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Research Article

The Cartesian product of wheel graph and path graph is antimagic

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Abstract: Suppose each edge of a simple connected undirected graph is given a unique number from the numbers 1, 2, . . . , q, where q is the number of edges of that graph. Then each vertex is labelled with sum of the labels of the edges incident to it. If no two vertices have the same label, then the graph is called an antimagic graph. We prove that the Cartesian product of wheel graph and path graph is antimagic.

Keywords: Graph labeling, antimagic labeling, magic labeling

AMS Subject classification: 05C78, 05C76

1. Introduction

An undirected, simple graph $G = (V, E)$ is said to be antimagic, if there exists a bijective labeling $f : E \to \{1, 2, 3, \ldots, |E|\}$ such that all vertex sums are distinct. Vertex sum is the sum of all the labels of edges incident to it. Antimagic labeling was introduced by Hartsfield and Ringel [6]. They had also conjectured that every connected graph different from K_2 is antimagic. This conjecture remains open. Nevertheless, some classes of graphs are shown to be antimagic. A short version of the conjecture is that all trees but K_2 are antimagic. The details and latest updates of various graph labelings can be found in Gallian's survey [5].
Wang proved that toroidal grids, $C_m \times C_n$, and higher dimensional toroidal grids, $C_{m_1} \times C_{m_2} \times C_{m_3} \times \cdots \times C_{m_t}$ are antimagic [10]. It is proved that lattice grid graphs, prism grid graphs, generalised prism grid graphs, generalised toroidal grid graphs, some lexicographic product graphs and some composition graphs are antimagic [4, 10].

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Zhang and Sun have proved that the Cartesian product of two or more paths and the Cartesian product of an antimagic regular graph and a connected graph are antimagic [11]. Phanalasy et al. constructed an antimagic labeling for the Cartesian product of regular graphs [8]. In 2011, Li gave the antimagic labeling of C_n^k for $k = 2, 3, 4$. [7]. In 2013, Wang et al. showed that join of G_1 and G_2 are antimagic [9], where G_1 is a graph with n vertices of minimum degree r, and G_2 is a graph with m vertices of maximum degree $2r - 1$ $(m \geq n)$. In 2014, Arumugam et al. defined a new family of graphs called generalized pyramid graph [1]. The authors gave the antimagic labeling for the family of generalized pyramid graphs. In the same year, Buset et al. provided antimagic labelings for a family of generalized antiprism graphs and generalized toroidal antiprism graphs [3]. Bača et al. (2015) constructed the antimagic labeling for some multipartite graphs. The antimagic labeling includes, the join graph $G + nK_1$, $G + K_m$ and $G + G_1$, with some conditions on G [2].
In this paper, we present the antimagic labeling of the Cartesian product of wheel graph and path graph. Also, we show that switching of priority of operations in graphs $(K_1 + C_{n-1}) \square P_m$ and $K_1 + (C_{n-1} \square P_m)$ does not have an impact on antimagicness.

2. Existence of the Antimagic Labelling

Theorem 1. *The Cartesian product of wheel graph and path graph is antimagic.*

Proof. A non-trivial Cartesian product of wheel graph and path, $W_n \square P_m$, where W_n is a wheel with $n \geq 4$ vertices and P_m is a path with $m \geq 1$ vertices, can be visualized as cylindrical in shape. See Figure 1. The Cartesian product graph $W_n \square P_m$ has $3mn - 2m - n$ edges.

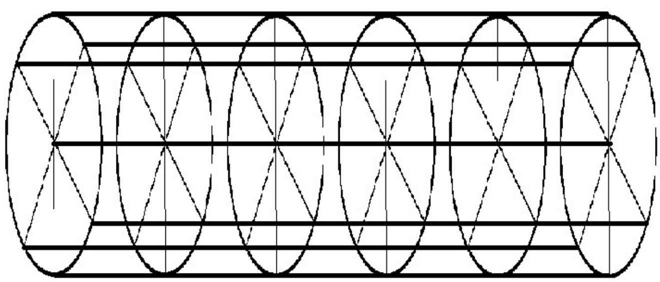

Figure 1. $W_6 \square P_6$

The proof is obtained in three cases. They are $m = 1$ and $n \geq 4$, $m > 1$ and $n > 4$ and $m > 1$ and $n = 4$.

Case 1. $m = 1$ and $n \geq 4$.

