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Research Article

Tetravalent half-arc-transitive graphs of order $12p$

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Abstract: A graph is half-arc-transitive if its automorphism group acts transitively on its vertex set, edge set, but not its arc set. In this paper, we study all tetravalent half-arc-transitive graphs of order $12p$, where p is a prime.

Keywords: Half-arc-transitive graph, Tightly attached, Regular covering projection, Solvable groups

AMS Subject classification: 05C25, 20B25

1. Introduction

In this study, all graphs considered are assumed to be finite, simple and connected. For a graph X, $V(X)$, $E(X)$, $A(X)$ and Aut(X) denote its vertex set, edge set, arc set, and full automorphism group, respectively. For $u, v \in V(X)$, $\{u, v\}$ denotes the edge incident to u and v in X, and $N_X(u)$ denotes the neighborhood of u in X, that is, the set of vertices adjacent to u in X.

A graph $\widetilde{X}$ is called a covering of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection $p : V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\widetilde{v})} : N_{\widetilde{X}}(\widetilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\widetilde{v} \in p^{-1}(v)$. A permutation group G on a set Ω is said to be semiregular if the stabilizer G_v of v in G is trivial for each $v \in \Omega$, and is regular if G is transitive, and semiregular. Let K be a subgroup of Aut(X) such that K is intransitive on $V(X)$. The quotient graph X/K induced by K is defined as the graph

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such that the set Ω of K-orbits in $V(X)$ is the vertex set of X/K and B, $C \in \Omega$ are adjacent if and only if there exists a $u \in B$ and $v \in C$ such that $\{u, v\} \in E(X)$. A covering $\widetilde{X}$ of X with a projection p is said to be regular (or K-covering) if there is a subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{\mathrm{X}})$ such that K is semiregular on both $V(\widetilde{X})$ and $E(\widetilde{X})$ and graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ph of p and h. The group of covering transformations CT(p) of $p : \widetilde{X} \to X$ is the group of all self equivalences of p, that is, of all automorphisms $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{\mathrm{X}})$ such that $p = \widetilde{\alpha}p$. If $\widetilde{X}$ is connected, K becomes the covering transformation group.

For a graph X and a subgroup G of Aut(X), X is said to be G-vertex-transitive, G-edge-transitive or G-arc-transitive if G is transitive on $V(X)$, $E(X)$ or $A(X)$, respectively, and G-arc-regular if G acts regularly on $A(X)$. A graph X is called vertex-transitive, edge-transitive, arc-transitive, or arc-regular if X is Aut(X)-vertex-transitive, Aut(X)-edge-transitive, Aut(X)-arc-transitive, or Aut(X)-arc-regular, respectively. Let X be a tetravalent G-half-arc-transitive graph for a subgroup G of Aut(X), that is G acts transitively on $V(X)$, $E(X)$, but not $A(X)$. Then under the natural action of G on $V(X) \times V(X)$, G has two orbits on the arc set $A(X)$, say A_1 and A_2, where $A_2 = \{(v, u)|(u, v) \in A_1\}$. Therefore, one may obtain two oriented graphs with the vertex set $V(X)$ and the arc sets A_1 and A_2. Assume that $D_G(X)$ be one of the two oriented graphs. Also in the special case, if $G = \mathrm{Aut(X)}$ then X is said to be 1/2-transitive or half-arc-transitive.

By Tutte [29], each connected vertex-transitive and edge-transitive graph of odd valency is arc-transitive. So half-arc-transitive graphs of odd valency do not exist. Bouwer [5] answered Tutte's question about existence of half-arc-transitive graphs of even valency. A number of authors later studied the construction of these graphs. See, for example [1, 2, 9, 11, 14, 20–23, 30, 33, 36]. Let p be a prime. There are no half-arc-transitive graphs of order p, p^2 and $2p$ (see [6, 8]). Feng, Kwak, Wang and Zhou [12] classified the connected tetravalent half-arc-transitive graphs of order $2pq$ for distinct odd primes p and q. The tetravalent half-arc-transitive graphs of order p^5, p^4, $2p^2$, p^3 and $2p^3$ are classified in [7, 13, 34, 37, 38] respectively. Wang et al. [32] studied tetravalent half-arc-transitive graphs of order a product of three primes. In [24], Liu studied tetravalent half-arc-transitive graphs of order p^2q^2 with p, q distinct odd primes. Feng et al. [15] classified the tetravalent half-arc-transitive graphs of order $4p$. In [10] a complete classification of tetravalent half-arc-transitive metacirculants of order 2-powers was given. In [35], a classification of all tetravalent half-arc-transitive graphs of order $8p$ was given. In this paper, we will study tetravalent half-arc-transitive graphs of order $12p$.

