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Roman domination number of signed graphs

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Abstract: A function $f : V \to \{0,1,2\}$ on a signed graph $S = (G,\sigma)$ where $G = (V,E)$ is a Roman dominating function (RDF) if $f(N[v]) = f(v) + \sum_{u \in N(v)} \sigma(uv) f(u) \geq 1$ for all $v \in V$ and for each vertex v with $f(v) = 0$ there is a vertex u in $N^+(v)$ such that $f(u) = 2$. The weight of an RDF f is given by $\omega(f) = \sum_{v \in V} f(v)$ and the minimum weight among all the RDFs on S is called the Roman domination number $\gamma_R(S)$. Any RDF on S with the minimum weight is known as a $\gamma_R(S)$-function. In this article we obtain certain bounds for γ_R and characterise the signed graphs attaining small values for γ_R.

Keywords: Signed graphs, Dominating function, Roman dominating function

AMS Subject classification: 05C69, 05C22

1. Introduction

A graph with its edges designated as positive or negative is called a signed graph. Formally, a signed graph is an ordered pair $S = (G,\sigma)$ where G is called the underlying graph and σ is a function from the edge set $E(G)$ to the set $\{+,-\}$ known as the *signing* of G or the *signature* of S. The concept of signed graphs was introduced by Harary [3, 6] in the context of modelling social psychological processes. Signed graphs have been studied extensively by researchers [13].

The positive and negative edges of a signed graph are usually depicted using solid and dashed lines. For any vertex v of S, $N^+(v) = \{u \in N(v) |\ \sigma(uv) = +\}$ and $N^-(v) = \{u \in N(v) |\ \sigma(uv) = -\}$. Further, $d^+(v) = |N^+(v)|$ and $d^-(v) = |N^-(v)|$. The maximum(minimum) value of $d^+(v)$ among all the vertices $v \in V(S)$ is denoted

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by $\Delta^+(S)(\delta^+(S))$, while $\Delta^-(S)$ and $\delta^-(S)$ denote the maximum and minimum value of $d^-(v)$ respectively. Unless mentioned otherwise, the underlying graph $G = (V, E)$ of a signed graph S is always simple and connected. The neighbourhood of a vertex v is $N_G(v)$ or $N(v)$ and the degree of v in G is denoted by $d_G(v)$. The closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. For definitions and notations in graph theory we refer [12].

Roman dominating functions is an interesting class of dominating functions which is in the literature for over more than 15 years [2, 4, 5, 7, 10, 11]. The concept of Roman domination was implicitly given in [11]. It was Emperor Constantine's defence strategy to assign two armies at any region which is adjacent to a region that is defenceless. Later Cockayne et al. [5] explicitly stated that a Roman dominating function is a labelling of the vertices of a graph with the labels $\{0, 1, 2\}$ in such a way that every vertex with label 0 is adjacent to at least one vertex with label 2. Joseph and Joseph [9] have examined the concept of Roman domination in the realm of signed graphs. The study of Roman dominating functions in signed graphs assumes importance as in any network the nature of the edges connecting the vertices need not be equally efficient or strong. A function $f : V \to \{0, 1, 2\}$ on a signed graph $S = (G, \sigma)$ is a *Roman dominating function(RDF)* if $f(N[v]) = f(v) + \sum_{u \in N(v)} \sigma(uv) f(u) \geq 1$ for all $v \in V$ and for each vertex v with $f(v) = 0$ there is a vertex u in $N^+(v)$ with $f(u) = 2$. The weight of an RDF f is given by $\omega(f) = \sum_{v \in V} f(v)$ and the minimum weight among all the *RDF*s on S is called the Roman domination number $\gamma_R(S)$. Any RDF on S with the minimum weight is known as a $\gamma_R(S)$-function.

