Communications in Combinatorics and Optimization
Vol. 9, No. 1 (2024), pp. 27-36
https://doi.org/10.22049/cco.2022.27898.1389

CCO
Commun. Comb. Optim.

Research Article

Some lower bounds on the Kirchhoff index

S. Stankov[†], **I. Milovanović**[*], **E. Milovanović**[‡] **and M. Matejić**[§]

Faculty of Electronic Engineering, University of Niš, Niš, Serbia
[†]stefan.stankov@elfak.ni.ac.rs
[*]igor@elfak.ni.ac.rs
[‡]ema@elfak.ni.ac.rs
[§]marjan.matejic@elfak.ni.ac.rs

Received: 25 June 2022; Accepted: 19 October 2022
Published Online: 23 October 2022

Abstract: Let $G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$, $E = \{e_1, e_2, \dots, e_m\}$, be a simple graph of order $n \geq 2$ and size m without isolated vertices. Denote with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ the Laplacian eigenvalues of G. The Kirchhoff index of a graph G, defined in terms of Laplacian eigenvalues, is given as $Kf(G) = n\sum_{i=1}^{n-1} \frac{1}{\mu_i}$. Some new lower bounds on $Kf(G)$ are obtained.

Keywords: Topological indices, Laplacian eigenvalues, Kirchhoff index

AMS Subject classification: 05C90, 15A18

1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$, $E = \{e_1, e_2, \dots, e_m\}$, be a simple graph of order $n \geq 2$ and size m without isolated vertices. Denote by $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(v_i)$, a sequence of vertex degrees given in nonincreasing order. If vertices v_i and v_j are adjacent in G, we denote it as $i \sim j$.
Let $N(i)$ be a set of all neighbours of the vertex v_i, i.e. $N(i) = \{v_j \mid v_j \in V, v_i \sim v_j\}$, and d_{ij} the distance between the vertices v_i and v_j. Denote by Γ_d a set of all d-regular graphs, $1 \leq d \leq n-1$, with diameter $D = 2$ and $|N(i) \cap N(j)| = d$, $i \nsim j$, [15].
Denote by $A(G)$ and $D(G) = diag(d_1, d_2, \dots, d_n)$ the adjacency and the diagonal degree matrix of G, respectively. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$. Eigenvalues of L, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$, form the Laplacian spectrum of G.

* *Corresponding Author*

The Wiener index, $W(G)$, originally termed as a "path index", is a topological graph index defined as [17]

$$W(G)=\sum_{i<j} d_{ij}\,,$$

where d_{ij} is the number of edges in the shortest path between the vertices v_i and v_j in graph G.
By analogy with Wiener index, in [8] the Kirchhoff index was introduced. It is defined as

$$Kf(G)=\sum_{i<j} r_{ij}\,,$$

where r_{ij} is the resistance distance between the vertices v_i and v_j of G, i.e. r_{ij} is equal to the resistance between two equivalent points on an associated electrical network, obtained by replacing each edge of G by a unit (1 ohm) resistor. In [6, 19] it was observed that the Kirchhoff index can be obtained from the non–zero eigenvalues of the Laplacian matrix, that is

$$Kf(G)=n\sum_{i=1}^{n-1}\frac{1}{\mu_i}\,.$$

If $G\in\Gamma_d$, then [15]

$$Kf(G)=\frac{n(n-1)-d}{d}\,.$$

The Kirchhoff index is investigated extensively in mathematical and chemical literatures [1, 3, 4, 9–13, 18] In the present paper we consider lower bounds on $Kf(G)$ as well as its relationship with some other topological indices.

2. Preliminaries

In this section we recall some analytical inequalities and lower bound on $Kf(G)$, reported in [18], that are of interest for the present paper.

