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Research Article

Cycle transit function and betweenness

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Abstract: Transit functions are introduced to study betweenness, intervals and convexity in an axiomatic setup on graphs and other discrete structures. Prime example of a transit function on graphs is the well studied interval function of a connected graph. In this paper, we study the Cycle transit function $\mathcal{C}(u, v)$ on graphs which is a transit function derived from the interval function. We study the betweenness properties and also characterize graphs in which the cycle transit function coincides with the interval function. We also characterize graphs where $|\mathcal{C}(u, v) \cap \mathcal{C}(v, w) \cap \mathcal{C}(u, w)| \leq 1$ as an analogue of median graphs.

Keywords: Interval function, Cycle transit function, Betweenness

AMS Subject classification: 05C12, 05C38

1. Introduction

Transit functions are introduced by Mulder [9] to generalize three classical notions, namely convexity, interval and betweenness. The theory of transit functions follows an axiomatic approach to study betweenness.

Formally, a *transit function* on a non-empty set V, is defined as a function $R : V \times V$ to 2^V satisfying the following three axioms:

$(t1)$ $u \in R(u, v)$, for all $u, v \in V$,

$(t2)$ $R(u, v) = R(v, u)$, for all $u, v \in V$,

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$(t3)$ $R(u, u) = \{u\}$, for all $u \in V$.

R is called as a transit function on V. If V is the vertex set of a graph G and R is a transit function on V, then we say that R is a transit function on G. Given a transit function R on a finite set V, a subset W of V is said to be R-convex if $R(u, v) \subseteq V$, for all $u, v \in W$. The collection of all R-convex sets in V is termed as the R-convexity on V, which is a collection of subsets of V closed under intersection and contains the empty set $\emptyset$ and V.
Throughout this paper, we consider only finite, simple and connected graphs. The underlying graph G_R of a transit function R on V is the graph with vertex set V, where two distinct vertices u and v are joined by an edge if and only if $R(u, v) = \{u, v\}$. A $u - v$ *shortest path* in a connected graph $G = (V, E)$ is a $u - v$ path in G of minimal length, in the sense that it contains minimum number of edges. The length of a shortest $u - v$ path P is the standard distance in G. We describe some graph theoretical notions and terminologies required in this paper. Let $G = (V, E)$ be a connected graph and let $u \in V(G)$ then, $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ is the neighborhood of u in G. $N_G[u] = N_G(u) \cup \{u\}$ is the closed neighborhood of u in G and $\delta_G(u) = |N_G(u)|$ is the degree of u in G. A subgraph H of G is called an *isometric* subgraph if the distance $d_H(u, v)$ between any pair of vertices, u, v in H coincides with that of the distance $d_G(u, v)$. H is called an *induced* subgraph if u, v are vertices in H such that uv is an edge in G, then uv must be an edge in H also. A path in G which is induced as a subgraph is an *induced path*. The complete graph on n vertices is denoted by K_n. A graph G is a *geodetic graph*, if any two vertices in G has a unique shortest path between them.
The n- *dimensional hypercube* Q_n is defined as the Cartesian product of two graphs, $Q_n = K_2 \times Q_{n-1}$ where $Q_1 = K_2$. A graph H has an isometric embedding into a graph G if and only if $H \subseteq G$ and for all $u, v \in V(H)$, $d_H(u, v) = d_G(u, v)$. A graph G is a *partial cube* if it can be isometrically embedded into a hypercube Q_n. A *Hamming graph* is the Cartesian product of complete graphs $H_{a_1,\cdots,a_n} = K_{a_1} \times \cdots \times K_{a_n}$. A *partial Hamming graph* is an isometric subgraph of a Hamming graph. For an edge $e = ab \in E$, we define the following

$W_{ab} = \{w \in V : d(w, a) < d(w, b)\}$.

$W = \{w \in V : d(w, a) = d(w, b)\}$.

$U_{ab} = \{w \in W_{ab} : w \textit{ is adjacent to a vertex in } W_{ba}\}$.