It is to be observed that the concept of domination in signed graphs have been viewed by researchers from different perspectives [1, 8]. We consider the concept of domination similar to the general concept of domination where a dominating set of a graph G is considered as a subset D of the vertex set $V(G)$ such that every vertex of G is either in D or any vertex not in D is adjacent to a vertex in D. In the context of signed graphs domination in terms of adjacency is relevant when there is a positive edge between the vertices under consideration. Therefore a set $D \subseteq V(S)$ of a signed graph $S = (G, \sigma)$ is called a dominating set [9] if for each vertex v in $V \setminus D$, there exists a vertex u in $N^+(v) \cap D$. The minimum cardinality among all the dominating sets on S is called the domination number $\gamma(S)$.

The functions $f : V \to \{0, 1, 2\}$ on a signed graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. There is always a one-one correspondence between these functions and the ordered partitions induced by them and we may write $f = (V_0, V_1, V_2)$. Moreover, whenever f is an RDF on a signed graph, $V_1 \cup V_2$ is a dominating set of the signed graph. This implies that,

$$\gamma(S) \leq |V_1| + |V_2|$$

and if f is a $\gamma_R(S)$-function then,

$$\gamma(S) \leq \gamma_R(S).$$

We will be using the notation (V_0, V_1, V_2) throughout this paper to depict a function $f : V \to \{0, 1, 2\}$. In this article we obtain some bounds for the Roman domination number and characterise the signed graphs having small values for their Roman domination number. We also characterise signed graphs with Roman domination number equal to the order of the signed graph.

2. Results

We have the following bound for the Roman domination number of a certain class of signed graphs in terms of the domination number.

Theorem 1. *Let S be a signed graph and D be a minimum dominating set in S such that any negative edge in S is only between the vertices in $V \setminus D$. Then*

$$\gamma_R(S) \leq 2\gamma(S).$$

Proof. Since D is a minimum dominating set, $\gamma(S) = |D|$. Define a function $f : V \to \{0, 1, 2\}$ by $f = (V \setminus D, \emptyset, D)$. Then, clearly $f(N[v]) \geq 1$ for all $v \in V$. Further, each vertex v in $V_0 = V \setminus D$ is adjacent to a vertex u in $V_2 = D$ such that $u \in N^+(v)$. Therefore f is an RDF on S and $\omega(f) = 2|D| = 2\gamma(S)$. Now we can easily see that, since f is an RDF,

$$\gamma_R(S) \leq \omega(f) = 2\gamma(S).$$

□

Next we obtain a bound for the Roman domination number of signed graphs S with $d^-(v) \leq d^+(v)$ for all $v \in V(S)$ in terms of the order.

Theorem 2. *If $S = (G, \sigma)$ is a signed graph of order n admitting an RDF such that $d^-(v) \leq d^+(v)$ for all $v \in V$, then $\gamma_R(S) \leq n$.*

Proof. We define a function $f : V \to \{0, 1, 2\}$ on S by $f = (\emptyset, V, \emptyset)$. We claim that f is an RDF on S. Since $V_0 = \emptyset$, it is enough to prove that $f(N[v]) \geq 1$ for all $v \in V$. Now, for any vertex $v \in V$,

$$\begin{aligned} f(N[v]) &= f(v) + \sum_{u \in N(v)} \sigma(uv) f(u) \\ &= f(v) + \sum_{u \in N^+(v)} f(u) - \sum_{u \in N^-(v)} f(u) \\ &= f(v) + d^+(v) - d^-(v) \\ &= 1 + d^+(v) - d^-(v) \geq 1, \end{aligned}$$

since by our hypothesis $d^-(v) \le d^+(v)$ for all $v \in V$. Therefore f is an RDF on S and $\omega(f) = |V| = n$. The function f being an RDF,

$$\gamma_R(S) \le \omega(f) = n.$$

□

Now we examine the signed graphs whose Roman domination number is less than 5. Note that for any signed graph S, $\gamma_R(S) = 1$ if and only if the underlying graph is K_1. Next let us examine signed graphs with $\gamma_R(S) = 2$.