Lemma 1. [14] *Let $p=(p_i)$, $i=1,2,\dots,n$, be a sequence of non–negative real numbers, and $a=(a_i)$, $i=1,2,\dots,n$, sequence of positive real numbers. Then, for any real r, $r\le 0$ or $r\ge 1$, holds*

$$\left(\sum_{i=1}^{n}p_i\right)^{r-1}\sum_{i=1}^{n}p_i a_i^r \ge \left(\sum_{i=1}^{n}p_i a_i\right)^{r}. \tag{1}$$

If $0\le r\le 1$, then the opposite inequality is valid. Equality holds if and only if either $r=0$, or $r=1$, or $a_1=a_2=\cdots=a_n$, or $p_1=\cdots=p_t=0$ and $a_{t+1}=\cdots=a_n$, or $a_1=\cdots=a_t$ and $p_{t+1}=\cdots=p_n=0$, for some t, $1\le t\le n-1$.

The inequality (1), originally proved in [7], is known as Jensen's inequality.
Let $a=(a_i)$, $i=1,2,\dots,n$ be a sequence of non–negative real numbers, and $p=(p_i)$ a sequence of positive real numbers. Denote by $I=\{1,2,\dots,n\}$ and $I_2=\{1,n\}$ two index sets, and

$$M_1(a,p;I)=\frac{\sum_{i=1}^{n}p_i a_i}{\sum_{i=1}^{n}p_i}\,.$$

In [5] the following result was proven.

Lemma 2. [5] *Let $a=(a_i)$ and $b=(b_i)$, $i=1,2,\dots,n$, be sequences of similar monotonicity of non–negative real numbers, and $p=(p_i)$, $i=1,2,\dots,n$, a sequence of positive real numbers. If the pairs $\left(M_1(a,p;I-I_2),M_1(a,p;I_2)\right)$ and $\left(M_1(b,p;I-I_2),M_1(b,p;I_2)\right)$ are similarly ordered, then*

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \geq \frac{p_1 p_n (a_1-a_n)(b_1-b_n)}{p_1+p_n} \sum_{i=1}^{n} p_i \,. \tag{2}$$

Equality holds if and only if $a_2=a_3=\cdots=a_{n-1}=\frac{a_1+a_n}{2}$, or $b_2=b_3=\cdots=b_{n-1}=\frac{b_1+b_n}{2}$. If the pairs $\left(M_1(a,p;I-I_2),M_1(a,p;I_2)\right)$ and $\left(M_1(b,p;I-I_2),M_1(b,p;I_2)\right)$ are oppositely ordered, then the sense of (2) reverses.

The inequality (2) is generalization of the inequality proven in [16]. On the other hand, it is a corollary of one more general result proven in [5].

Lemma 3. [2] *Let $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ be real numbers. Then*

$$\sum_{i=1}^{n} a_i \geq n \left(\prod_{i=1}^{n} a_i \right)^{1/n} + \left(\sqrt{a_1}-\sqrt{a_n}\right)^2 \,, \tag{3}$$

with equality if $a_2=\cdots=a_{n-1}=\sqrt{a_1 a_n}$.

A lower bound for $Kf(G)$ that depends on all vertex degrees of G was determined in [18].

Lemma 4. [18] *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$Kf(G) \geq -1 + (n-1) \sum_{i=1}^{n} \frac{1}{d_i} \,. \tag{4}$$

Equality holds if and only if either $G \cong K_n$, or $G \cong K_{t,n-t}$, $1 \leq t \leq \left[\frac{n}{2}\right]$, or $G \in \Gamma_d$.

3. Main results

In the next theorem we determine a relationship between the Kirchhoff index and the first Zagreb index, $M_1(G)$.