2. Preliminaries

Let X be a graph and K be a finite group. By a^{-1} we mean the reverse arc to an arc a. A voltage assignment (or K-voltage assignment) of X is a function $\xi : A(X) \to K$ with the property that $\xi(a^{-1}) = \xi(a)^{-1}$ for each arc $a \in A(X)$. The values of ξ are called voltages, and K is the voltage group. The graph $X \times_\xi K$ derived from a voltage assignment $\xi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$, so that an edge (e, g) of $X \times$ K joins a vertex (u, g) to $(v, \xi(a)g)$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = \{u, v\}$. Clearly, the derived graph $X \times_\xi K$ is a covering of X with the first coordinate projection $p : X \times_\xi K \to X$, which is called the natural projection. By defining $(u, g')^g = (u, g'g)$ for any $g \in K$ and $(u, g') \in V(X \times_\xi K)$, K becomes a subgroup of $\mathrm{Aut(X \times_\xi K)}$ which acts semiregularly on $V(X \times_\xi K)$. Therefore, $X \times_\xi K$ can be viewed as a K-covering. For each $u \in V(X)$ and $\{u, v\} \in E(X)$, the vertex set $\{(u, g)|g \in K\}$ is the fibre of u and the edge set $\{(u, g)(v, \xi(a)g)|g \in K\}$ is the fibre of $\{u, v\}$, where $a = (u, v)$. The group K of automorphisms of X fixing every fibre setwise is called the covering transformation group. Conversely, each regular covering $\widetilde{X}$ of X with a covering transformation group K can be derived from a K-voltage assignment. Given a spanning tree T of the graph X, a voltage assignment ξ is said to be T-reduced if the voltages on the tree arcs are the identity. Gross and Tucker in [18] showed that every regular covering $\widetilde{X}$ of a graph X can be derived from a T-reduced voltage assignment $\widetilde{X}$ with respect to an arbitrary fixed spanning tree T of X.

Let $\widetilde{X}$ be a K-covering of X with a projection p. If $\alpha \in \mathrm{Aut(X)}$ and $\widetilde{\alpha} \in \mathrm{Aut(\widetilde{X})}$ satisfy $\widetilde{\alpha}p = p\alpha$, we call $\widetilde{\alpha}$ a lift of α, and α the projection of $\widetilde{\alpha}$. The lifts and projections of such subgroups are of course subgroups in $\mathrm{Aut(\widetilde{X})}$ and $\mathrm{Aut(X)}$, respectively.

Let G be a group, and let $S \subseteq G$ be a set of group elements such that the identity element 1 not in S. The Cayley graph associated with (G, S) is defined as the graph having one vertex associated with each group element, edges (g, h) whenever hg^{-1} in S. The Cayley graph X is denoted by $\mathrm{Cay(G, S)}$. In graph theory, the lexicographic product or (graph) composition $G[H]$ of graphs G and H is a graph such that the vertex set of $G[H]$ is the cartesian product $V(G) \times V(H)$; and any two vertices (x, y) and (v, w) are adjacent in $G[H]$ if and only if either x is adjacent with v in G or $v = x$ and w is adjacent with y in H. Clearly, if G and H are arc-transitive then $G[H]$ is arc-transitive.

Let X be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut(X)}$. Then no element of G can interchange a pair of adjacent vertices in X. By [19], there is no half-arc-transitive graph with less then 27 vertices. Half-arc-transitive graphs have even valencies. An even length cycle C in X is a G-alternating cycle if every other vertex of C is the head and every other vertex of C is the tail of its two incident edges in $D_G(X)$. All G-alternating cycles in X have the same length. The radius of graph is half of the length of an alternating cycle. Any two adjacent G-alternating cycles in X intersect in the same number of vertices, called the G-attachment number of X. The intersection of two adjacent G-alternating cycles is called a G-attachment set. We say that X is tightly attached if the attachment number of X equal with its radius.

Now we introduce graph $X(r; m, n)$ and a result due to Marušič.

Suppose that $m \geq 3$ be an integer, $n \geq 3$ an odd integer and let $r \in \mathbb{Z}_n^*$ satisfy $r^m = \pm 1$. The graph $X(r; m, n)$ is defined to have vertex set $V = \{u_i^j \mid i \in \mathbb{Z}_m, j \in$

$\mathbb{Z}_n\}$ and edge set $E = \{\{u_i^j, u_{i+1}^{j\pm r^i}\} \mid i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$.