$W_n \square P_1$ is just a wheel. It has already been shown that wheel graphs are antimagic [6]. Hence the theorem is true for $n \geq 4$ and $m = 1$.

Case 2. $m > 1$ and $n > 4$.
For the wheel W_n, let the central vertex be u_0 and u_1, u_2, u_3, $\ldots$, $u_{(n-1)}$ be the other vertices. For the path P_m, let v_1, v_2, v_3, $\ldots$, v_m be the vertices. Let the function $s : V(W_n \square P_m) \to \mathbb{N}$ be the vertex sum induced by the edge labeling $f : E(W_n \square P_m) \to \{1, 2, 3, \ldots, (3mn - 2m - n)\}$. The Cartesian product gives rise to two parts, one contributed by W_n and the other by P_m. Let the vertices of the graph $W_n \square P_m$ be (u_i, v_j) for $0 \leq i \leq n-1$ and $1 \leq j \leq m$.

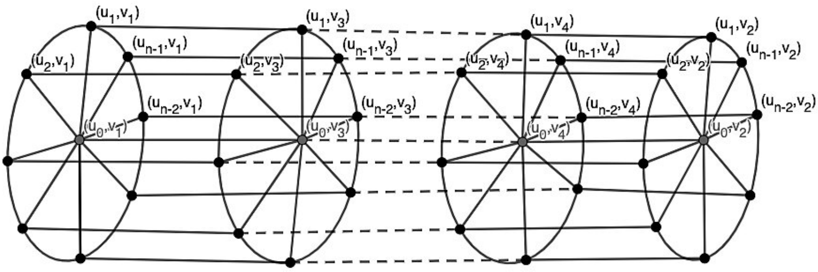

Figure 2. $W_n \square P_m$

We define the edge labeling for the graph $W_n \square P_m$ as follows.
$f((u_1, v_j), (u_{(n-1)}, v_j)) = 1 + (2n-2)(j-1)$, for $j = 1, 2, 3, \ldots, m$,
$f((u_i, v_j), (u_{i+1}, v_j)) = 2i + 1 + (2n-2)(j-1)$ for $i = 1, 2, \ldots, n-2$ and $j = 1, 2, 3, \ldots, m$,
$f((u_i, v_j), (u_i, v_{j+2})) = (2n-2i) + (2n-2)(j-1)$, for $i = 1, 2, \ldots, n-1$ and $j = 1, 2, 3, \ldots, m-2$,
$f((u_i, v_{m-1}), (u_i, v_m)) = (2n-2i) + (2n-2)(m-2)$ for $i = 1, 2, 3, \ldots, n-1$,
$f((u_0, v_1), (u_i, v_1)) = (2n-2)m - 2(n-(2+i))$ for $i = 1, 2, 3, \ldots, n-2$,
$f((u_0, v_1), (u_{n-1}, v_1)) = 2((n-1)m - n + 2)$
$f((u_0, v_j), (u_i, v_j)) = (2n-2)m + 1 + i + (n-1)(j-2)$ for $i = 1, 2, 3, \ldots, n-2$ and $j = 2, 3, \ldots, m$,
$f((u_0, v_j), (u_{n-1}, v_j)) = (2n-2)m + 1 + (n-1)(j-2)$ for $j = 2, 3, \ldots, m$,
$f((u_0, v_j), (u_0, v_{j+2})) = (2n-2)m + (n-1)(m-1) + j$ for $j = 1, 2, 3, \ldots, m-2$ and
$f((u_0, v_{m-1}), (u_0, v_m)) = (2n-2)m + (m-1)n$.
Using these labelings the vertex sum is as follows.
$s(u_{n-1}, v_1) = (2n-2)m + 4$,
$s(u_i, v_1) = (2n-2)m + 4 + 4i$ for $i = 1, 2, 3, \ldots, n-2$,
$s(u_{n-1}, v_2) = (2n-2)(m+3) + 2n + 1$,
$s(u_i, v_2) = 3i + 2n + 1 + (2n-2)(m+3)$ for $i = 1, 2, 3, \ldots, n-2$,
$s(u_{n-1}, v_j) = 3 + 2n + (n-1)(9j + 2m - 14)$ for $j = 3, 4 \ldots, m-1$,