Theorem 1. *Let G be a connected graph of order $n \geq 4$ and size m with p, $0 \leq p \leq n-2$, pendant vertices. If G is a d-regular graph, $2 \leq d \leq n-1$, then*

$$Kf(G) \geq \frac{n(n-1)-d}{d} \,, \tag{5}$$

with equality if and only if $G \cong K_n$, or $G \in \Gamma_d$. If $d_i \neq \Delta$ for at least one i, $2 \leq i \leq n-p$, then

$$Kf(G) \geq (n-1)\left(p - \frac{1}{n-1} + \frac{1}{\Delta}\left(n-p+\frac{(n\Delta-2m-p(\Delta-1))^2}{2m\Delta - M_1(G) - p(\Delta-1)} \right) \right) \,. \tag{6}$$

Equality holds if and only if $G \cong K_{t,n-t}$, $2 \leq t < \frac{n}{2}$.

Proof. Let p, $0 \le p \le n-2$, be an integer. Then, for any real r, $r \le 0$ or $r \ge 1$, the inequality (1) can be observed in a following form

$$\left(\sum_{i=1}^{n-p} p_i\right)^{r-1} \sum_{i=1}^{n-p} p_i a_i^r \ge \left(\sum_{i=1}^{n-p} p_i a_i\right)^r . \tag{7}$$

For $r=2$, $p=0$, $p_i = \frac{1}{d_i}$, $a_i = d_i$, $i=1,2,\dots,n$, from (7), that is (1), we obtain

$$\sum_{i=1}^{n} \frac{1}{d_i} \ge \frac{n^2}{2m} .$$

Now, from the above and (4) we obtain that

$$Kf(G) \ge \frac{n^2(n-1)-2m}{2m} . \tag{8}$$

If G is d-regular, $2 \le d \le n-1$, from (8) and identity $2m = nd$, we arrive at (5). In [15] (see also [12]) it was proven that equality in (5) holds if and only if $G \cong K_n$, or $G \in \Gamma_d$.
For $r=2$, $p_i = \frac{\Delta - d_i}{d_i}$, $a_i = d_i$, $i = 1,2,\dots,n-p$, the inequality (7) becomes

$$\sum_{i=1}^{n-p} \frac{\Delta - d_i}{d_i} \sum_{i=1}^{n-p} (\Delta - d_i) d_i \ge \left(\sum_{i=1}^{n-p} (\Delta - d_i)\right)^2 . \tag{9}$$

Since

$$\sum_{i=1}^{n-p} \frac{\Delta - d_i}{d_i} = \Delta \sum_{i=1}^{n-p} \frac{1}{d_i} - n + p ,$$
$$\sum_{i=1}^{n-p} (\Delta - d_i) d_i = \Delta \sum_{i=1}^{n-p} d_i - \sum_{i=1}^{n-p} d_i^2 = 2m\Delta - M_1(G) - p(\Delta - 1) ,$$
$$\sum_{i=1}^{n-p} (\Delta - d_i) = n\Delta - 2m - p(\Delta - 1) ,$$

from the above identities and (9) we obtain

$$\left(\Delta \sum_{i=1}^{n-p} \frac{1}{d_i} - n + p\right) (2m\Delta - M_1(G) - p(\Delta - 1)) \ge (n\Delta - 2m - p(\Delta - 1))^2 . \tag{10}$$

If $d_i = \Delta$, for every i, $i = 1,2,\dots,n-p$, then in (10) equality occurs. Therefore, without affecting the generality, assume that $d_i \ne \Delta$ for at least one i, $2 \le i \le n-p$. In that case

$$2m\Delta - M_1(G) - p(\Delta - 1) > 0 ,$$

and hence from (10) we get

$$\sum_{i=1}^{n-p}\frac{1}{d_i}\geq\frac{1}{\Delta}\left(n-p+\frac{(n\Delta-2m-p(\Delta-1))^2}{2m\Delta-M_1(G)-p(\Delta-1)}\right). \tag{11}$$

On the other hand, if G has p, $0\leq p\leq n-2$, pendant vertices, the inequality (4) can be considered in the following way

$$Kf(G)\geq -1+(n-1)\sum_{i=1}^{n}\frac{1}{d_i}=(n-1)\left(p-\frac{1}{n-1}+\sum_{i=1}^{n-p}\frac{1}{d_i}\right). \tag{12}$$