Proposition 1. [25, Theorem 3.4] *A connected tetravalent graph X is a tightly attached half-arc-transitive graph of odd radius n if and only if $X \cong X(r; m, n)$, where $m \geq 3$, and $r \in \mathbb{Z}_n^*$ satisfying $r^m = \pm 1$, and moreover none of the following conditions is fulfilled:*
(1) $r^2 = \pm 1$;
(2) $(r; m, n) = (2; 3, 7)$;
(3) $(r; m, n) = (r; 6, 7k)$, where $k \geq 1$ is odd, $(7, k) = 1$, $r^6 = 1$, and there exists a unique solution $q \in \{r, -r, r^{-1}, -r^{-1}\}$ of the equation $x^2 + x - 2 = 0$ such that $7(q - 1) = 0$ and $q \equiv 5 \pmod 7$.

The following is the main result of the paper tetravalent half-transitive graphs of order $4p$.

Proposition 2. [15, Theorem 3.3] *Let p be a prime and X a tetravalent graph of order $4p$. Then, X is half-transitive if and only if $p \equiv 1 \pmod 8$ and $X \cong X(r; 4, p)$(denote by $X(4, p)$ the graph $X(r; 4, p)$).*

Now we express an observations about tetravalent half-arc-transitive graphs.

Proposition 3. [26, Lemma 3.5] *Let X be a connected tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut(X)}$, and let Δ be a G-attachment set of X. If $|\Delta| \geq 3$, then the vertex-stabilizer of $v \in V(X)$ in G is of order 2.*

Proposition 4. [17] *A non-abelian simple group whose order has at most three prime divisors is isomorphic to one of the following groups:*

$$\mathrm{A}_5, \mathrm{A}_6, \mathrm{PSL}(2,7), \mathrm{PSL}(2,8), \mathrm{PSL}(2,17), \mathrm{PSL}(3,3), \mathrm{PSU}(3,3), \mathrm{PSU}(4,2),$$

whose orders are $2^2 \times 3 \times 5$, $2^3 \times 3^2 \times 5$, $2^3 \times 3 \times 7$, $2^3 \times 3^2 \times 7$, $2^4 \times 3^2 \times 17$, $2^4 \times 3^3 \times 13$, $2^5 \times 3^3 \times 7$, $2^6 \times 3^4 \times 5$, respectively.

The following result is extracted from [4, Theorem 1].

Proposition 5. *Let X be a tetravalent arc-transitive graph of order $2pq$ where p and q are odd and distinct primes. Then one of the following holds:*
(1) X is arc-regular and appears in [40];
(2) X is isomorphic to the lexicographic product $C_{pq}[2K_1]$ of the cycle C_{pq} and the edgeless graph on two vertices $2K_1$.

In the following, we describe the structure of the graphs required in this paper [[27], [28], [39]].

The Rose Window graph $R_6(5, 4)$ is a tetravalent graph with 12 vertices. Its vertex set is $\{S_i, Q_i | i \in Z_6\}$. The graph has four kinds of edges: kind of edges: S_iS_{i+1} (rim edges), S_iQ_i (inspoke edges), $S_{i+5}Q_i$ (outspoke edges) and Q_iQ_{i+4} (hub edges). $|\mathrm{Aut}(\mathrm{R}_6(5, 4))| = 48$. Figure 1 shows $R_6(5, 4)$.

A general Wreath graph $W(6,2)$ has 12 vertices and it is regular of valency 4. Its vertex set is $\{E_i, F_i | i \in Z_6\}$, where $E_i = (i,0)$ and $F_i = (i,1)$. Its edges are $\{E_i, E_{i+1}\}$, $\{E_i, F_{i+1}\}$, $\{F_i, E_{i+1}\}$ and $\{F_i, F_{i+1}\}$. $|\mathrm{Aut(W(6,2))}| = 768$. See Figure 2.

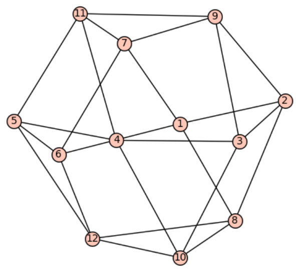

Figure 1. **The Rose Window graph** $R_6(5,4)$

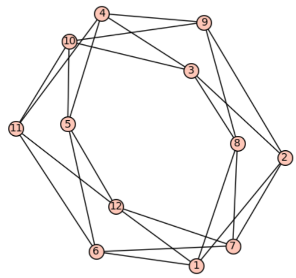

Figure 2. **The Wreath graph** $W(6,2)$

The graph $C(2;p,2)$ was first defined by Praeger and Xu [28, Definition 2.1 (b)]. Let p be an odd prime. The graph $C(2;p,2)$ has vertex set $\mathbb{Z}_p \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and its edges are defined by $\{(i,(x,y)),(i+1,(y,z))\} \in E(C(2;p,2))$ for all $i \in \mathbb{Z}_p$ and $x,y,z \in \mathbb{Z}_2$. $\mathrm{Aut(C(2;p,2))} \cong \mathrm{D_{2p}} \ltimes \mathbb{Z}_2^{\mathrm{p}}$.