Theorem 3. *Let S be a signed graph of order $n \ge 2$. Then $\gamma_R(S) = 2$ if and only if there exists a vertex v with $d^+(v) = n - 1$.*

Proof. We first prove the sufficiency part. Suppose that there exists a vertex $v \in V$ such that $d^+(v) = n - 1$. Observe that $\gamma_R(S) > 1$.
Now define the function $f = (V \setminus \{v\}, \emptyset, \{v\})$ on S. We claim that f is a $\gamma_R(S)$-function. Observe that all the vertices in $V \setminus \{v\}$ are adjacent to the vertex v by positive edges. Further, $f(N[v]) = 2$ and $f(N[w]) = 2$ for all $w \in V \setminus \{v\}$. Thus f is an RDF on S. Since $V_0 = V \setminus \{v\}$, $V_1 = \emptyset$ and $V_2 = \{v\}$, $\omega(f) = 2|V_2| = 2$ and therefore $\gamma_R(S) = 2$.

Conversely, suppose that $\gamma_R(S) = 2$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(S)$-function, then either $|V_1| = 2$, $|V_2| = 0$ or $|V_1| = 0$, $|V_2| = 1$.

Case 1: $|V_1| = 2$, $|V_2| = 0$. Since $V_2 = \emptyset$, $V_0 = \emptyset$. This implies that $V = V_1$ and hence S is a signed graph with 2 vertices and a positive edge.

Case 2: $|V_1| = 0$, $|V_2| = 1$. Then $|V_0| = n - 1$. For all the remaining vertices in $u \in V \setminus V_2$, $f(u) = 0$ and are adjacent to the vertex in V_2 by positive edges. This shows that, if $V_2 = \{v\}$ then $d^+(v) = n - 1$.

From the above two cases we can conclude that S contains a vertex v with $d^+(v) = n - 1$.

□

The following result characterises the signed graphs for which $\gamma_R(S) = 3$.

Theorem 4. *For a signed graph S of order $n \ge 3$, $\gamma_R(S) = 3$ if and only if $\Delta^+(S) = n-2$ and there exists a vertex v with $d^+(v) = n - 2$ and $d^-(v) = 0$.*

Proof. To prove the sufficiency part suppose that $\Delta^+(S) = n - 2$ and there is a vertex v with $d^+(v) = n-2$ and $d^-(v) = 0$. Then $N(v) = N^+(v)$ and $|V \setminus N^+[v]| = 1$.

We define the function $f = (N^+(v), V \setminus N^+[v], \{v\})$ on S and claim that f is a $\gamma_R(S)$-function. Here, $V_0 = N^+(v)$, $V_1 = V \setminus N^+[v]$ and $V_2 = \{v\}$ and clearly all the vertices in V_0 are adjacent to v by positive edges. We can see that, $f(N[w]) = 2 \pm 1 \geq 1$ for all $w \in N^+(v)$, $f(N[v]) = 2$ and $f(N[u]) = 1, u \in V \setminus N^+[v]$. Thus f is an RDF on S and $\omega(f) = |V_1| + 2|V_2| = 3$. Since there is no vertex v of S with $|N^+(v)| = n - 1$, $\gamma_R(S) > 2$. Therefore $\gamma_R(S) = 3$.

To prove the converse part, suppose that $\gamma_R(S) = 3$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(S)$-function. Then either $|V_1| = 3$, $|V_2| = 0$ or $|V_1| = 1$, $|V_2| = 1$.

Case 1: $|V_1| = 3$, $|V_2| = 0$. Then $|V_0| = 0$ and $V_1 = V$ i.e. $|V| = 3$. But, for no signed graph on 3 vertices has a $\gamma_R(S)$-function f with $|V_0| = 0$, $|V_1| = 3$ and $|V_2| = 0$.

Case 2: $|V_1| = 1$, $|V_2| = 1$. Then $|V_0| = n - 2$ and all the vertices in V_0 must be adjacent to the vertex in V_2 by positive edges. If $V_2 = \{v\}$, then $d^+(v) = n - 2$.
Now we show that, $d^-(v) = 0$. Suppose on the contrary that $d^-(v) \neq 0$. If $V_1 = \{u\}$, then v must be adjacent to u and $\sigma(uv) = -1$. But, then we can see that $f(N[u]) = f(u) - f(v) = -1 < 1$, which is a contradiction to the fact that f is an RDF on S. Therefore our assumption is wrong and hence $d^-(v) = 0$.