$U_{ba} = \{w \in W_{ba} : w \textit{ is adjacent to a vertex in } W_{ab}\}$.

The interval function I_G of a connected graph G is the function $I_G : V \times V \longrightarrow 2^V$ defined as

$$\begin{aligned} I_G(u, v) &= \{w \in V : w \text{ lies on some shortest } u, v - \text{path in } G\} \\ &= \{w \in V : d(u, w) + d(w, v) = d(u, v)\}. \end{aligned}$$

The interval function I_G is the classical instance of a transit function on G. If the graph G is clear from the context, we may simply, denote the interval function I_G by I. It is easy to see that the underlying graph G_I of I is isomorphic to G. The interval function and the corresponding I-convexity, namely the geodesic convexity of a connected graph is an important tool in metric graph theory and extensively studied by several authors including Mulder in [8], Nebeský [11, 12], Chepoi [6] and others.

Nebeský in a series of papers, [11–16], followed by Mulder and Nebeský in [10] obtained an axiomatic characterization of the interval function $I(u, v)$ in terms of a set of first order axioms defined on an arbitrary transit function. The axiomatic characterization of the interval function is further extended to arbitrary graphs including disconnected graphs in [4]. The following five axioms, namely, the transit axioms $(t1)$, $(t2)$ and $(b2)$, $(b3)$, and $(b4)$ are essential in all these characterizations.

$(b2)$ if $x \in R(u, v)$ and $y \in R(u, x)$, then $y \in R(u, v)$,
$(b3)$ if $x \in R(u, v)$ and $y \in R(u, x)$ then $x \in R(y, v)$,
$(b4)$ if $x \in R(u, v)$ then $R(u, x) \cap R(x, v) = \{x\}$.

We quote two more axioms and the Mulder-Nebeský Theorem from [10] characterizing the interval function of a connected graph.

$(s1)$: $R(u, \bar{u}) = \{u, \bar{u}\}, R(v, \bar{v}) = \{v, \bar{v}\}, u \in R(\bar{u}, \bar{v})$ and $\bar{u}, \bar{v} \in R(u, v)$, then $v \in R(\bar{u}, \bar{v})$.

$(s2)$: $R(u, \bar{u}) = \{u, \bar{u}\}, R(v, \bar{v}) = \{v, \bar{v}\}, \bar{u} \in R(u, v)$, $v \notin R(\bar{u}, \bar{v}), \bar{v} \notin R(u, v)$, then $\bar{u} \in R(u, \bar{v})$.

Theorem 1. *[10] Let $R : V \times V \longrightarrow 2^V$ be a function on V, satisfying the axioms $(t1), (t2), (b2), (b3), (b4)$ with the underlying graph G_R and let I be the interval function of G_R. The following statements are equivalent.*

(a) $R = I$.

(b) *R satisfies axioms $(s1)$ and $(s2)$.*

Motivated by the interest on the interval function I of a graph G, one can look at transit functions on G derived from the I. Examples of such transit functions are the functions I_j, I^{Δ} and pre-fiber transit function I_F. The functions I_j and I^{Δ} are introduced in [3] and the function I^{Δ} is studied in detail in [5]. Recently, the transit function I_F and its betweenness properties are discussed in [2]. For the sake of clarity, we define these transit functions on the vertex set V of a connected graph G.

$I_j(u, v) = \{w \in V\colon w$ lies on some $u - v$ path in G of length $\leq d(u, v) + j$, for a positive integer $j\}$.