Now, from the above and inequality (11) we arrive at (6).
Equality in (12) holds if and only if either $G\cong K_n$, or $G\cong K_{t,n-t}$, $1\leq t\leq\left[\frac{n}{2}\right]$, or $G\in\Gamma_d$. Since $d_i\neq\Delta$, for at least one i, $2\leq i\leq n-p$ ($p\neq n-1$), equality in (11) holds if $G\cong K_{t,n-t}$, $2\leq t<\frac{n}{2}$. This implies that equality in (6) holds if $G\cong K_{t,n-t}$, $2\leq t<\frac{n}{2}$. □

Remark 1. The inequality (5) was proven in [15], whereas (8) in [12].
If the condition for pendant vertices is omitted, then similarly as in Theorem 1, when $d_i\neq\Delta$, for at least one i, $2\leq i\leq n$, it was proven that [9]

$$Kf(G)\geq\frac{n(n-1)-\Delta}{\Delta}+\frac{(n-1)(n\Delta-2m)^2}{\Delta(2m\Delta-M_1(G))},$$

with equality if and only if $G\cong K_{t,n-t}$, $2\leq t<\frac{n}{2}$.

Corollary 1. *Let G be a connected graph of order $n\geq 3$ and size m, with p, $0\leq p\leq n-1$, pendant vertices. Then*

$$Kf(G)\geq (n-1)p+\frac{(n-1)(n-p)^2-(2m-p)}{2m-p}. \tag{13}$$

Equality holds if and only if either $G\cong K_n$, or $G\cong K_{1,n-1}$, or $G\in\Gamma_d$.

Proof. For $r=2$, $p_i=1$, $a_i=d_i$, $i=1,2,\ldots,n-p$, the inequality (7) becomes

$$\sum_{i=1}^{n-p}1\sum_{i=1}^{n-p}d_i^2\geq\left(\sum_{i=1}^{n-p}d_i\right)^2,$$

That is

$$(n-p)(M_1(G)-p)\geq (2m-p)^2. \tag{14}$$

From the above and inequality (6) we obtain (13). □

In the next theorem we determine a lower bound on $Kf(G)$ in terms of number of vertices, edges, pendant vertices, and vertex degrees $\Delta_2=d_2$ and $\delta_p=d_{n-p}$.

Theorem 2. *Let G be a connected graph of order $n \geq 4$ and size m with p pendant vertices. If $p = n-1$, then*

$$Kf(G) = (n-1)^2 .$$

If $0 \leq p \leq n-2$, then

$$Kf(G) \geq (n-1)\left(\frac{n-1-\Delta}{\Delta(n-1)} + p + \frac{(n-p-1)^2}{2m-\Delta-p} + \frac{(\Delta_2-\delta_p)^2}{\Delta_2\delta_p(\Delta_2+\delta_p)}\right) . \tag{15}$$

Equality holds if and only if $G \cong K_n$, or $G \in \Gamma_d$.

Proof. The inequality (2) can be considered as

$$\sum_{i=2}^{n-p} p_i \sum_{i=2}^{n-p} p_i a_i b_i - \sum_{i=2}^{n-p} p_i a_i \sum_{i=2}^{n-p} p_i b_i \geq \frac{p_2 p_{n-p}(a_2-a_{n-p})(b_2-b_{n-p})}{p_2+p_{n-p}} \sum_{i=2}^{n-p} p_i ,$$

where p is an integer such that $0 \leq p \leq n-2$.
For $p_i = d_i$, $a_i = b_i = \frac{1}{d_i}$, $i = 2,3,\dots,n-p$, the above inequality transforms into

$$\sum_{i=2}^{n-p} d_i \sum_{i=2}^{n-p} \frac{1}{d_i} - \left(\sum_{i=2}^{n-p} 1\right)^2 \geq \frac{\Delta_2\delta_p\left(\frac{1}{\delta_p}-\frac{1}{\Delta_2}\right)^2}{\Delta_2+\delta_p} \sum_{i=2}^{n-p} d_i ,$$

that is

$$(2m-\Delta-p)\sum_{i=2}^{n-p}\frac{1}{d_i} \geq (n-p-1)^2 + \frac{(\Delta_2-\delta_p)^2}{\Delta_2\delta_p(\Delta_2+\delta_p)}(2m-\Delta-p) . \tag{16}$$