$s(u_i, v_j) = 1 + 4n + i + (n-1)(9j + 2m - 14)$ for $i = 1, 2, 3, \ldots, n-2$ and $j = 3, 4, \ldots, m-1$
$s(u_{n-1}, v_m) = 5 + (n-1)(11m - 14)$,
$s(u_i, v_m) = 3 + i + 2n + (n-1)(11m - 14)$ for $i = 1, 2, 3, \ldots, n-2$,
$s(u_0, v_1) = 2(n-1)^2(m-1) + (n-1)(3m + n - 1) + 1$,
$s(u_0, v_2) = 2(n-1)^2 m + (n-1)(\frac{n}{2} + 3m - 1) + 2$,
$s(u_0, v_j) = (n-1)^2(2m + j - 2) + \frac{n(n-1)}{2} + (n-1)(6m - 2) + 2j - 2$ for $j = 3, 4, \ldots, m-1$ and
$s(u_0, v_m) = (n-1)^2(3m - 2) + \frac{n(n-1)}{2} + (n-1)(6m - 2) + 2m - 3$.

Subcase 2a. $m = 2$ and $n > 4$

The above labeling remains the same while the vertex sum changes as there is only one part contributed by P_2 to the graph $W_n \Box P_2$. Hence the vertex sum for $(u_1, v_2), (u_2, v_2), (u_3, v_2) \ldots (u_{n-1}, v_2)$ and (u_0, v_2) will be $s(u_i, v_2) = 1 + 3i + 2n + (n-1)(11m - 14)$ for $i = 1, 2, 3, \ldots, n-2$.
Also, $s(u_{n-1}, v_2) = 1 + 2n + (n-1)(11m - 14)$ and $s(u_0, v_2) = 2(n-1)^2 m + (n-1)(\frac{n}{2} + 3m - 1) + 1$. The vertex sum remains the same for $(u_1, v_1), (u_2, v_1), (u_3, v_1) \ldots (u_{n-1}, v_1)$ and (u_0, v_1).
From the above defined vertex sum it is obvious that $s(u_{n-1}, v_1) < s(u_1, v_1) < s(u_2, v_1) < s(u_3, v_1) < \cdots < s(u_{n-2}, v_1) < s(u_{n-1}, v_2) < s(u_1, v_2) < s(u_2, v_2) < s(u_3, v_2) < \cdots < s(u_{n-2}, v_2) < s(u_0, v_1) < s(u_0, v_2)$
We now show that the above labeling makes the graph $W_n \Box P_m$ antimagic.
$s(u_{n-1}, v_1) < s(u_1, v_1) < s(u_2, v_1) < \cdots < s(u_{n-2}, v_1) <$
$s(u_{n-1}, v_2) < s(u_1, v_2) < s(u_2, v_2) < \ldots, s(u_{n-2}, v_2) <$
$s(u_{n-1}, v_3) < s(u_1, v_3) < s(u_2, v_3) < \ldots, s(u_{n-2}, v_3) <$
$\vdots$
$s(u_{n-1}, v_{m-1}) < s(u_1, v_{m-1}) < s(u_2, v_{m-1}) < \ldots, < s(u_{n-2}, v_{m-1}) < s(u_{n-1}, v_m) < s(u_1, v_m) < s(u_2, v_m) < s(u_3, v_m) \ldots, < s(u_{n-2}, v_m) < s(u_0, v_1) < s(u_0, v_2) < s(u_0, v_3) < \ldots, < s(u_0, v_{m-1}) < s(u_0, v_m)$.
We know that, $4 < 4i + 4$ for $i = 1, 2, \ldots, n-2$ and thus
$(2n-2)m + 4 < (2n-2)m + 4i + 4$ and hence,
$s(u_{n-1}, v_1) < s(u_i, v_1)$.
Since $s(u_i, v_1) = (2n-2)m + 4 + 4i$ for $i = 1, 2, 3, \ldots, n-2$, we have, $s(u_1, v_1) < s(u_2, v_1) < \cdots < s(u_{n-2}, v_1)$.
Hence, $s(u_{n-1}, v_1) < s(u_1, v_1) < s(u_2, v_1) < \cdots < s(u_{n-2}, v_1)$.
Also, $4n - 4 < 8n - 5$.
Hence, $(2n-2)m + 4n - 4 < (2n-2)m + 8n - 5$.
i.e., $(2n-2)m + 4 + 4(n-2) < (2n-2)(m+3) + 2n + 1$.
Hence, $s(u_{n-2}, v_1) < s(u_{n-1}, v_2)$ and