Let $p \equiv 1 \pmod 4$, where p is a prime and w is an element of order 4 in $\mathbb{Z}_p^*$. The graph CA_{4p}^0 is $\mathrm{Cay(G, \{a, a^{-1}, a^{w^2}b, a^{-w^2}b\})}$ and the graph CA_{4p}^1 is $\mathrm{Cay(G, \{a, a^{-1}, a^{w}b, a^{-w}b\})}$, where $G = < a > \times < b > \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$.

3. Main Results

In this section, we study all tetravalent half-arc-transitive graphs of order $12p$ where p is a prime. To do this, we prove the following results.

Lemma 1. *Let X be a graph, $G \leqslant \mathrm{Aut(X)}$, $N \trianglelefteq G$ and X be N-regular covering of X_N. Then X is G-half-arc-transitive if and only if X_N is G/N-half-arc-transitive.*

Proof. Suppose that $N \trianglelefteq G$ and X is G-half-arc-transitive. Since X is N-regular covering of X_N, it follows that $K = N$ and $G/N \leqslant \mathrm{Aut(X_N)}$, where K is the kernel of G acting on orbits of N. Let x^N, y^N be two arbitrary vertices of graph X_N. By our assumption there exits $g \in G$ such that $x^g = y$. Now $(x^N)^{Ng} = (x^g)^N = y^N$. It implies that X_N is G/N-vertex-transitive. Now, suppose that $\{x^N, y^N\}$ and $\{u^N, v^N\}$ are two arbitrary edges of X_N. Without loss of generality, we may suppose that $\{x, y\}$ and $\{u, v\}$ are two edges of X. By our assumption there exits $g \in G$ such that $\{x, y\}^g = \{u, v\}$. Then we may assume that $x^g = u$ and $y^g = v$. Hence $(x^N)^{Ng} = x^{Ng} = x^{gN} = u^N$ and $(y^N)^{Ng} = y^{Ng} = y^{gN} = v^N$. Then X_N is G/N-edge-transitive. Suppose to contrary that X_N is G/N-arc-transitive. Let (x, y) and (u, v) are two arcs of graph X. Now (x^N, y^N) and (u^N, v^N) are two arcs of graph X_N. By our assumption, there exits $Ng \in G/N$ such that $(x^N, y^N)^{Ng} = (u^N, v^N)$. Therefore $(x^N)^{Ng} = u^N$ and $(y^N)^{Ng} = v^N$. Thus $x^{Ng} = u^N$ and $y^{Ng} = v^N$. Then $x^g = u^n$ and $y^g = v^{n'}$ for $n, n' \in N$ and so $(x, y)^g = (u^n, v^{n'})$. There exits $n'' \in N$ such that $(u^n, v^{n'})^{n''} = (u, v)$. Then $(x, y)^{gn''} = (u^n, v^{n'})^{n''} = (u, v)$. Therefore X is G-arc-transitive, a contradiction. Then X_N is G/N-half-arc-transitive.

Now suppose that X_N is G/N-half-arc-transitive. Thus G/N acts transitively on $V(X_N)$. Let $u, v \in V(X)$ and $u^N, v^N \in V(X_N)$. Then there is $Ng \in G/N$ such that $(u^N)^{Ng} = v^N$ and hence, there is $n' \in N$ such that $u^g = v^{n'}$ and $u^{g(n')^{-1}} = v$. Then since $g(n')^{-1} \in G$, it implies that X is vertex-transitive. For any $\{u, v\}, \{x, y\} \in E(X)$, we have $\{u^N, v^N\}, \{x^N, y^N\} \in E(X_N)$. Since X_N is G/N-edge-transitive, we have $Ng \in G/N$ such that $\{u^N, v^N\}^{Ng} = \{x^N, y^N\}$ and $\{(u^N)^{Ng}, (v^N)^{Ng}\} = \{x^N, y^N\}$. Without loss of generality, we may suppose that $(u^N)^{Ng} = (u)^{Ng} = x^N$ and $(v^N)^{Ng} = (v)^{Ng} = y^N$. There exits $n', n'' \in N$ such that $\{u, v\}^g = \{x^{n'}, y^{n''}\}$. Also there exits $n \in N$ such that $\{x^{n'}, y^{n''}\}^n = \{x, y\}$. Thus we may assume that $\{u, v\}^{gn} = \{x, y\}$ and so X is G-edge-transitive. Similar to the previous, it can be shown that if X_N is not G/N-arc-transitive then X is not G-arc-transitive. Therefore X is G-half-arc-transitive. □

The following lemma is basic for the main result.