$I^{\Delta}(u, v) = \{w \in V\colon w$ lies on some shortest $u - v$ path in G or w is adjacent to two adjacent vertices in some shortest $u - v$ path in $G\}$.

$$I_F(u,v) = \{w \in V\colon\ I(u,w) \cap I(w,v) = \{w\}\}.$$

From the definition of these transit functions, it can be observed that $I(u,v) \subseteq I_j(u,v)$, $I_G(u,v) \subseteq I^{\Delta}(u,v)$ and $I(u,v) \subseteq I_F(u,v)$, for every $u,v \in V$.
In this paper, we attempt to study another derived transit function from I, namely the $\mathcal{C}-$ transit function of a graph G, which is a transit function finer than I_G. This transit function is obtained from the $\mathcal{C}$-convexity introduced by Norbert Polat in [17] in the context of the so called netlike partial cubes.
In Section 2, we formally define the $\mathcal{C}-$ transit function and discuss its betweenness properties. In Section 3, we discuss graphs in which the set $\mathcal{C}(u,v,w) = \mathcal{C}(u,v) \cap \mathcal{C}(v,w) \cap \mathcal{C}(u,w)$ contain at most one element, for every triple of vertices u,v,w in G.

2. $\mathcal{C}-$ transit function and betweenness axioms

In [17], Polat introduced the $\mathcal{C}$-convexity as a convexity finer than the geodesic convexity as follows. On a graph G with the interval function I, define the map $\mathcal{I} : P(V) \to P(V)$ as $\mathcal{I}(A) := \bigcup_{u,v \in A} I(u,v)$, for each $A \subseteq V$. The set of vertices of G which belong to a cycle of G is denoted by $CV(G)$ and the set of vertices of the subgraph of G induced by $\mathcal{I}_G(A)$ is denoted by $CV(G[\mathcal{I}_G(A)])$. A set $A \subseteq V(G)$ is $\mathcal{C}$-convex if $CV(G[\mathcal{I}_G(A)]) \subseteq A$.
Now, we define $\mathcal{C}(u,v)$ as $\mathcal{C}(u,v) = \{u,v\} \cup \{w : w \in I(x,y)$ where $x,y \in I(u,v)$ and $I(x,y)$ is a cycle (which is clearly isometric) $\}$. If $I(u,v)$ doesn't contain any cycle, then $\mathcal{C}(u,v) = \{u,v\}$. It can be easily observed that if the set $A \subset V(G)$ is $\mathcal{C}$- convex, then $\mathcal{C}(u,v) \subseteq A$, for all $u,v \in A$.
Now let $u,\ v \in A$ be arbitrary and let $w \in \mathcal{C}(u,v)$, then if $w = u$ or $w = v$ then clearly $w \in A$. Otherwise, by the definition of $\mathcal{C}(u,v)$, w belongs to some $I(x,y)$, where $x,y \in I(u,v)$ and $I(x,y)$ induces a cycle. Then $w \in CV(G[I(u,v)])$. Since A is $\mathcal{C}$-convex, $w \in CV(G[I(u,v)]) \subseteq CV(G[\mathcal{I}_G(A)]) \subseteq A$. Thus $w \in A$. It is interesting to observe that the function $\mathcal{C}(u,v)$ is a well defined transit function in the sense that it satisfies all the axioms $(t1),(t2)$, and $(t3)$ and that the $\mathcal{C}$-convexity is a convexity induced by the transit function $\mathcal{C}(u,v)$.
From the definition of $\mathcal{C}$-transit function, it is clear that $\mathcal{C}(u,v) \subseteq I(u,v)$. We are interested to find graphs G in which $\mathcal{C}(u,v) = \{u,v\}$ and $\mathcal{C}(u,v) = I(u,v)$, for every $u,v \in V(G)$. We will also prove that in general, the $\mathcal{C}$-transit function does not satisfy all the axioms that are satisfied by the interval function I.
We have the following straightforward proposition.

Proposition 1. *A connected graph G is a geodetic graph if and only if $\mathcal{C}(u,v) = \{u,v\}$ for all $u,v \in V(G)$.*

It is easy to observe that the interval function I_G of a graph G satisfies a weaker axiom than the axiom $(b3)$, named as $(b1)$, defined as:

$(b1)$ $x \in R(u,v), x \neq v \Rightarrow v \notin R(u,x)$, $\forall u,v \in V$.

