problem is NP-complete even for the star convex bipartite graphs.*

Proof. Clearly, the TRD problem is in NP because for a given graph G, a positive integer t and a function f on G we can check in polynomial-time whether f is an TRDF of G with $w(f) \leq t$. In the rest of the proof, we transfer an instance $\Phi = \{C, X\}$ of the 3-SAT problem to an instance $(G_\Phi, 8k)$ of the TRD problem. Let $i \in [1, k]$ and $j \in [1, l]$.

- Add a path $u_i^f u_i u_i^t$ such that u_i is adjacent to new pendants $b_i^1, b_i^2, b_i^3, b_i^4$ and both u_i^f and u_i^t are adjacent to new vertices $a_i^1, a_i^2, a_i^3, a_i^4$ for each $x_i \in X$.
- Add a vertex z_j for each $c_j \in C$.
- Add an edge $u_i^t z_j$ if $x_i \in c_j$ for each $c_j \in C$.
- Add an edge $u_i^f z_j$ if $\neg x_i \in c_j$, where $\neg x_i$ is the negation of x_i, for each $c_j \in C$.
- Add a new vertex o such that is adjacent to both u_i^t and u_i^f for each $i \in [1, k]$.

Let G_Φ be the resulting graph. See Figure 1(a). The graph $G_\Phi = (A, B, E)$ is a star convex bipartite graph with an associated star tree $T = (A, F)$, see Figure 1(b), where $A = \{o, z_j, u_i, a_i^1, a_i^2, a_i^3, a_i^4 : i \in [1, k], j \in [1, l]\}$, $B = \{u_i^f, u_i^t, b_i^1, b_i^2, b_i^3, b_i^4 : i \in [1, k]\}$ and $F = \{oy : y \in A \setminus \{o\}\}$.

Claim 1. The boolean formula Φ is satisfiable if and only if there is an TRDF f on G_Φ with $w(f) \leq 8k$.

Proof of Claim. Assume that Φ is satisfiable. Let T be a truth assignment for variables in X for which Φ evaluates to *true*. We construct sets V_0 and V_4 on the vertex set of G_Φ as follows. Initialize V_0 to be $\{o, z_j, a_i^1, \ldots, a_i^4, b_i^1, \ldots, b_i^4 : i \in [1, k], j \in [1, l]\}$

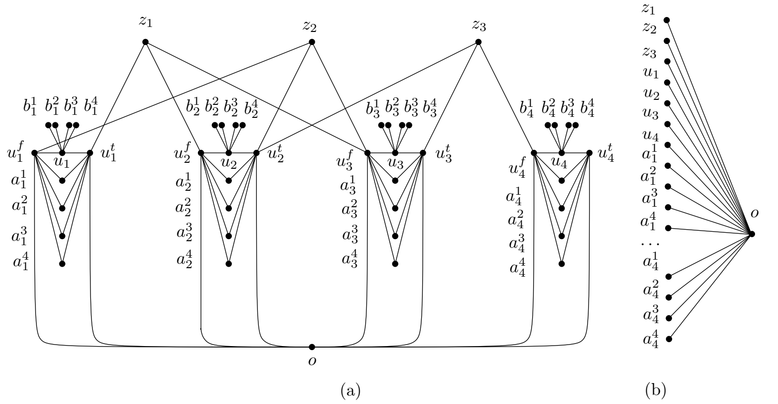

Figure 1. (a) Constructing a star convex bipartite graph G_Φ from a given instance $\Phi = \{C, X\}$ of the 3-SAT problem, where $X = \{x_1, x_2, x_3, x_4\}$, $C = \{c_1, c_2, c_3\}$, $c_1 = \{x_1, \neg x_2, \neg x_3\}$, $c_2 = \{\neg x_1, x_2, \neg x_3\}$ and $c_3 = \{x_2, x_3, \neg x_4\}$ and (b) an associated star tree with G_Φ.

and V_4 to be $\{u_i : i \in [1, k]\}$. If T assigns the value *false* (resp., *true*) to x_i, then we add the vertex u_i^f (resp., u_i^t) to V_4 and u_i^t (resp., u_i^f) to V_0. Function $f = (V_0, \emptyset, \emptyset, \emptyset, V_4)$ is an TRDF on G_Φ with $w(f) = 8k$.

Conversely, let $f = (V_0, V_1, V_2, V_3, V_4)$ be an TRDF on G_Φ with $w(f) \leq 8k$. We fix indices i and j, where $1 \leq i \leq k$ and $1 \leq j \leq l$. It gets that $f(u_i) + \sum_{s=1}^{4} f(b_i^s) \geq 4$ and $f(u_i^f) + f(u_i^t) + \sum_{s=1}^{4} f(a_i^s) \geq 4$ and so $S_i = f(u_i^f) + f(u_i) + f(u_i^t) + \sum_{s=1}^{4}(f(a_i^s) + f(b_i^s)) \geq 8$. Since f is an TRDF on G_Φ with $w(f) \leq 8k$, it obtains that $S_i = 8$ and so $f(o) = f(z_j) = 0$. This holds only when $\sum_{s=1}^{4}(f(a_i^s) + f(b_i^s)) = 0$, $f(u_i) = 4$ and either $f(u_i^f) = 4$ and $f(u_i^t) = 0$ or $f(u_i^f) = 0$ and $f(u_i^t) = 4$. If $f(u_i^f) = 4$ and $f(u_i^t) = 0$ (resp., $f(u_i^f) = 0$ and $f(u_i^t) = 4$), then we assign the value *false* (resp., *true*) to the variable x_i. We claim that this assignment satisfies Φ. Assume $c_j = \{y_{1j}, y_{2j}, y_{3j}\}$. By constructing G_Φ, for each $s \in \{1, 2, 3\}$, if $y_{sj} = x_i$, for some $1 \leq i \leq k$, then z_j is adjacent to u_i^t and otherwise, adjacent to u_i^f. Since $f(z_j) = f(o) = 0$ and $|N_{G_\Phi}(z_j)| = 3$, $f(y) = 4$ for some $y \in N_{G_\Phi}(z_j)$. Assume without loss of generality that the vertex y is corresponding to y_{1j} and $y_{1j} \in \{x_i, \neg x_i\}$, for some $i \in [1, k]$, where $\neg x_i$ is the negation of x_i. If $y_{1j} = \neg x_i$ (resp., $y_{1j} = x_i$), then $f(u_i^f) = 4$ and $f(u_i^t) = 0$ (resp., $f(u_i^f) = 0$ and $f(u_i^t) = 4$) and so x_i has the value *false* (resp., *true*). It causes to satisfy the clause c_j and so the boolean formula Φ is satisfiable. This completes the proof of the claim. $\lhd$