Thus from the above two cases we can see that whenever $\gamma_R(S) = 3$ there exists a vertex v with $d^+(v) = n - 2$ and $N^-(v) = \emptyset$. Also note that, $\Delta^+(S) = n - 2$. For, if $\Delta^+(S) = n - 1$, then by Theorem 3, $\gamma_R(S) = 2$ which is a contradiction. Hence the result. □

We can find that, there are no signed graphs of order $n \leq 3$ and $\gamma_R = 4$. The only signed graphs on 4 vertices with $\gamma_R = 4$ are the signed path and cycle with alternating positive and negative edges such that a pendant edge of the path is positive. Now we characterise the signed graphs of order $n \geq 5$ with $\gamma_R = 4$.

Theorem 5. *A signed graph S of order $n \geq 5$ have $\gamma_R(S) = 4$ if and only if $\Delta^+(S) \leq n-2$, there is no vertex x of S with $d^+(x) = n-2$, $d^-(x) = 0$ and satisfies any one of the following:*

(i) there is $\gamma(S)$-set $D = \{u, v\}$ such that $d^-(u) = d^-(v) = 0$

(ii) there is a vertex $u \in V$ with $d^+(u) = n - 3$, $d^-(u) = 0$ such that the two vertices in $V \setminus N[u]$ are not adjacent by negative edge and $|N^-(w) \cap V \setminus N[u]| \leq 1$ for all $w \in N(u)$.

Proof. *Sufficiency:* Assume that $\Delta^+(S) \leq n - 2$, there is no vertex x of S with $d^+(x) = n - 2$ and $d^-(x) = 0$, and S satisfies any one of the conditions (i), (ii). Then by Theorem 3 and 4, $\gamma_R(S) > 3$.

Case 1: S contains a $\gamma(S)$-set $D = \{u, v\}$ such that $d^-(u) = d^-(v) = 0$. We define the function $f = (V \setminus D, \emptyset, D)$ on S, where $V_0 = V \setminus D$, $V_1 = \emptyset$, $V_2 = D$. Since D is

a dominating set of S, every vertex in $V \setminus D$ is adjacent to at least one vertex in D by a positive edge. Therefore, for any vertex v with $f(v) = 0$ there exists a vertex $u \in N^+(v)$ such that $f(u) = 2$ and $f(N[w]) \geq 2$ for all $w \in V \setminus D$. Since for any $x \in D$ is either adjacent to a vertex in $V \setminus D$ or a vertex in D itself by positive edges only, $f(N[x]) \geq 2$ for all $x \in D$.
Thus f is an RDF on S. Observe that $\omega(f) = 2|D| = 4$ and since $\gamma_R(S) > 3$, we get $\gamma_R(S) = 4$.

Case 2: There is a vertex $u \in V$ with $d^+(u) = n - 3$, $d^-(u) = 0$ such that the two vertices in $V \setminus N[u]$ are not adjacent by negative edge and
$|N^-(w) \cap V \setminus N[u]| \leq 1$ for all $w \in N(u)$.

Define the function $f = (N(u), V \setminus N[u], \{u\})$ on S, where $V_0 = N(u)$, $V_1 = V \setminus N[u]$, $V_2 = \{u\}$. Clearly, all the vertices in $N(u)$ are adjacent to u by positive edges. Now, by our assumption that a vertex in $N(u)$ is adjacent to at most one vertex in $V \setminus N[u]$ by negative edge and the definition of f, $f(N[w]) \geq 2 \pm 1 \geq 1$ for all $w \in N(u)$. Now observe that, since a vertex in $V \setminus N[u]$ is either adjacent to a vertex in $N(u)$ or a vertex in $V \setminus N[u]$, $f(N[x]) \geq 1$ for all $x \in V \setminus N[u]$. Finally, since $N(u) = N^+(u)$, $f(N[u]) = 2$. This proves that f is an RDF on S. Therefore $\gamma_R(S) = 4$, since $\gamma_R(S) > 3$ and $\omega(f) = |V \setminus N[u]| + 2|\{u\}| = 4$.