Lemma 2. *Let X be a half-arc-transitive graph, p is a prime and $N \trianglelefteq \mathrm{Aut(X)}$, where $N \cong \mathbb{Z}_p$. If the quotient graph X_N is a Cayley graph and has the same valency with X then X is a N-regular covering of X_N and X is a Cayley graph.*

Proof. Let N be a normal subgroup of $A := \mathrm{Aut(X)}$ and X_N be the quotient graph of X with respect to the orbits of N on $V(X)$. Assume that K is the kernel of A acting on $V(X_N)$. The stabilizer K_v of $v \in V(X)$ in K fixes the neighborhood of v in X. The connectivity of X implies $K_v = 1$ for any $v \in V(X)$ and hence $N_v = 1$. If $N_{\{\alpha,\beta\}} \neq 1$ then $N_{\{\alpha,\beta\}} = N$, because $N \cong \mathbb{Z}_p$. Since X is connected, there is a $\{\beta, \gamma\} \in E(X)$ where $\beta, \gamma \in V(X)$. Then we have $g \in A$ such that $\{\alpha, \beta\} = \{\beta, \gamma\}^g$ because X is an edge-transitive graph. Hence $N_{\{\alpha,\beta\}} = N_{\{\beta,\gamma\}^g} = g^{-1}N_{\{\beta,\gamma\}}g = N_{\{\beta,\gamma\}}$. It is a contradiction and so $N_{\{\alpha,\beta\}} = 1$. Therefore X is a $\mathbb{Z}_p$-regular covering of X_N. Now we prove that X is a Cayley graph. Let $X_N \cong \mathrm{Cay(G,S)}$, $X \cong X_N \times_\xi \mathbb{Z}_p$ where ξ is the T-reduced voltage assignment and $\tilde{G}$ is a lift of G such that $\tilde{\alpha}p = p\alpha$ where $p : X \to X_N$ is regular covering projection, $\alpha \in \mathrm{Aut(X_N)}$ and $\tilde{\alpha} \in A$. For any $(x, k), (y, k') \in V(X)$ where $k, k' \in \mathbb{Z}_p$ and $x, y \in V(X_N)$, we have $\alpha \in \mathrm{Aut(X_N)}$ such that $x^\alpha = y$. For $k^{''} \in \mathbb{Z}_p$, $(x, k)^{\tilde{\alpha}p} = (z, k^{''})^p = z$ where $(x, k)^{\tilde{\alpha}} = (z, k^{''})$. Also $(x, k)^{p\alpha} = x^\alpha = y$. Then $y = z$ and hence $(y, k), (y, k^{''}) \in p^{-1}(y)$. Therefore $\tilde{G}$ is transitive on $V(X)$. Now, we prove that $\tilde{G}$ is semiregular. Suppose that $(x, k)^{\tilde{\alpha}} = (x, k)$. Now, since G is semiregular and $\tilde{\alpha}p = p\alpha$, it implies that $x = (x, k)^{\tilde{\alpha}p} = (x, k)^{p\alpha} = x^\alpha$. Then $\alpha = 1$ and hence $\tilde{\alpha}p = p$. Therefore $\tilde{\alpha} \in \mathrm{CT(p)} = \mathrm{\mathbb{Z}_p}$ and since $\mathrm{CT(p)}$ is semiregular, it follows that $\tilde{\alpha} = 1$. □

By [27], all tetravalent half-arc-transitive graphs of order $12p$ where $p \leq 53$ is a prime, are classified. Then in the following, we may assume that $p > 53$.

Lemma 3. *Let X be a tetravalent half-arc-transitive graph of order $12p$, where p is a prime. Then $\mathrm{Aut(X)}$ has a normal Sylow p-subgroup or X is $\mathbb{Z}_3$-regular cover of $C(2; p, 2)$ or $C_{2p}[2K_1]$.*

Proof. Let X be a tetravalent half-arc-transitive graph of order $12p$ where p is a prime. Let $A := \mathrm{Aut(X)}$. Since the stabilizer A_v of $v \in V(X)$ is a 2-group, we have $|A| = 2^{m+2}.3.p$, for some nonegative integer m. Suppose to the contrary that A has no normal Sylow p-subgroups. Let N be a minimal normal subgroup of A. We claim that N is solvable. Otherwise, by Proposition 4 and since $p > 53$, we get a contradiction. Then N is solvable and hence it is an elementary abelian 2-,3- or p-group.

Case I. N is a 2-group.
Let X_N be the quotient graph of X corresponding to the orbits of N on $V(X)$. Then $|V(X_N)| = 6p$ or $3p$.