We can compute G_Φ in polynomial time with respect to the size of $|X|$ and $|C|$. This completes the proof of the theorem. □

Theorem 2. *The TRD problem is NP-complete even for the comb convex bipartite graphs.*

Proof. We transfer an instance $\Phi = \{C, X\}$ of the 3-SAT problem to an instance $(H_\Phi, 8k)$ of the TRD problem as follows. Let $i \in [1, k]$ and $j \in [1, l]$.

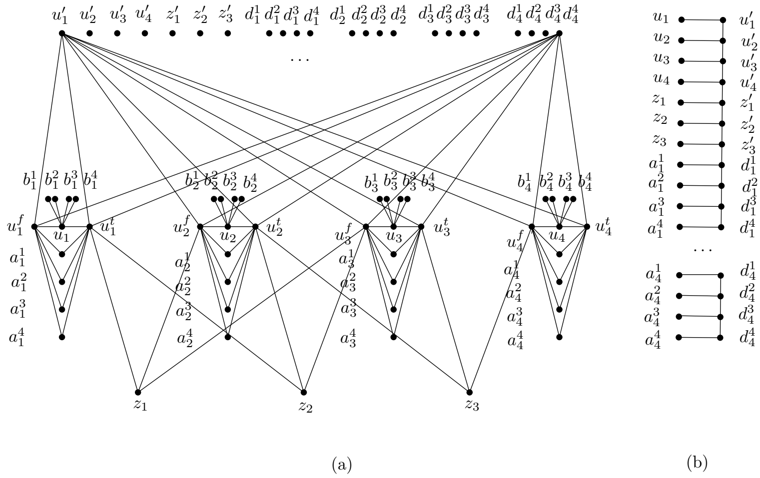

Figure 2. (a) Constructing a comb convex bipartite graph H_Φ from a given instance $\Phi = \{C, X\}$ of the 3-SAT problem, where $X = \{x_1, x_2, x_3, x_4\}$, $C = \{c_1, c_2, c_3\}$, $c_1 = \{x_1, \neg x_2, \neg x_3\}$, $c_2 = \{\neg x_1, x_2, \neg x_3\}$ and $c_3 = \{x_2, x_3, \neg x_4\}$, (b) an associated comb tree with H_Φ. Note that some edges of H_Φ are not drawn.

- Add a path $u_i^f u_i u_i^t$ such that u_i is adjacent to new pendants $b_i^1, b_i^2, b_i^3, b_i^4$ and both u_i^f and u_i^t are adjacent to new vertices $a_i^1, a_i^2, a_i^3, a_i^4$ for each $x_i \in X$.
- Add a vertex z_j for each $c_j \in C$.
- Add an edge $u_i^t z_j$ if $x_i \in c_j$ for each $c_j \in C$.
- Add an edge $u_i^f z_j$ if $\neg x_i \in c_j$, where $\neg x_i$ is the negation of x_i, for each $c_j \in C$.
- Add new vertices $u_i', z_j', d_i^1, d_i^2, d_i^3, d_i^4$ for each $i \in [1, k]$ and $j \in [1, l]$ such that each of these vertices is adjacent to u_i^f and u_i^t for all $i \in [1, k]$.

Let H_Φ be the resulting graph, see Figure 2(a), and let $A = \{z_j, z_j', u_i, u_i', a_i^1, a_i^2, a_i^3, a_i^4, d_i^1, d_i^2, d_i^3, d_i^4 : i \in [1, k], j \in [1, l]\}$ and $B = \{u_i^f, u_i^t, b_i^1, b_i^2, b_i^3, b_i^4 : i \in [1, k]\}$. The graph $H_\Phi = (A, B, E)$ is a comb convex bipartite graph with an associated comb tree $T = (A, F)$, see Figure 2(b), where T includes the path $u_1' \dots u_k' z_1' \dots z_l' d_1^1 d_1^2 d_1^3 d_1^4 \dots d_k^1 d_k^2 d_k^3 d_k^4$ such that u_i' is adjacent to u_i, z_j' is adjacent to z_j, and d_i^s is adjacent to a_i^s for all $i \in [1, k]$, $j \in [1, l]$ and $s \in [1, 4]$. Similar to Claim 1, we can obtain the following result.

Claim 2. The boolean formula Φ is satisfiable if and only if there is an TRDF f on H_Φ such that $w(f) \leq 8k$.