Necessity: Suppose that $\gamma_R(S) = 4$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(S)$-function. Then, either $|V_1| = 4$, $|V_2| = 0$ or $|V_1| = 2, |V_2| = 1$ or $|V_1| = 0$, $|V_2| = 2$.

Case 1: $|V_1| = 4$, $|V_2| = 0$. Then $|V_0| = 0$ and therefore $V = V_1$. This implies that S is a signed graph on 4 vertices, which is not possible.

Case 2: $|V_1| = 0$, $|V_2| = 2$. We know that $V_1 \cup V_2 = V_2$ is a dominating set in S. Since there is no vertex x with $d^+(x) = n - 1$, $\gamma(S) > 1$. This shows that V_2 is a $\gamma(S)$-set.
Let $V_2 = \{u, v\}$, we claim that $d^-(u) = d^-(v) = 0$. Suppose that $d^-(u) \neq 0$, then $v \in N^-(u)$ or there is a vertex $w \in N^-(u) \cap V_0$. Then $f(N[u]) < 1$ or $f(N[w]) < 1$, which is a contradiction. Therefore our assumption is wrong and $d^-(u) = 0$. Similarly, we can show that $d^-(v) = 0$. Therefore, $V_2 = \{u, v\}$ is a $\gamma(S)$-set such that $d^-(u) = d^-(v) = 0$.

Case 3: $|V_1| = 2$, $|V_2| = 1$. Let $V_2 = \{u\}$. Clearly, $|V_0| = n-3$ and all the vertices in V_0 are adjacent to the vertex in V_2 by positive edges. This implies that $d^+(u) = n-3$. Now we claim that $d^-(u) = 0$. Suppose on the contrary that $d^-(u) \neq 0$, then there exists a vertex $v \in V_1$ such that $v \in N^-(u)$. But observe that, $f(N[v]) \leq 1-2+1 = 0$, which is a contradiction to the fact that f is an RDF on S. Therefore $d^-(u) = 0$.
Now, let $V_1 = \{x, y\}$ and suppose that $\sigma(xy) = -1$. Then, $f(N[x]) = 0$, which is a contradiction and therefore $\sigma(xy) \neq -1$.
It remains to prove that $|N^-(w) \cap (V \setminus N[u])| \leq 1$ for all $w \in N(u)$. On the contrary

suppose that there is a vertex $w \in N(u)$ such that $|N^-(w) \cap (V \setminus N[u])| = 2$. Note that, $V \setminus N[u] = V_1$ and $N(u) = V_0$. This shows that $|N^-(w) \cap V_1| = 2$ and $f(N[w]) = 0$, which is a contradiction and hence our assumption is wrong.
Therefore $u \in V$ is a vertex with $d^+(u) = n - 3$, $d^-(u) = 0$ such that two vertices in $V_1 = V \setminus N[u]$ are not adjacent by a negative edge and $|N^-(w) \cap (V \setminus N[u])| \leq 1$ for all $w \in N(u)$. □

For a signed graph of order n and $\gamma_R = n$, we have the following characterisation.

Theorem 6. *Let S be a signed graph of order n that admits an RDF, then $\gamma_R(S) = n$ if and only if there exists a $\gamma_R(S)$-function $f = (V_0, V_1, V_2)$ such that $|V_0| = |V_2|$.*

Proof. First assume that there exist a $\gamma_R(S)$-function $f = (V_0, V_1, V_2)$ with $|V_0| = |V_2|$. Since (V_0, V_1, V_2) is a partition of the vertex set, $|V_0| + |V_1| + |V_2| = n$. This implies that $|V_1| + 2|V_2| = n$. Therefore $\gamma_R(S) = \omega(f) = |V_1| + 2|V_2| = n$.

Conversely, suppose that $\gamma_R(S) = n$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_R(S)$-function. Then $|V_1| + 2|V_2| = n$, which shows that either $|V_1| = n, |V_2| = 0$ or $|V_1| = 0, |V_2| = \frac{n}{2}$ or $|V_1| = \frac{n}{3}, |V_2| = \frac{n}{3}$. In all the cases $|V_0| = |V_2|$. □

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