Subcase 1. $|V(X_N)| = 6p$.
Since X is edge-transitive, X_N has valency 2 or 4. Suppose that X_N has valency 2. Then $X \cong C_{6p}[2K_1]$, which is arc-transitive. It is a contradiction. Assume now that X_N has valency 4. If X_N is half-arc-transitive then by [12, Theorem 4.1], $|\mathrm{Aut(X_N)}| = 2^2.3.\mathrm{p}$. Let K be the kernel of A acting on $V(X_N)$. Since K fixes each orbit of N, the stabilizer $K_v = 1$ for any $v \in V(X)$. Then $|N| = |K|$. On the other hand $A/K \leqslant \mathrm{Aut(X_N)}$. Since A/K acts transitively on $V(X_N)$ and $E(X_N)$, $|A| = 24p$. Then $1 + np \mid 24$. Since $p > 53$ then $P \trianglelefteq A$, a contradiction. Now, suppose that X_N is arc-transitive. Let X_N has valency 4. By Proposition 5, if X_N is arc-regular then $|\mathrm{Aut(X_N)}| = 24\mathrm{p}$. By lemma 1, A/K is half-arc-transitive and hence $|A| = 24p$. Then $P \trianglelefteq A$ because $p > 53$. It is a contradiction. If X_N do not be arc-regular then by Proposition 5, $Y = X_N \cong C_{3p}[2K_1]$ and $B = \mathrm{Aut(Y)}$. $|B| = 2^{3p+1}.3.p$. Assume that M is a minimal normal subgroup of B. By the same argument as in the first paragraph, M is solvable and hence it is an elementary abelian

2-,3- or p-group. First, assume that M is a 2-group and Y_M is the quotient graph of Y corresponding to the orbits of M on $V(Y)$. The quotient graph Y_M has order $3p$ and valency 2 or 4. If Y_M has valency 4 then $M_v = 1$ for $v \in V(Y)$. Assume that K_1 be the kernel of B acting on $V(Y_M)$. Hence $|K_1| = |M|$. Thus $B/K_1 \leqslant \mathrm{Aut(Y_M)}$. It is a contradiction because $|\mathrm{Aut(Y_M)}| = \mathrm{12p}$ by [31, Theorem 5]. If Y_M has valency 2 then $Y_M \cong C_{3p}$ and $\mathrm{Aut(Y_M)} \cong \mathrm{D_{6p}}$. Since $|K_1| \leq 2$, we have $|B| \leq 12p$. We get a contradiction because $p > 53$. Now, suppose that M be a 3-group. Then $|V(Y_M)| = 2p$. Since $M_v = 1$ for $v \in V(Y)$ by using [16, Theorem 1.1(4)], Y_M has valency 4. By [8, Table 1], $Y_M \cong G(2,p,r)$ or $G(2p,r)$. Then $|K_1| = |M|$ and hence $B/K_1 \leqslant \mathrm{Aut(Y_M)}$. It is a contradiction because $|\mathrm{Aut(Y_M)}| = 2^{\mathrm{p}+1}.\mathrm{p}$ or $8p$ and $p > 53$. Let M be a p-group. Then $|Y_M| = 6$. Since $M_v \leq M$ we have $|M_v| = 1$. By [16, Theorem 1.1(4)], Y_M has valency 4. By [27], $|\mathrm{Aut(Y_M)}| = 48$. Hence $B/K_1 \leqslant \mathrm{Aut(Y_M)}$. It is a contradiction.

Subcase 2. $|V(X_N)| = 3p$.
Let $|V(X_N)| = 3p$ and X_N has valency 2. Then $X \cong C_{3p}[2K_1]$. This leads to a contradiction. If X_N has valency 4 and it is half-arc-transitive then by [1, Theorem 2.5], $|\mathrm{Aut(X_N)}| = \mathrm{6p}$. Since X_N is an edge-transitive graph, $6p \mid |A/K| \,|6p$. Then $|A| = 24p$ and hence $P \trianglelefteq A$. It is a contradiction. Suppose now that X_N is arc-transitive. By [31, Theorem 5], $|\mathrm{Aut(X_N)}| = \mathrm{12p}$. Then with the same arguments as before, a contradiction can be obtained.