Recall that the TRD problem is in NP. We can compute H_Φ in polynomial time with respect to the size of $|X|$ and $|C|$. This completes the proof of the theorem. □

Algorithm 3.1: TRDNPIG$(G, 1, \ldots, n)$

Input: A proper interval graph $G = (V, E)$ with $|V| = n$ such that $(1, \ldots, n)$ is a consecutive ordering of vertices in V.

Output: The triple Roman domination number of G.

Compute $\texttt{MIN}(1), \ldots, \texttt{MIN}(n)$;
$\gamma^0_{[3R]}(1) \leftarrow \infty$; $\gamma^3_{[3R]}(1) \leftarrow 3$; $\gamma^4_{[3R]}(1) \leftarrow 4$; $i \leftarrow 1$;
while $i < n$ **do**
  $i \leftarrow i + 1$;
  $v \leftarrow \texttt{MIN}(i)$;
  $\gamma^0_{[3R]}(i) \leftarrow \gamma^4_{[3R]}(v)$;
  $\gamma^3_{[3R]}(i) \leftarrow \gamma_{[3R]}(i + 1) + 3$;
  $\gamma^4_{[3R]}(i) \leftarrow \gamma_{[3R]}(v - 1) + 4$;
  $\gamma_{[3R]}(i) \leftarrow \min\{\gamma^0_{[3R]}(i), \gamma^3_{[3R]}(i), \gamma^4_{[3R]}(i)\}$;
end while
return $\gamma_{[3R]}(n)$;

3. Proper interval graphs

In this section we propose a linear algorithm (Algorithm 3.1) for computing the triple Roman domination number of a given proper interval graph. Let G be a graph of order n. An ordering $(v_1, v_2, \ldots, v_n)$ of vertices of G is a *consecutive ordering* if $v_i v_k \in E$ for some $1 \le i < k \le n$ implies both $v_i v_j \in E$ and $v_j v_k \in E$ for every $i < j < k$.

Theorem 3 ([13]). *A graph G is a proper interval graph if and only if G has a consecutive ordering of its vertices.*

Booth and Lueker [7] proposed a linear-time algorithm for testing whether a graph is a proper interval graph, and give a consecutive ordering if the answer is positive. For a given proper interval $G = (V, E)$ of order n, let $V = \{1, \ldots, n\}$ and let $1 \le i \le j \le n$ and $a \in \{0, 1, 2, 3, 4\}$.

- $[i, j) = \{i \le k < j\}$,
- $(i, j] = \{i < k \le j\}$,
- $(i, j) = \{i < k < j\}$,
- $G[i, j] = G[\{i \le k \le j\}]$,
- $\texttt{MIN}(i) = \min N_G[i]$,
- $\gamma^a_{[3R]}(i) = \min\{w(f) : f \text{ is an TRDF on } G[1, i] \text{ with } f(i) = a\}$.

To prove that Algorithm 3.1 works correctly we need the following results.

Proposition 1. *Given a proper interval graph $G = (V, E)$ with $|V| = n$ and a consecutive ordering $(1, \ldots, n)$ of vertices of G, let $1 \le i \le j \le n$.*

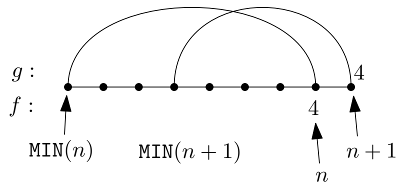

Figure 3. Illustrating an TRDF g on H such that $g(n+1) = 4$ and an TRDF f on $H[1, n]$ such that $f(n) = 4$; note that some edges of H are not drawn.

(i) For all $S \subseteq V$, the induced subgraph $G[S]$ is also a proper interval graph.

(ii) If $ij \in E$, then $[i, j]$ is a clique of G.

(iii) $\texttt{MIN}(i) \leq \texttt{MIN}(j)$.

Lemma 1. *Given a proper interval graph $G = (V, E)$ with $|V| = n$ and a consecutive ordering $(1, \ldots, n)$ of vertices of G, $\gamma^4_{[3R]}(1) \leq \gamma^4_{[3R]}(2) \leq \cdots \leq \gamma^4_{[3R]}(n)$.*

Proof. The proof is by induction on n. Clearly, $\gamma^4_{[3R]}(1) = \gamma^4_{[3R]}(2) = 4$ and so the claim holds for $n = 2$. Assume the claim holds for all proper interval graphs of order $n \geq 2$. Let H be a proper interval graph of order $n+1$ with a consecutive ordering $(1, \ldots, n+1)$ of vertices of H. By Proposition 1, the induced subgraph $H[1, n]$ is a proper interval graph of order n and so

$$\gamma^4_{[3R]}(1) \leq \cdots \leq \gamma^4_{[3R]}(n). \tag{1}$$

Let g be an TRDF on H such that its weight is minimum and $g(n+1) = 4$. So, $w(g) = \gamma^4_{[3R]}(n+1)$. See Figure 3. By Proposition 1 and since H is connected, $\texttt{MIN}(n) \leq \texttt{MIN}(n+1) \leq n$ and $[\texttt{MIN}(n), n]$ is a clique of H. So, each vertex adjacent to $n+1$ is also adjacent to n. Let f be a new function from $[1, n]$ to $[0, 4]$ as follows: $f(i) = g(i)$ for all $i \in [1, n-1]$ and $f(n) = 4$. Clearly, $w(f) = w(g) - g(n)$, where $g(n) \in [0, 4]$, and so $w(f) \leq w(g)$. Since each vertex adjacent to $n+1$ is also adjacent to n, f is an TRDF on $H[1, n]$ such that $f(n) = 4$. Hence, $\gamma^4_{[3R]}(n) \leq w(f) \leq \gamma^4_{[3R]}(n+1)$. This, together with Inequality (1), completes the proof of the lemma. □

Lemma 2. *Given a proper interval graph $G = (V, E)$ with $|V| = n$ and a consecutive ordering $(1, \ldots, n)$ of vertices of G, if $i \in [2, n]$; then $\gamma^0_{[3R]}(i) = \gamma^4_{[3R]}(\texttt{MIN}(i))$.*

Proof. Let $f = (V_0, V_1, V_2, V_3, V_4)$ be an TRDF on $G[1, i]$ such that its weight is minimum and $f(i) = 0$. So, $w(f) = \gamma^0_{[3R]}(i)$. Since $f(i) = 0$, the vertex i must have either (i) one neighbour in V_4, or either (ii) two neighbours in $V_2 \cup V_3$ (one neighbour in V_3) or either (iii) three neighbours in V_2.