Next preposition shows that in addition to the transit axioms, $(t1),(t2),(t3)$, the $\mathcal{C}-$ transit function satisfy the axioms $(b1)$, $(b2)$, $(b4)$ and $(s1)$ that are satisfied by the interval function.

Proposition 2. *Let G be a connected graph. Then the $\mathcal{C}$- transit function satisfy the axioms $(b1)$, $(b2)$, $(b4)$ and $(s1)$.*

Proof. For $(b1)$: Let $x \in \mathcal{C}(u,v), x \neq v$. If possible assume $v \in \mathcal{C}(u,x)$, then since $\mathcal{C}(u,v) \subseteq I(u,v)$, $v \in I(u,x)$, a contradiction.
For $(b2)$: Let $x \in \mathcal{C}(u,v)$ then $x \in I(u,v)$. If $\mathcal{C}(u,x) = \{u,x\}$, then clearly, $\mathcal{C}(u,x) \subseteq \mathcal{C}(u,v)$. If $\mathcal{C}(u,x) \neq \{u,x\}$, then there exist a $y \neq x$ such that $y \in \mathcal{C}(u,x)$ and then $y \in I(u,x)$. That is y is a vertex of some cycle C which is induced by the vertices in $I(u,x)$. That is, $y \in \mathcal{C}(u,x) \subseteq I(u,x) \subseteq I(u,v)$. Hence $y \in I(u,v)$ and y is a vertex of some cycle C whose vertices belongs to $I(u,x) \subseteq I(u,v)$. Thus $y \in \mathcal{C}(u,v)$.
For $(b4)$: Let $x \in \mathcal{C}(u,v)$. We have to show that $\mathcal{C}(u,x) \cap \mathcal{C}(x,v) = \{x\}$. Suppose not. That is, there exists $y \neq x$ such that $y \in \mathcal{C}(u,x) \cap \mathcal{C}(x,v)$. But $\mathcal{C}(u,x) \subseteq I(u,x)$ and $\mathcal{C}(x,v) \subseteq I(x,v)$. Therefore $y \in I(u,x)$ and $y \in I(x,v)$ implies $y \in I(u,x) \cap I(x,v)$ and $y \neq x$, a contradiction. Therefore $\mathcal{C}(u,x) \cap \mathcal{C}(x,v) = \{x\}$.
For $(s1)$: We have I satisfies the axiom $(s1)$, which means that if $d(u,v) = k$ then $d(\bar{u},\bar{v}) = k$. That is the vertices $u,v,\bar{u},\bar{v}$ lies on some cycle on G. That is, $v \in I(\bar{u},\bar{v})$ and v lies on some cycle on G. Therefore, $v \in \mathcal{C}(\bar{u},\bar{v})$. □

The graphs in Figure 1 shows that $\mathcal{C}(u,v)$ need not satisfy the axioms $(b3)$ and $(s2)$. The next theorem characterize the graphs in which $\mathcal{C}$ coincide with I.

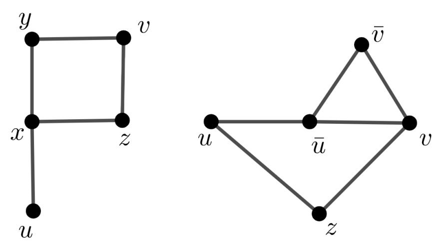

Figure 1. **Example showing that $\mathcal{C}(u,v)$ does not satisfy the axioms $(b3)$ and $(s2)$**

Theorem 2. *Let G be a connected graph. Then $\mathcal{C}(u,v) = I(u,v)$ for all $u,v \in V(G)$ if and only if for any two vertices $x,y \in V(G)$ with $d(x,y) = 2$ there exists at least two shortest path between x and y.*