Case II. N is 3-group.
If $|V(X_N)| = 4p$ and X_N has valency 2, then $X_N \cong C_{4p}$ and hence $\mathrm{Aut(X_N)} \cong \mathrm{D_{8p}}$. Since $K = K_vN$ for any $v \in V(X)$ and K acts faithfully on $V(X)$, we have $K \leqslant S_3$ and hence $K_v \leq 2$. Then $|A| \mid 48p$. Therefore $P \trianglelefteq A$ according to assumption $p > 53$. This leads to a contradiction. Now let $|V(X_N)| = 4p$ and X_N has valency 4. Then X_N is arc-transitive or half-arc-transitive. By [39, Table 1] and Proposition 2, $X_N \cong C(2;p,2)$, $C_{2p}[2K_1]$, CA^0_{4p}, CA^1_{4p} or $X(4,p)$. Let $X_N \cong C(2;p,2)$ or $C_{2p}[2K_1]$. Since X_N has valency 4, N acts semiregularly on $V(X)$ and so X is a $\mathbb{Z}_3$-regular cover of $C(2;p,2)$ or $C_{2p}[2K_1]$. Assume that $Y = X_N \cong CA^0_{4p}$ or CA^1_{4p} and $B = \mathrm{Aut(Y)}$. Since $|K| = |N|$, we have $A/K \leq B$ and hence $|A| \leq 48p$. Then $P \trianglelefteq A$. Suppose that $Y = X_N \cong X(4,p)$ and $B = \mathrm{Aut(Y)}$. Since Y is half-arc-transitive, we have $|B| = 2^{m+2}.p$, for some nonegative integer m. Let M be a minimal normal subgroup of B. Thus M is an elementary abelian 2- or p-group. First, assume that M be a p-group and Y_M be the quotient graph of Y corresponding to the orbits of M on $V(Y)$. Then $|V(Y_M)| = 4$. Since Y is an edge-transitive graph and $M_v = 1$ for $v \in V(Y)$, Y_M has valency 4, a contradiction. Suppose that M is a 2-group. Therefore $|V(Y_M)| = 2p$ or p and Y_M has valency 2 or 4.

Subcase 1. $|V(Y_M)| = 2p$.
If Y_M has valency 2 then $Y \cong C_{2p}[2K_1]$, which is arc-transitive. Since Y is half-arc-transitive, we get a contradiction. Suppose now that Y_M has valency 4. By [8, Table 1], $Y_M \cong G(2p,4)$ or $G(2,p,2)$. Assume that $Y_M \cong G(2p,4)$. Since $(K_1)_v = 1$, $|B/K_1| \leqslant 8p$ and hence $|A| \leq 48p$. It is a contradiction because $p > 53$. Suppose that $Y_M \cong G(2,p,2)$. Let $Z = Y_M \cong G(2,p,2)$ and $C = \mathrm{Aut(Z)}$. Let H be a minimal

normal subgroup of C and let Z_H be the quotient graph of Z with respect to the orbits of H. Since $|C| = 2^{p+1}.p$, H is 2- or p-group. Assume that H is a 2-group. Thus $|Z_H| = p$ and Z_H has valency 2 or 4. By [6, Theorem 3], $|\mathrm{Aut(Z_H)}| = 2\mathrm{p}$ or $4p$. Assume that K_1 be the kernel of C acting on $V(Z_H)$. If Z_H has valency 4 then $|K_1| = |H| = 2$ because $|(K_1)_v| = 1$. Then $C/K_1 \leqslant 16p$ and hence $2^{p+1} \leq 8p$. We get a contradiction because $p > 53$. If Z_H has valency 2 then $|K_1| \leq 8$ because $|(K_1)_v| \leq 2$. Thus $C/K_1 \leqslant 16p$ and hence $2^{p+1} \leq 8p$, a contradiction can be obtained. Now, suppose that H is a p-group. Then $|Z_H| = 2$ with valency 2, 4, a contradiction.

Subcase 2: $|V(Y_M)| = p$.

If Y_M has valency 4 then by lemma 2, Y is $\mathbb{Z}_2$-regular cover of Y_M and Y is a Cayley graph. But by [15], $X(4,p)$ is not a Cayley graph, a contradiction. Suppose that Y_M has valency 2 and hence $Y_M \cong C_p$. Assume that K_1 is the kernel of B acting on $V(Y_M)$ and $(K_1)_v = 1$. Then $B/K_1 \leqslant \mathrm{Aut(Y_M)}$ and so $|B| \leq 8p$. Therefore $|A| \leq 24p$ and hence $P \trianglelefteq A$ because $p > 53$. Then $(K_1)_v \neq 1$. Let $V(Y_M) = \{\Omega_0, \Omega_1, \Omega_2, ..., \Omega_{p-1}\}$. The subgraph induced by any two adjacent orbits is either a cycle of length 8 or a union of two cycles of length 4. Suppose that $\langle \Omega_i \cup \Omega_{i+1} \rangle$ is an 8-cycle. Thus K_1 acts faithfully on each Ω_i and hence $(K_1)_v \cong \mathbb{Z}_2$. It implies that $|K_1| = 8$. Since M is transitive on each Ω_i and $(K_1)_v > 1$, all edges in the induced subgraph $\langle \Omega_i \cup \Omega_{i+1} \rangle$ have the same direction either from Ω_i to Ω_{i+1} or from Ω_{i+1} to Ω_i in the oriented graph $D_B(Y)$. It follows that $B/K_1 \cong \mathbb{Z}_p$ and $|B| \leq 8p$. Therefore $|A| \leq 24p$ and hence $P \trianglelefteq A$ because $p > 53$. Assume that $\langle \Omega_i \cup \Omega_{i+1} \rangle$ is a union of two 4-cycles. Let $\Omega_i = \{u_i^0, u_i^1, u_i^2, u_i^3\}$ for any i in $\mathbb{Z}_p$. Then B has an automorphism α of order p such that for any i in $\mathbb{Z}_p$, $\Omega_i^\alpha = \Omega_{i+1}$. Let $(u_i^j)^\alpha = u_{i+1}^j$ for i in $\mathbb{Z}_p$ and j in $\mathbb{Z}_4$. Consider a 4-cycle C in the induced subgraph $\langle \Omega_0 \cup \Omega_1 \rangle$ and let n be the number of edges of C which are in some orbit of α. Then $n = 0, 1$, or 2. Consequently, the induced subgraph $\langle \Omega_0 \cup \Omega_1 \rangle$ is one of the of the following three cases.