Since $m \geq n-1$ and $0 \leq p \leq n-2$, the following is valid

$$2m \geq 2(n-1) = n-1+n-1 > \Delta + n - 2 \geq \Delta + p ,$$

that is

$$2m - \Delta - p > 0 .$$

Now, from the above and inequality (16) we have that

$$\sum_{i=2}^{n-p}\frac{1}{d_i} \geq \frac{(n-p-1)^2}{2m-\Delta-p} + \frac{(\Delta_2-\delta_p)^2}{\Delta_2\delta_p(\Delta_2+\delta_p)} . \tag{17}$$

Since G has p pendant vertices, the inequality (4) can be considered as

$$\begin{aligned} Kf(G) &\geq -1 + (n-1)\sum_{i=1}^{n}\frac{1}{d_i} = -1 + (n-1)\left(\sum_{i=2}^{n-p}\frac{1}{d_i} + \frac{1}{\Delta} + p\right) = \\ &= (n-1)\left(\frac{n-1-\Delta}{\Delta(n-1)} + p + \sum_{i=2}^{n-p}\frac{1}{d_i}\right) . \end{aligned} \tag{18}$$

From the above and inequality (17) we obtain (15).
Equality in (18) holds if and only if either $G \cong K_n$, or $G \cong K_{t,n-t}$, $1 \leq t \leq \left[\frac{n}{2}\right]$, or $G \in \Gamma_d$. Equality in (17) holds if and only if $\Delta_2 = d_2 = \cdots = d_{n-p} = \delta_p$, or $\frac{1}{d_3} = \cdots = \frac{1}{d_{n-p-1}} = \left(\frac{1}{\Delta_2} + \frac{1}{\delta_p}\right)/2$, $0 \leq p \leq n-2$. Since G is connected and $G \ncong K_{1,n-1}$, $p \neq n-1$, equality in (15) holds if and only if $G \cong K_n$ or $G \in \Gamma_d$. □

Corollary 2. *Let G be a connected graph of order $n \geq 4$ and size m with p pendant vertices. If $G \cong K_{1,n-1}$, then*

$$Kf(G) = (n-1)^2 .$$

If $0 \leq p \leq n-2$, then

$$Kf(G) \geq (n-1)\left(\frac{n-1-\Delta}{\Delta(n-1)} + p + \frac{(n-p-1)^2}{2m-\Delta-p}\right) .$$

Equality holds if and only if $G \cong K_n$, or $G \in \Gamma_d$.

Remark 2. If the condition for pendant vertices is omitted, then similarly as in Theorem 2, the following results can be proven.
Let G be a connected graph with $n \geq 3$ vertices and m edges. Then

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + \frac{(n-1)^3}{2m-\Delta} + \frac{(n-1)(\Delta_2-\delta)^2}{\Delta_2\delta(\Delta_2+\delta)} . \tag{19}$$

Equality holds if and only if either $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$.
Let G be a connected graph with $n \geq 3$ vertices and m edges. Then

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + \frac{(n-1)^3}{2m-\Delta} . \tag{20}$$

Equality holds if and only if either $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$.
The inequality (20) was proven in [12].