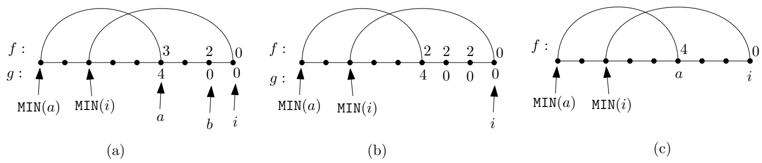

Figure 4. Illustrating two TRDFs f and g on $G[1,i]$; note that some edges of G are not drawn.

We claim that Cases (ii) and (iii) do not occur. Assume that i has two neighbours in $V_2 \cup V_3$ (one neighbour in V_3), that is, $f(a) = 3$ and $f(b) = 2$ for some $a, b \in [\texttt{MIN}(i), i)$. We first consider the case that $a < b$. By Proposition 1, $\texttt{MIN}(a) \leq \texttt{MIN}(b) \leq \texttt{MIN}(i)$ and $[\texttt{MIN}(x), x]$ is a clique of G for each vertex x. So, each vertex adjacent to both vertices a, b is also adjacent to a. Let g be a new function on $G[1, i]$ as $g(k) = f(k)$ for all $k \in [1, i] \setminus \{a, b\}$, $g(a) = 4$ and $g(b) = 0$. We get that g is an TRDF on $G[1, i]$ such that $g(i) = 0$ and $w(g) < w(f)$, contradicting that f is an TRDF on $G[1, i]$ such that its weight is minimum and $f(i) = 0$. See Figure 4(a). Similarly, if $b < a$ or i has three neighbours in V_2, then we can obtain a new TRDF g on $G[1, i]$ such that $g(i) = 0$ and $w(g) < w(f)$, see Figure 4(b), a contradiction. This proves the claim.
Now, assume that i has one neighbour in V_4. So, $f(a) = 4$ for some $a \in [\texttt{MIN}(i), i)$. See Figure 4(c). By Proposition 1, $\texttt{MIN}(a) \leq \texttt{MIN}(i)$ and $[\texttt{MIN}(a), a]$ is a clique of G. So, each vertex adjacent to i is also adjacent to a in the induced subgraph $G[1, i]$. Let f' be the restriction of f to $G[1, a]$. We get that $w(f') \leq w(f)$ and f' is an TRDF on $G[1, a]$ such that $f'(a) = 4$. Thus, $\gamma^4_{[3R]}(a) \leq w(f') \leq w(f) = \gamma^0_{[3R]}(i)$. Since $\texttt{MIN}(i) \leq a$, by Lemma 1, $\gamma^4_{[3R]}(\texttt{MIN}(i)) \leq \gamma^4_{[3R]}(a)$ and so $\gamma^4_{[3R]}(\texttt{MIN}(i)) \leq \gamma^0_{[3R]}(i)$.
Conversely, let g be an TRDF on $G[1, \texttt{MIN}(i)]$ such that its weight is minimum and $g(\texttt{MIN}(i)) = 4$. So, $w(g) = \gamma^4_{[3R]}(\texttt{MIN}(i))$. Let h be a function on $G[1, i]$ as $h(k) = g(k)$ for all $k \in [1, \texttt{MIN}(i)]$ and $h(j) = 0$ for all $j \in (\texttt{MIN}(i), i]$. We have $w(h) = w(g) = \gamma^4_{[3R]}(\texttt{MIN}(i))$. Because $[\texttt{MIN}(i), i]$ is a clique of G, h is an TRDF on $G[1, i]$ such that $h(i) = 0$. Hence, $\gamma^0_{[3R]}(i) \leq w(h) = \gamma^4_{[3R]}(\texttt{MIN}(i))$. This implies that $\gamma^0_{[3R]}(i) = \gamma^4_{[3R]}(\texttt{MIN}(i))$ and completes the proof of the lemma. □

Theorem 4 ([1]). *For a given graph G, there exists an TRDF on G with minimum weight that does not assign an 1 to any vertex in G.*

Lemma 3. *Given a proper interval graph $G = (V, E)$ with $|V| = n$ and a consecutive ordering $(1, \ldots, n)$ of vertices of G, let $i \in [2, n]$; then*

(i) $\gamma^0_{[3R]}(i) \leq \gamma^2_{[3R]}(i)$ *and*

(ii) $\gamma^3_{[3R]}(i) = \gamma_{[3R]}(i-1) + 3$ *or* $\gamma^0_{[3R]}(i) \leq \gamma^3_{[3R]}(i)$.