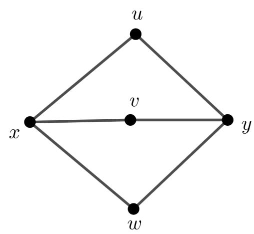

Figure 2. $K_{3,2}$

Proof. First assume that $\mathcal{C}(u,v) = I(u,v)$, for every $u,v \in V(G)$. Let u,v be two vertices in G with distance $d(u,v) = 2$ and have exactly one shortest path between them. Then the vertices in $I(u,v)$ does not induce a cycle. That implies $\mathcal{C}(u,v) = \{u,v\} \neq I(u,v)$, contradiction. Therefore there exist at least two shortest path between any two vertices with distance two.
Conversely, assume that there exist at least two shortest path between any two vertices with distance two. We have to show that $\mathcal{C}(u,v) = I(u,v)$. Suppose $w \in I(u,v)$. Let P be a $u-v$ shortest path containing w and let w' and w'' be two vertices in the shortest path P and adjacent to w. Then $d(w',w'') = 2$. By our assumption there exist another $w'-w''$ shortest path say R which does not contain the vertex w. Let y be the vertex between w' and w'' in the path R. Then the vertices $w'yw''ww'$ induces a cycle and all these vertices belongs to $I(u,v)$. Therefore these vertices belongs to $\mathcal{C}(u,v)$. In particular $w \in \mathcal{C}(u,v)$. Thus $\mathcal{C}(u,v) = I(u,v)$. □

A graph G, with interval function I and distance function d is *interval- regular* if, $|I(u,v) \cap N(u)| = d_G(u,v)$ or $|I(u,v) \cap N(v)| = d_G(u,v)$. We have the following remark.

Remark 1. In an interval- regular graph G, $\mathcal{C}(u,v) = I(u,v)$, since for any two vertices with distance 2 in G there is exactly two shortest paths. Hypercubes and Hamming graphs are examples of interval- regular graphs.

There exist graphs which are not interval-regular but $\mathcal{C}(u,v) = I(u,v)$. For example $K_{n,m}, n \geq 2, m > 2$. In $K_{n,m}, n \geq 2, m > 2$ for any two vertices $u,v \in V(G)$ with $d(u,v) = 2$ there exists at least two shortest paths between u and v but $|I(u,v) \cap N(u)| \neq d(u,v)$.
Let G be a connected graph and R be a transit function on G. Then generally the underlying graph G_R and G need not be isomorphic. In case of interval function I, G and G_I are isomorphic but in the case of $\mathcal{C}$- transit function, $G_{\mathcal{C}}$ and G need not be isomorphic. From Theorem 2, we have the following remark.

Remark 2. Let G be a connected graph and let $\mathcal{C}$ be the $\mathcal{C}$-transit function then G is isomorphic to $G_{\mathcal{C}}$ if and only if for any two vertices u, v with $d(u, v) = 2$ has at least two shortest path between them.

The following are some axioms satisfied by the $\mathcal{C}$ - transit function.

$(c1)$: if $x \in R(u, v)$, $y \in R(u, x)$ and $x \notin R(y, v)$ then $R(x, y) = \{x, y\}$.

$(c2)$: if $x \in R(u, v)$, $x \neq u, v$ then there exist $y \in R(u, v)$ with $y \notin R(u, x)$ and $y \notin R(v, x)$.

Proposition 3. *Let G be a connected graph. Then the $\mathcal{C}$- transit function satisfy the axioms $(c1)$ and $(c2)$.*