In the Case 1, Y is disconnected, a contradiction. In the Case 2, $Y \cong C_{2p}[2K_1]$. We get a contradiction because $Y \cong X(4,p)$. In the Case 3, $Y \cong C(2;p,2)$ that is arc-transitive. It is a contradiction because $X(4,p)$ is a half-arc-transitive graph.

Case III. N is p-group.

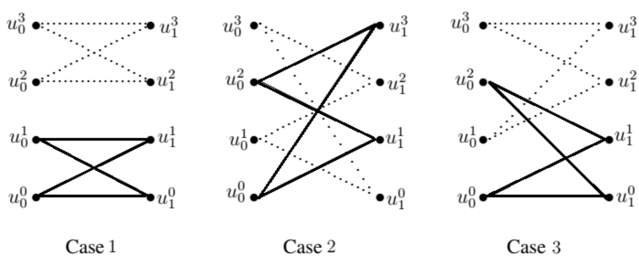

Figure 3. **The induced subgraph** $\langle \Omega_0 \cup \Omega_1 \rangle$

If $|N| = p$ then N is a normal Sylow p-subgroup of A as claimed. □

Theorem 1. *Let X be a connected tetravalent half-arc-transitive graph of order $12p$, where $p > 53$ is a prime. Then one of the following statements holds:*
(1) X is $\mathbb{Z}_3$-regular cover of $C(2; p, 2)$.
(2) X is $\mathbb{Z}_3$-regular cover of $C_{2p}[2K_1]$ and in this case X is a Cayley graph.
(3) $X \cong X(r; 12, p)$ such that $r \in \mathbb{Z}_p^*$ satisfying $r^{12} = \pm 1$.
(4) X is $\mathbb{Z}_p$-regular cover of $W(6, 2)$ or $R_6(5, 4)$ and in this case X is a Cayley graph.

Proof. Let X be a tetravalent half-arc-transitive graph of order $12p$ and hence $|A| = 2^{m+2}.3.p$ for some integer $m \geq 0$. By Lemma 3, either $P \trianglelefteq A$ or X is a $\mathbb{Z}_3$-regular cover of $C(2; p, 2)$ or $C_{2p}[2K_1]$. If X is $\mathbb{Z}_3$-regular cover of $C(2; p, 2)$ then we have Case 1. Also, if X is $\mathbb{Z}_3$-regular cover of $C_{2p}[2K_1]$ then by Lemma 2, X is a Cayley graph and we have Case 2. Now, suppose that $P \trianglelefteq A$. Let X_P be the quotient graph of X corresponding to the orbits of P. Assume that K is the kernel of A acting on $V(X_P)$. Then $V(X_P) = 12$ and X_P has valency 2 or 4. If X_P has valency 2 then $X_P \cong C_{12}$ and hence $\mathrm{Aut(X_P)} \cong \mathrm{D_{24}}$. By Proposition 3, $A_v \cong \mathbb{Z}_2$ and hence $|A| = 24p$. The attachment number of X is equal to its radius. So X is a tetravalent tightly attached half-arc-transitive graph of odd radius p. By Proposition 1, $X \cong X(r; 12, p)$ where $r \in \mathbb{Z}_p^*$ and $r^{12} = \pm 1$, which is Case 3. Assume that X_P has valency 4 and X_P is arc-transitive or half-arc-transitive. There is no half-arc-transitive graph of order 12. Suppose that X_P is an arc-transitive graph. By [27], $W(6, 2)$ and $R_6(5, 4)$ are the only two arc-transitive graphs of order 12. These graphs are Cayley graphs by [3]. Since P acts semiregular on $V(X)$ and $E(X)$, by Lemma 2, X is a $\mathbb{Z}_p$-regular cover of X_P and X is a Cayley graph, which is Case 4. □

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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