Proof. We first prove (i). Let $f = (V_0, V_1, V_2, V_3, V_4)$ be an TRDF on $G[1, i]$ such that its weight is minimum and $f(i) = 2$, that is, $w(f) = \gamma^2_{[3R]}(i)$. Since $f(i) = 2$,

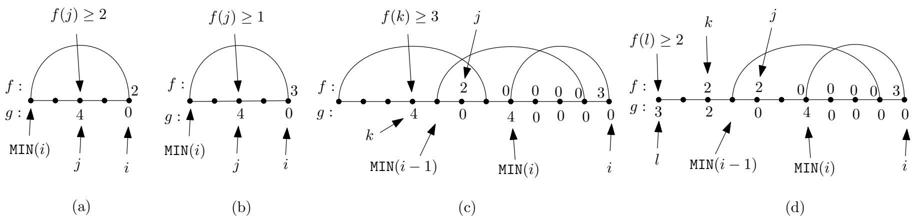

Figure 5. Illustrating two TRDFs f and g on $G[1, i]$; note that some edges of G are not drawn.

the vertex i has one neighbour j in $V_2 \cup V_3 \cup V_4$, that is, $j \in [\texttt{MIN}(i), i)$ such that $f(j) \geq 2$. See Figure 5(a). Let g be a new function on $G[1, i]$ as $g(x) = f(x)$ for all $x \in [1, i) \setminus \{j\}$, $g(j) = 4$ and $g(i) = 0$. We obtain that g is an TRDF on $G[1, i]$ such that $g(i) = 0$ and $w(g) \leq w(f)$. Thus, $\gamma^0_{[3R]}(i) \leq w(g) \leq \gamma^2_{[3R]}(i)$. This completes the proof of (i).

Now, we prove (ii). Let $f = (V_0, V_1, V_2, V_3, V_4)$ be an TRDF on $G[1, i]$ such that its weight is minimum and $f(i) = 3$, that is, $w(f) = \gamma^3_{[3R]}(i)$. We distinguish two cases depending on whether $f(x) \geq 1$ for some $x \in [\texttt{MIN}(i), i)$.

Case 1. Assume $f(j) \geq 1$ for some $j \in [\texttt{MIN}(i), i)$. Let g be a new function on $G[1, i]$ as $g(x) = f(x)$ for all $x \in [1, i) \setminus \{j\}$, $g(j) = 4$ and $g(i) = 0$. See Figure 5(b). We see that g is an TRDF on $G[1, i]$ such that $g(i) = 0$ and $w(g) \leq w(f)$. Hence, $\gamma^0_{[3R]}(i) \leq w(g) \leq \gamma^3_{[3R]}(i)$.

Case 2. Assume that $f(x) = 0$ for all $x \in [\texttt{MIN}(i), i)$. We have $\texttt{MIN}(i) < i$ and so $f(i-1) = 0$. Since f is an TRDF on $G[1, i]$, $f(j) \geq 2$ for some $j \in [\texttt{MIN}(i-1), \texttt{MIN}(i))$ and so $\texttt{MIN}(i-1) < \texttt{MIN}(i)$.

- Assume that $f(j) = 2$. Since f is an TRDF on $G[1, i]$, the vertex j has one neighbour k in $V_2 \cup V_3 \cup V_4$.

  If $f(k) \geq 3$, then let g be a new function on $G[1, i]$ as $g(x) = f(x)$ for all $x \in [1, i) \setminus \{k, j, \texttt{MIN}(i)\}$, $g(k) = g(\texttt{MIN}(i)) = 4$ and $g(j) = g(i) = 0$. See Figure 5(c). We get that g is an TRDF on $G[1, i]$ such that $g(i) = 0$ and $w(g) \leq w(f)$. Thus, $\gamma^0_{[3R]}(i) \leq \gamma^3_{[3R]}(i)$.

  Now, assume that $f(k) = 2$. Let l be a vertex such that $f(l) \geq 2$, $l < k$, and $f(x) \leq 1$ for all $x \in (l, k)$, that is, $l = \max\{x < k : f(x) \geq 2\}$. (If such vertex does not exist, then we obtain that $k = 1$ and $j = 2 = \texttt{MIN}(i-1)$ and $\gamma^0_{[3R]}(i) \leq \gamma^3_{[3R]}(i)$.) We construct a new function g on $G[1, i]$ as $g(x) = f(x)$ for all $x \in [1, i) \setminus \{l, j, \texttt{MIN}(i)\}$, $g(l) = 3$, $g(j) = g(i) = 0$, and $g(\texttt{MIN}(i)) = 4$. See Figure 5(d). We get that g is an TRDF on $G[1, i]$ such that $g(i) = 0$ and $w(g) \leq w(f)$. Hence, $\gamma^0_{[3R]}(i) \leq \gamma^3_{[3R]}(i)$.

- Assume that $f(j) = 3$. Let k be a vertex such that $f(k) \geq 2$, $k < j$, and $f(x) \leq 1$ for all $x \in (k, j)$, that is, $k = \max\{x < j : f(x) \geq 2\}$. (If such

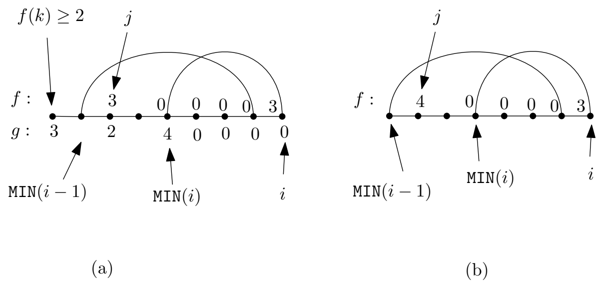

Figure 6. Illustrating an TRDF f on $G[1, i]$ with $f(i) = f(j) = 2$ such that j is adjacent to i; note that some edges of G are not drawn.

vertex does not exist, then we obtain that $j = 1 = \texttt{MIN}(i-1)$ and $\gamma^0_{[3R]}(i) \leq \gamma^3_{[3R]}(i)$.) We construct a new function g on $G[1, i]$ as $g(x) = f(x)$ for all $x \in [1, i) \setminus \{k, j, \texttt{MIN}(i)\}$, $g(k) = 3$, $g(j) = 2$, $g(\texttt{MIN}(i)) = 4$, and $g(i) = 0$. See Figure 6(a). We get that g is an TRDF on $G[1, i]$ such that $g(i) = 0$ and $w(g) \leq w(f)$. Thus, $\gamma^0_{[3R]}(i) \leq \gamma^3_{[3R]}(i)$.