Proof. For $(c1)$: Suppose $x \in \mathcal{C}(u, v)$. Then x is a vertex of some cycle say C_n which is induced by some vertices in $I(u, v)$. Also $y \in \mathcal{C}(u, x)$ implies that y is a vertex of some cycle say C_m which is induced by some vertices in $I(u, x)$. Now $x \notin \mathcal{C}(y, v)$ implies that the subgraph induced by the vertices in $I(y, v)$ contain no cycle with x as its vertex, which implies that x and y are the vertices in same cycle induced by some vertices in $I(u, v)$ and hence there is only one $y - x$ shortest path and $\mathcal{C}(x, y) = \{x, y\}$.
For $(c2)$: Suppose $x \in \mathcal{C}(u, v)$. Then x is a vertex of some cycle say C_n which is induced by some vertices in $I(u, v)$. Let P and Q be the two disjoint subpaths of the $u - v$ shortest path in the cycle. Assume x is in the subpath P and let $d(u, x) = m$. Then there exist a vertex y in the subpath Q with $d(u, y) = m$. Therefore $y \in \mathcal{C}(u, v)$ and $y \notin \mathcal{C}(u, x)$ and $y \notin \mathcal{C}(v, x)$. □

3. Graphs with $|\mathcal{C}(u, v, w)| \leq 1$

The graphs G having the property that $|I(u, v, w)| \leq 1$, for every triple of vertices $u, v, w \in V(G)$ are well studied. These graphs include the class of median graphs, where $|I(u, v, w)| = 1$ and the so called weakly median graphs, see [1]. In this section, we attempt the analogue of graphs G having $|I(u, v, w)| \leq 1$, that is, $|\mathcal{C}(u, v, w)| \leq 1$, for every triple of vertices $u, v, w \in V(G)$. We need the following result by Chepoi [7].

Theorem 3. *[7] For the connected graph G the following conditions are equivalent.*

1. *G is isometrically embeddable in a Hamming graph.*
2. *For every edge $e = ab$, the sets W_{ab}, W_{ba} and W are geodesic convex sets.*

We have the following Theorem.

Theorem 4. *Let G be a partial Hamming graph, then $|\ \mathcal{C}(u, v, w)\ | \leq 1$, for any triple of vertices u, v, w in G.*

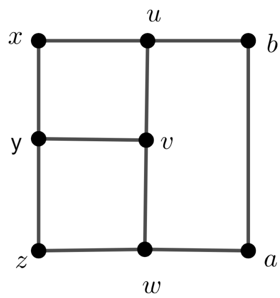

Figure 3. **Example for $\mid \mathcal{C}(u,v,w) \mid \leq 1$ but not a partial Hamming graph.**