Theorem 3. *Let G be a connected graph of order $n \geq 4$ with p, $0 \leq p \leq n-2$, pendant vertices. Then*

$$Kf(G) \geq (n-1)\left(\frac{n-1-\Delta}{\Delta(n-1)} + p + (n-p-1)\left(\frac{\Delta}{\det D}\right)^{\frac{1}{n-p-1}} + \frac{(\sqrt{\Delta_2}-\sqrt{\delta_p})^2}{\Delta_2\delta_p}\right) . \tag{21}$$

Equality holds when $G \cong K_n$, or $G \in \Gamma_d$.

Proof. The inequality (3) can be considered as

$$\sum_{i=2}^{n-p} a_i \geq (n-p-1)\left(\prod_{i=2}^{n-p} a_i\right)^{\frac{1}{n-p-1}} + (\sqrt{a_2}-\sqrt{a_{n-p}})^2 ,$$

where p is an integer such that $0 \leq p \leq n-2$.
For $a_i = \frac{1}{d_i}$, $i = 2, 3, \ldots, n-p$, the above inequality becomes

$$\sum_{i=2}^{n-p} \frac{1}{d_i} \geq (n-p-1)\left(\prod_{i=2}^{n-p} \frac{1}{d_i}\right)^{\frac{1}{n-p-1}} + \left(\frac{1}{\sqrt{\delta_p}} - \frac{1}{\sqrt{\Delta_2}}\right)^2 ,$$

that is

$$\sum_{i=2}^{n-p} \frac{1}{d_i} \geq (n-p-1)\left(\frac{\Delta}{\det D}\right)^{\frac{1}{n-p-1}} + \frac{(\sqrt{\Delta_2}-\sqrt{\delta_p})^2}{\Delta_2\delta_p}\,. \tag{22}$$

From the above and inequality (18) we obtain (21).
Equality in (22) holds when $\Delta_2 = d_2 = \cdots = d_{n-p} = \delta_p$, or $d_3 = \cdots = d_{n-p-1} = \sqrt{\Delta_2\delta_p}$. Equality in (18) holds if and only if either $G \cong K_n$, or $G \cong K_{t,n-t}$, $1 \leq t \leq \left[\frac{n}{2}\right]$, or $G \in \Gamma_d$. This implies that equality in (21) holds if $G \cong K_n$, or $G \in \Gamma_d$. □

Corollary 3. *Let G be a connected graph of order $n \geq 3$ with p, $0 \leq p \leq n-2$, pendant vertices. Then*

$$Kf(G) \geq (n-1)\left(\frac{n-1-\Delta}{\Delta(n-1)} + p + (n-p-1)\left(\frac{\Delta}{\det D}\right)^{\frac{1}{n-p-1}}\right).$$

Equality holds when $G \cong K_n$, or $G \in \Gamma_d$.

Remark 3. If the condition for pendant vertices is omitted, then similarly as in Theorem 3, the following results can be proven.
Let G be a connected graph with $n \geq 4$ vertices. Then

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + (n-1)^2\left(\frac{\Delta}{\det D}\right)^{\frac{1}{n-1}} + (n-1)\frac{(\sqrt{\Delta_2}-\sqrt{\delta})^2}{\Delta_2\delta}\,. \tag{23}$$

Equality holds if either $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$.
Let G be a connected graph with $n \geq 3$ vertices. Then

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + (n-1)^2\left(\frac{\Delta}{\det D}\right)^{\frac{1}{n-1}}\,. \tag{24}$$

Equality holds if and only if either $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$.

Acknowledgement. Research supported by the Serbian Ministry of Education, Science and Technological Development, grant No. 451-03-68/2022-14/ 200102

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] Ş.B. Bozkurt Altındağ, M. Matejić, I. Milovanović, and E. Milovanović, *Improved bounds for Kirchhoff index of graphs*, Commun. Comb. Optim. **8** (2023), no. 1, 243–251
https://doi.org/10.22049/cco.2022.27492.1275.

[2] V. Cirtoaje, *The best lower bound depended on two fixed variables for Jensen's inequality with ordered variables*, J. Ineq. Appl. **2010** (2010), Article ID: 128258 https://doi.org/10.1155/2010/128258.