- Assume that $f(j) = 4$. See Figure 6(b). Let f' be the restriction of f to $G[1, i-1]$. Since $f'(j) = 4$, we see that f' is an TRDF on $G[1, i-1]$. So, $\gamma_{[3R]}(i-1) \leq w(f') = w(f) - 3 = \gamma^3_{[3R]}(i) - 3$. Conversely, assume that g is an TRDF on $G[1, i-1]$ such that its weight is minimum. So, $w(g) = \gamma_{[3R]}(i-1)$. Let $h = g \cup \{(i, 3)\}$. We get that h is an TRDF on $G[1, i]$ such that $h(i) = 3$. Hence, $\gamma^3_{[3R]}(i) \leq w(h) = w(g) + 3 = \gamma_{[3R]}(i-1) + 3$. This, together with $\gamma_{[3R]}(i-1) \leq \gamma^3_{[3R]}(i) - 3$, implies $\gamma^3_{[3R]}(i) = \gamma_{[3R]}(i-1) + 3$.

This completes the proof of (ii) and so the proof of the lemma. □

Lemma 4. *Given a proper interval graph $G = (V, E)$ with $|V| = n$ and a consecutive ordering $(1, \ldots, n)$ of vertices of G, let $i \in [2, n]$. If $\texttt{MIN}(i) \geq 2$, then $\gamma^4_{[3R]}(i) = \gamma_{[3R]}(\texttt{MIN}(i) - 1) + 4$, otherwise, $\gamma^4_{[3R]}(i) = 4$.*

Proof. If $\texttt{MIN}(i) = 1$, that is, $[1, i]$ is a clique of G, then clearly $\gamma^4_{[3R]}(i) = 4$. Note that $i \geq 2$. In the rest of the proof, we assume that $\texttt{MIN}(i) \geq 2$. Let $f = (V_0, V_1, V_2, V_3, V_4)$ be an TRDF on $G[1, i]$ such that its weight is minimum and $f(i) = 4$, that is, $w(f) = \gamma^4_{[3R]}(i)$. Clearly, $f(x) \neq 1$ for all $x \in [\texttt{min}(i), i)$. If $f(u), f(v) \geq 2$ for some $u, v \in [\texttt{min}(i), i)$ with $u < v$, then let g be a new function on $G[1, i]$ as $g(x) = f(x)$ for all $x \in [1, i] \setminus \{u, v\}$, $g(u) = 4$ and $g(v) = 0$. We get that g is an TRDF on $G[1, i]$ such that $g(i) = 4$ and $w(g) \leq w(f)$. See Figure 7(a). If $f(u) \geq 2$ for exactly one vertex $u \in [\texttt{min}(i), i)$, then let g be a new function on $G[1, i]$ as $g(x) = f(x)$ for all $x \in [1, i] \setminus \{\texttt{min}(i) - 1, u\}$, $g(\texttt{min}(i) - 1) = f(u)$ and $g(u) = 0$. We get that g is an TRDF on $G[1, i]$ such that $g(i) = 4$ and $w(g) \leq w(f)$. See Figure 7(b). So, in the rest of the proof, we assume that $f(u) = 0$ for all $u \in [\texttt{min}(i), i)$.

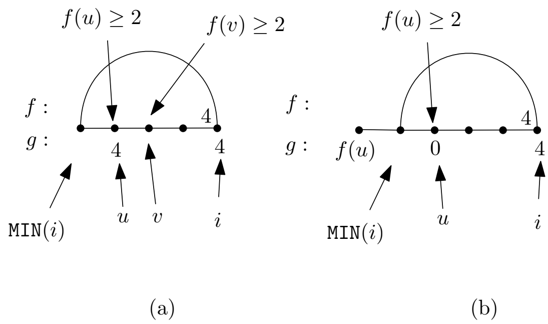

Figure 7. Illustrating two TRDFs f and g on $G[1, i]$; note that some edges of G are not drawn.

Let f' be the restriction of f to $G[1, \mathtt{min}(i))$. We see that f' is an TRDF on $G[1, \mathtt{min}(i))$. So, $\gamma_{[3R]}(\mathtt{min}(i) - 1) \leq w(f') = w(f) - 4 = \gamma^4_{[3R]}(i) - 4$. Conversely, assume that g is an TRDF on $G[1, \mathtt{min}(i))$ such that its weight is minimum, that is, $w(g) = \gamma_{[3R]}(\mathtt{min}(i) - 1)$. Let h be a function on $G[1, i]$ as $h(x) = g(x)$ for all $x \in [1, \mathtt{min}(i))$, $h(y) = 0$ for all $y \in [\mathtt{min}(i), i)$ and $h(i) = 4$. We see that h is an TRDF on $G[1, i]$ such that $h(i) = 4$. Hence, $\gamma^4_{[3R]}(i) \leq w(h) = w(g) + 4 = \gamma_{[3R]}(\mathtt{min}(i) - 1) + 4$. □