Proof. First assume that either one of $u \in \mathcal{C}(v,w)$ or $v \in \mathcal{C}(u,w)$ or $w \in \mathcal{C}(u,v)$. Without loss of generality assume that $u \in \mathcal{C}(v,w)$. Then there exist vertices v' (may be v itself) and w' (may be w itself) with $v', w' \in \mathcal{C}(v,w)$ such that v' and w' are the diametric vertices of a cycle and $u \in \mathcal{C}(v', w')$. Since $\mathcal{C}$ satisfies axiom $(b4)$ and $u \in \mathcal{C}(v,w)$ implies that $\mathcal{C}(u,v) \cap \mathcal{C}(v,w) \cap \mathcal{C}(u,w) = \{v\}$. Therefore $\mid \mathcal{C}(u,v,w) \mid = 1$. Now assume that $u \notin \mathcal{C}(v,w)$ and $v \notin \mathcal{C}(u,w)$ and $w \notin \mathcal{C}(u,v)$. If $\mathcal{C}(u,v) = \{u,v\}$, $\mathcal{C}(u,w) = \{u,w\}$ and $\mathcal{C}(v,w) = \{v,w\}$ then $\mid \mathcal{C}(u,v,w) \mid = \phi$. Suppose that $\mid \mathcal{C}(u,v,w) \mid > 1$. That is, there exist vertices x, y with $x \neq y$ and $x, y \notin \{u,v,w\}$ such that $x, y \in \mathcal{C}(u,v,w)$. Since $x, y \in \mathcal{C}(u,v,w)$, the subgraph induced by $I(u,v)$ contains a cycle say C_{uv} such that x and y are two vertices in C_{uv}. That is, there exist vertices u' (may be u itself) and v'(may be v itself) with $u', v' \in \mathcal{C}(u,v)$ such that u' and v' are the diametric vertices of the cycle C_{uv} and $x, y \in \mathcal{C}(u', v')$. Also Since $x, y \in \mathcal{C}(u,w)$, the subgraph induced by $I(u,w)$ contains a cycle say C_{uw} such that x and y are any two vertices of C_{uw} and since $x, y \in \mathcal{C}(v,w)$, the subgraph induced by $I(v,w)$ contains a cycle say C_{vw} such that x and y are any two vertices of C_{vw}. Also we can find diametric vertices u' and w' (resp v' and w') corresponding to the cycle C_{uw} (resp C_{vw}). Let P and Q be the disjoint $u' - v'$ path in C_{uv}. Then we have $\ell(P) = \ell(Q)$. Now suppose that both the vertices x and y lies on the path P. Let u'' be the neighbour of u' on the path Q. Then $W_{u''u'}$ contains all the vertices in the path Q and it does not contain any vertices of the path P. That is $x, y \notin W_{u''u'}$ and $v \in W_{u''u'}$. Also, since $x, y \in \mathcal{C}(u,w)$ and u' and w' are diametric vertices of the cycle C_{uw}, we have $w \in W_{u''u'}$. That is, both v and w are in $W_{u''u'}$. But $x, y \in I(v,w)$ and $x, y \notin W_{u''u'}$. That is, $W_{u''u'}$ is not geodesic convex and G is not a partial Hamming graph, a contradiction. Now suppose that vertex x lies on the path P and vertex y lies on the path Q. Now $x \notin W_{u''u'}$ and $y, v, w \in W_{u''u'}$. Here also, $x \in I(v,w)$, but $x \notin W_{u''u'}$. That is, $W_{u''u'}$ is not geodesic convex and G is not a partial Hamming graph, a contradiction. Therefore in a partial Hamming graph $\mid \mathcal{C}(u,v,w) \mid \leq 1$ for any triple of vertices u, v, w. $\square$

Consider the graph in Figure 3. In this graph, $\mid \mathcal{C}(u,v,w) \mid \leq 1$, for any three vertices u, v, w. But it is not a partial Hamming graph. The next theorem characterize the graph class in which $|\mathcal{C}(u,v,w)| \leq 1$.

Theorem 5. *For a graph G, $|\mathcal{C}(u,v,w)| \leq 1$ for all $u,v,w \in V(G)$ if and only if G does not contain $K_{2,3}$ and its particular subdivisions (subdivision of $K_{2,3}$ by keeping the distance $d(u,x) = d(u,y), d(v,x) = d(v,y)$ and $d(w,x) = d(w,y)$ in Figure 2) as isometric subgraphs.*