[3] K.C. Das and I. Gutman, *On Laplacian energy, Laplacian-energy-like invariant and Kirchhoff index of graphs*, Linear Algebra Appl **554** (2018), 170–184 https://doi.org/10.1016/j.laa.2018.05.030.

[4] ———, *Comparing Laplacian energy and Kirchhoff index*, MATCH Commun. Math. Comput. Chem. **81** (2019), no. 2, 419–424.

[5] I. Gutman, K.C. Das, B. Furtula, E. Milovanović, and I. Milovanović, *Generalizations of Szőkefalvi Nagy and Chebyshev inequalities with applications in spectral graph theory*, Applied Math. Comput. **313** (2017), 235–244 https://doi.org/10.1016/j.amc.2017.05.064.

[6] I. Gutman and B. Mohar, *The quasi-Wiener and the Kirchhoff indices coincide*, J. Chem. Inf. Comput. Sci. **36** (1996), no. 5, 982–985 https://doi.org/10.1021/ci960007t.

[7] J.L.W.V. Jensen, *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Math. **30** (1906), no. 1, 175–193 https://doi.org/10.1007/BF02418571.

[8] D.J. Klein and M. Randić, *Resistance distance*, J. Math. Chem. **12** (1993), no. 1, 81–95 https://doi.org/10.1007/BF01164627.

[9] M. Matejić, E. Milovanović, P.D. Milošević, and I. Milovanović, *A note on the Kirchhoff index of graphs*, Open J. Discr. Appl. Math. **2** (2019), no. 3, 1–6 https://doi.org/10.30538/psrp-odam2019.0017.

[10] E. Milovanović, I. Milovanović, and M. Matejić, *On relation between the Kirchhoff index and Laplacian-energy-like invariant of graphs*, Math. Inter. Research **2** (2017), no. 2, 141–154 https://doi.org/10.22052/mir.2017.85687.1063.

[11] I. Milovanović and E. Milovanović, *Bounds of Kirchhoff and degree Kirchhoff indices*, Bounds in Chemical graph theory – Mainstreams (I. Gutman, B. Furtula, K.C. Das, E. Milovanović, and I. Milovanović, eds.), Univ. Kragujevac, Kragujevac, 2017, pp. 93–119.

[12] ———, *On some lower bounds of the Kirchhoff index*, MATCH Commun. Math. Comput. Chem. **78** (2017), no. 1, 169–180.

[13] I. Milovanović, E. Milovanović, E. Glogić, and M. Matejić, *On Kirchhoff index, Laplacian energy and their relations*, MATCH Commun. Math. Comput. Chem. **81** (2019), no. 2, 405–418.

[14] D.S. Mitrinović, J. Pečarić, and A.M. Fink, *Classical and new inequalities in analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[15] J.L. Palacios, *Some additional bounds for the Kirchhoff index*, MATCH Commun. Math. Comput. Chem. **75** (2016), no. 2, 365–372.

[16] J.V. Szökefalvi Nagy, *Über algebraishe gleichungen mit lauter reellen wurzeln*, Jahresbericht der deutschen Mathematiker – Vereingung **24** (1918), 37–43.

[17] H. Wiener, *Structural determination of paraffin boiling points*, J. Amer. Chem.

Soc. **69** (1947), no. 1, 17–20
https://doi.org/10.1021/ja01193a005.

[18] B. Zhou and N. Trinajstić, *A note on Kirchhoff index*, Chem. Phys. Lett. **455** (2008), no. 1-3, 120–123
https://doi.org/10.1016/j.cplett.2008.02.060.

[19] H.Y. Zhu, D.J. Klein, and I. Lukovits, *Extensions of the Wiener number*, J. Chem. Inf. Comput. Sci. **36** (1996), no. 3, 420–428
https://doi.org/10.1021/ci950116s.