Theorem 5. *Given a proper interval graph $G = (V, E)$ with $|V| = n$ and a consecutive ordering $(1, \ldots, n)$ of vertices of G, Algorithm 3.1 computes $\gamma_{[3R]}(G)$ in $O(n)$ time.*

Proof. Let $i \in [2, n]$. By Theorem 4, let $f = (V_0, \emptyset, V_2, V_3, V_4)$ be an TRDF on $G[1, i]$ such that its weight is minimum. So, $w(f) = \gamma_{[3R]}(G[1, i])$. Clearly, $f(i) \in \{0, 2, 3, 4\}$ and so $\gamma_{[3R]}(G[1, i]) = \gamma_{[3R]}(i) = \min\{\gamma^0_{[3R]}(i), \gamma^2_{[3R]}(i), \gamma^3_{[3R]}(i), \gamma^4_{[3R]}(i)\}$. By Lemma 3, $\gamma_{[3R]}(i) = \min\{\gamma^0_{[3R]}(i), \gamma^3_{[3R]}(i), \gamma^4_{[3R]}(i)\}$. It obtains that $\gamma^0_{[3R]}(1)$ is not defined, $\gamma^3_{[3R]}(1) = 3$ and $\gamma^4_{[3R]}(1) = 4$. By Lemmas 2, 3 and 4, the output of Algorithm 3.1 on input $(G, 1, \ldots, n)$ is $\gamma^3_{[3R]}(G)$. It follows from (Algorithm 2 of) [5] that we can compute all values $\mathtt{MIN}(1), \ldots, \mathtt{MIN}(n)$ in $O(n)$ time. So, it deduces that the running time of Algorithm 3.1 is $O(n)$. □

4. Lower bound on the approximation ratio

In this section, a lower bound on the approximation factor of the triple Roman domination problem is established. Before we give our lower bound on the approximation factor of the triple Roman domination problem, we have to introduce the Min Dom Set and the Min Triple Roman Dom Function problems, formalized as follows.

Min Dom Set
Instance: A graph $G = (V, E)$.
Solution: A DS S of G, where a subset $S \subseteq V$ is called a dominating set (DS) of G

Algorithm 4.1: B

Input: A graph $G = (V, E)$.
Output: An DS of G.
1 Compute an TRDF $f = (V_0, V_1, V_2, V_3, V_4)$ of G using algorithm A;
2 $D = V_1 \cup V_2 \cup V_3 \cup V_4$;
3 **return** D;

if each vertex not in S is adjacent to one vertex of S.
Measure: $|S|$.

MIN TRIPLE ROMAN DOM FUNCTION
Instance: A graph $G = (V, E)$.
Solution: An TRDF f of G.
Measure: $w(f)$.

Theorem 6 ([8]). *For a given bipartite or split graph $G = (V, E)$, there is no $(1-\varepsilon)\ln|V|$-approximation polynomial-time algorithm for any $\varepsilon > 0$ to solve the* MIN DOM SET *problem, unless NP $\subseteq$ DTIME $(|V|^{O(\log\log|V|)})$.*

Theorem 7 ([1]). *For a given graph G, $2\gamma(G) \le \gamma_{[3R]}(G) \le 4\gamma(G)$.*

Theorem 8. *For a given bipartite or split graph $G = (V, E)$, there is no $(1/4-\varepsilon)\ln|V|$-approximation polynomial-time algorithm for any $\varepsilon > 0$ to solve the* MIN TRIPLE ROMAN DOM FUNCTION *problem, unless NP $\subseteq$ DTIME $(|V|^{O(\log\log|V|)})$.*

Proof. Assume there is an algorithm A that can approximate within ratio $\alpha > 0$ the MIN TRIPLE ROMAN DOM FUNCTION problem. Let f^* be an TRDF of G such that $w(f^*) = \gamma_{[3R]}(G)$ and let D^* be an DS of G such that $|D^*| = \gamma(G)$. By Theorem 7, $2\gamma(G) \le \gamma_{[3R]}(G) \le 4\gamma(G)$ and so $w(f^*) \le 4|D^*|$. Algorithm 4.1 on input G returns an DS D of G such that $|D| = |V_1| + |V_2| + |V_3| + |V_4| \le |V_1| + 2|V_2| + 3|V_3| + 4|V_4| = w(f) \le \alpha \times w(f^*) \le 4\alpha|D^*|$. Thus, Algorithm 4.1 can approximate the MIN DOM SET problem within the ratio 4α. Assume that there is some fixed $\varepsilon > 0$ such that can approximate the MIN TRIPLE ROMAN DOM FUNCTION problem within ratio $\alpha = (1/4-\varepsilon)\ln|V|$ using algorithm A. Then, the MIN DOM SET problem can be approximated within ratio $(1-\varepsilon')\ln|V|$ by Algorithm 4.1, where $\varepsilon' = 4\varepsilon$, which is a contradiction to Theorem 6. This completes the proof of the theorem. □

5. An approximation algorithm and APX-Completeness

In this section, we first give an approximation algorithm for computing the triple Roman domination number of graphs. Next, we prove that the triple Roman domination problem is APX-complete for graphs of degree at most 4. To present an approximation algorithm for computing the triple Roman domination number of graphs, we need the following results.