Proof. Suppose G contain $K_{2,3}$ or its above mentioned subdivision as an isometric subgraph as labeled in Figure 2, then clearly $|\mathcal{C}(u,v,w)| = 2$. Now assume G does not contain $K_{2,3}$ and its above mentioned subdivision as an isometric subgraph. We have to show that $|\mathcal{C}(u,v,w)| \leq 1$. Assume $|\mathcal{C}(u,v,w)| > 1$. That is there exist at least two vertices x,y with $x \neq y$ and $x,y \notin \{u,v,w\}$ such that $x,y \in \mathcal{C}(u,v,w)$. Since $x,y \in \mathcal{C}(u,v)$, the subgraph induced by $I(u,v)$ contains a cycle say C_{uv} such that x and y are any two vertices of C_{uv}. Then there exist vertices u' (may be u itself) and v'(may be v itself) with $u',v' \in \mathcal{C}(u,v)$ such that u' and v' are the diametric vertices of the cycle C_{uv} and $x,y \in \mathcal{C}(u',v')$. Since $x,y \in \mathcal{C}(u,w)$, the subgraph induced by $I(u,w)$ contains a cycle say C_{uw} with diametric vertices u' and w' such that x and y are any two vertices of C_{uw}. Also since $x,y \in \mathcal{C}(v,w)$, the subgraph induced by $I(v,w)$ contains a cycle say C_{vw} with diametric vertices v' and w' such that x and y are any two vertices of C_{vw}.
Claim: $d(u',x) = d(u',y), d(v',x) = d(v',y)$ and $d(w',x) = d(w',y)$. Suppose $d(u',x) \neq d(u',y)$. That is either $d(u',x) < d(u',y)$ or $d(u',x) > d(u',y)$. We may assume $d(u',x) < d(u',y)$. Then $d(v',x) > d(v',y)$, since $d(u',x) + d(x,v') = d(u',v')$, $d(u',y) + d(y,v') = d(u',v')$ and u' and v' are the diametric vertices of the cycle C_{uv}. Also $d(w',x) > d(w',y)$, since $d(u',x) + d(x,w') = d(u',w')$, $d(u',y) + d(y,w') = d(u',w')$ and u' and w' are the diametric vertices of the cycle C_{uw}. But both $d(v',x) > d(v',y)$ and $d(w',x) > d(w',y)$ are not possible. If $d(v',x) > d(v',y)$ then $d(w',x) < d(w',y)$, since $d(v',x) + d(x,w') = d(v',w')$, $d(v',y) + d(y,w') = d(v',w')$ and v' and w' are the diametric vertices of the cycle C_{vw}. Therefore $d(u',x) = d(u',y), d(v',x) = d(v',y)$ and $d(w',x) = d(w',y)$ and hence the claim. That is the vertex u' is common to both the cycle C_{uv} and C_{uw}, the vertex v' is common to both the cycle C_{vw} and C_{uv} and the vertex w' is common to both the cycles C_{uw} and C_{vw}. Also $d(u',x) = d(u',y), d(v',x) = d(v',y)$ and $d(w',x) = d(w',y)$. The above arguments implies that G contain either a $K_{2,3}$ or its subdivisions (subdivision of $K_{2,3}$ by keeping the distance $d(u,x) = d(u,y), d(v,x) = d(v,y)$ and $d(w,x) = d(w,y)$ in Figure 2) as isometric subgraphs. □

Theorem 6. *For a graph G, $\mathcal{C}(u,v,w) = \emptyset$ for all $u,v,w \in V(G)$ if and only if G is a geodetic graph.*

Proof. Let G be a geodetic graph. Then $\mathcal{C}(x,y) = \{x,y\}$ for all $x,y \in V(G)$ so that $\mathcal{C}(u,v,w) = \mathcal{C}(u,w) \cap \mathcal{C}(w,v) \cap \mathcal{C}(u,v) = \emptyset$. Now suppose $\mathcal{C}(u,v,w) = \emptyset$ for all $u,v,w \in V(G)$. We have to prove that G is a geodetic graph. Assume G is not a geodetic graph. Then there exist vertices say x and y with $d(x,y) > 1$ such that there are at least two shortest path (not necessarily disjoint) between them. This means that $\mathcal{C}(x,y) \neq \{x,y\}$. That is a vertex say z which is disjoint from x and y such that $z \in \mathcal{C}(x,y)$ and then $z \in \mathcal{C}(x,y,z)$. Which is a contradiction to our assumption that $\mathcal{C}(u,v,w) = \emptyset$. So G is a geodetic graph. □

If the graph G is a hypercube, then $\mathcal{C}(u,v) = I(u,v)$ for all $u,v \in V(G)$ by Theorem 2. Also in hypercube, $\mid I(u,v,w) \mid = 1$. Then we have the following remark.

Remark 3. Let G be a hypercube then $\mid \mathcal{C}(u,v,w) \mid = 1$ for all $u,v,w \in V(G)$.

Concluding Remarks: In this paper, we have obtained some preliminary results on the cycle transit function and we have checked the status of the betweenness axioms of the interval function on the cycle transit function. An interesting problem is to characterize the cycle transit function using a set of first order axioms, which we may pursue in the near future.

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