Theorem 9 ([8]). *There is an $(H(\Delta(G)+1)-1/2)$-approximation algorithm for computing a minimum DS of any given graph G, where $H(d)=\sum_{i=1}^{d} 1/i$.*

Theorem 10. *For a given graph $G=(V,E)$, there is an $(2H(\Delta(G)+1)-1)$-approximation algorithm for computing an TRDF of G with minimum weight.*

Proof. By Theorem 9, let A be an approximation algorithm that computes an DS of G and let D be the output of Algorithm A on input A. So, $|D| \leq (H(\Delta(G)+1)-1/2)|D^*|$, where D^* is a minimum DS of G. Assume that $f=(V\setminus D,\emptyset,\emptyset,\emptyset,D)$. We get that f is an TRDF of G such that $w(f)=4|D|$. Thus, $w(f) \leq 4(H(\Delta(G)+1)-1/2)|D^*|$. Let f^* be an TRDF of G such that $w(f^*)=\gamma_{[3R]}(G)$. Hence, by Theorem 7, $w(f) \leq 4(H(\Delta(G)+1)-1/2)|D^*| \leq 2(H(\Delta(G)+1)-1/2)w(f^*)$. This completes the proof of the theorem. □

To show that the triple Roman domination problem is APX-complete, we use the L-reduction notation, see [3, 15]. Let F and G be two NP optimization problems. An *L-reduction* is a polynomial time transformation h from instances of F to instances of G, if for some positive constants α and β and each instance x of F

1. $\mathrm{OPT}_G(h(x)) \leq \alpha \cdot \mathrm{OPT}_F(x)$, and
2. we can find a solution y' of x with $\mathrm{m}_F(x,y')=c_1$ in polynomial time such that $|\mathrm{OPT}_F(x)-c_1| \leq \beta|\mathrm{OPT}_G(h(x))-c_2|$ for every feasible solution y of $h(x)$ with objective value $\mathrm{m}_G(h(x),y)=c_2$.

To prove that a problem $P \in$ APX is APX-complete, we need to give an L-reduction from some APX-complete problem to P. We formalize the considered problems as follows.

Min Dom Set-B
Instance: A graph $G=(V,E)$ with degree at most B.
Solution: A DS D of G.
Measure: $|D|$.

Min Triple Roman Dom Function-B
Instance: A graph $G=(V,E)$ with degree at most B.
Solution: A TRDF f of G.
Measure: $w(f)$.

Theorem 11 ([3]). Min Dom Set-3 *is APX-complete.*

Theorem 12. Min Triple Roman Dom Function-4 *is APX-complete.*

Proof. By Theorem 10, Min Triple Roman Dom Function-4 is in APX and by Theorem 11, Min Dom Set-3 is APX-complete. It is enough to construct an L-reduction h from Min Dom Set-3 to Min Triple Roman Dom Function-4. Let $G' = (V', E')$ be a graph constructed from a given graph $G = (V, E)$ with degree at most 3 as $V' = V \cup \{a_v : v \in V\}$ and $E' = E \cup \{va_v : v \in V\}$. We get that G' is a graph with degree at most 4. For a given DS D of G, let $f = (V_0, V_1 = \emptyset, V_2 = \emptyset, V_3, V_4)$, where $V_4 = D$, $V_3 = \{a_v : v \in V \setminus D\}$, and $V_0 = V' \setminus (V_3 \cup V_4)$. We get that $w(f) = 3|V_3| + 4|V_4| = 3(|V| - |D|) + 4|D| = |D| + 3|V|$. Since each vertex in V_0 is adjacent to a vertex in V_4 and $V_1 = V_2 = \emptyset$, the function f is an TRDF of G' such that $w(f) \leq |D| + 3|V|$. In particular, $w(f^*) \leq |D^*| + 3|V|$, where D^* is an DS of G with $|D^*| = \gamma(G)$ and f^* is an TRDF of G' with $w(f^*) = \gamma_{[3R]}(G')$. Since G is a graph with degree at most 3 and D^* is an DS of G, $|V| \leq \sum_{v \in D^*}(d_G(v)+1) \leq 4|D^*|$. Hence, $w(f^*) \leq |D^*| + 3|V| \leq |D^*| + 12|D^*| = 13|D^*|$.
Conversely, let g be an TRDF of G'. Assume that for some $v \in V$, we have either $g(a_v) = 4$, or either $g(a_v) = 3$ and $g(v) \geq 1$, or either $g(a_v) \in \{1, 2\}$. We obtain that $g(a_v) + g(v) \geq 4$. Let $g' = (V'_0, V'_1, V'_2, V'_3, V'_4)$ be a new function of G' such that for all $v \in V$ if either $g(a_v) = 4$, or either $g(a_v) = 3$ and $g(v) \geq 1$, or either $g(a_v) \in \{1, 2\}$, then $g'(a_v) = 0$ and $g'(v) = 4$, otherwise, $g'(v) = g(v)$ and $g'(a_v) = g(a_v)$. We obtain that g' is an TRDF of G' such that $w(g') \leq w(g)$, $g'(a_v) \in \{0, 3\}$ for all $v \in V$, and if $g'(a_v) = 3$ for some $v \in V$, then $g'(v) = 0$. Hence, $|V'_1| = |V'_2| = 0$, $|V'_3| + |V'_4| = |V|$, and $V'_4 \subseteq V$ and so $w(g') = 3|V'_3| + 4|V'_4| = 3|V| + |V'_4|$. Since each vertex $v \in V'_0 \cap V$ is adjacent to a vertex in $S = V'_4$, the set S is an DS of G such that $|S| = |V'_4| = w(g') - 3|V| \leq w(g) - 3|V|$. In particular, $|D^*| \leq w(f^*) - 3|V|$ and so $w(f^*) = |D^*| + 3|V|$. We obtain that $|S| - |D^*| \leq w(g) - 3|V| - (w(f^*) - 3|V|) = w(g) - w(f^*)$. As a result, h is an L-reduction such that $\alpha = 13$ and $\beta = 1$. □

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