$2n + 1 < 2n + 1 + 3i$ for $i = 1, 2, \ldots, n-2$,
$(2n-2)(m+3) + 2n + 1 < (2n-2)(m+3) + 2n + 1 + 3i$
$s(u_{n-1}, v_2) < s(u_i, v_2)$.
Since $s(u_i, v_2) = 3i + 2n + 1 + (2n-2)(m+3)$ for $i = 1, 2, 3, \ldots, n-2$, we have,

$s(u_1, v_2) < s(u_2, v_2) < \ldots, s(u_{n-2}, v_2)$.
Hence, $s(u_{n-1}, v_2) < s(u_1, v_2) < s(u_2, v_2) < \ldots, s(u_{n-2}, v_2)$.
We have, $11n - 11 < 15n - 10$.
Thus, $2m(n-1) + 11n - 11 < 15n - 10 + 2m(n-1)$ and
$5n - 5 + 2(n-1)(m+3) < 3 + 2n + (n-1)(2m+13)$.

Hence, $s(u_{n-2}, v_2) < s(u_{n-1}, v_3)$ and

$3 + 2n < 1 + 4n + i \quad$ for $i = 1, 2, 3, \ldots, n-2$ and
$(n-1)(2m+13) + 3 + 2n < (n-1)(2m+13) + 1 + 4n + i$
$s(u_{n-1}, v_3) < s(u_i, v_3)$.
Since $s(u_i, v_3) = (n-1)(2m+13) + 1 + 4n + i \quad$ for $i = 1, 2, 3, \ldots, n-2$, we have,
$s(u_1, v_3) < s(u_2, v_3) < \ldots, s(u_{n-2}, v_3)$.
Hence, $s(u_{n-1}, v_3) < s(u_1, v_3) < s(u_2, v_3) < \ldots, s(u_{n-2}, v_3)$.
Let k be any number in $j = 3, 4, \ldots, m-1$.

Claim: $s(u_{n-2}, v_{k-1}) < s(u_{n-1}, v_k) < s(u_i, v_k) \quad$ for $i = 1, 2, 3, \ldots, n-2$.
We have, $-18n - 22 < -12n + 17$.
Thus, $9k(n-1) + 2m(n-1) - 18n - 22 < -12n + 17 + 9k(n-1) + 2m(n-1)$
$1 + 4n + n - 2 + (n-1)(9(k-1) + 2m - 14) < 3 + 2n + (n-1)(9k + 2m - 14)$.
Hence, $s(u_{n-2}, v_{k-1}) < s(u_{n-1}, v_k)$.

We have, $3 + 2n < 1 + 4n + i$.
Thus $3 + 2n + (n-1)(9k + 2m - 14) < 1 + 4n + i + (n-1)(9k + 2m - 14)$.
Hence, $s(u_{n-1}, v_k) < s(u_i, v_k)$.
Therefore, $s(u_{n-2}, v_{k-1}) < s(u_{n-1}, v_k) < s(u_1, v_k) < s(u_2, v_k) < s(u_3, v_k) < \cdots < s(u_{n-2}, v_k)$.

We have, $-18n + 22 < -14n + 19$.
Thus $-18n + 22 + 11m(n-1) < 11m(n-1) - 14n + 19$ and
$1 + 4n + n - 2 + (n-1)(9(m-1) + 2m - 14) < 5 + (n-1)(11m - 14)$
$s(u_{n-2}, v_{m-1}) < s(u_{n-1}, v_m)$
$5 < 2n + 3 + i \quad$ for $i = 1, 2, 3, \ldots, n-2$ and
$5 + (n-1)(11m - 14) < (n-1)(11m - 14) + 2n + 3 + i$
$s(u_{n-1}, v_m) < s(u_i, v_m)$.
Since, $s(u_i, v_m) = (n-1)(11m - 14) + 2n + 3 + i \quad$ for $i = 1, 2, 3, \ldots, n-2$,
$s(u_1, v_m) < s(u_2, v_m) < s(u_3, v_m) < \cdots < s(u_{n-2}, v_m)$.

Therefore, $s(u_{n-1}, v_m) < s(u_1, v_m) < s(u_2, v_m) < s(u_3, v_m) < \cdots < s(u_{n-2}, v_m)$
$3n + 1 + 3m(n-1) + (n-1)(8m - 14) < (n-1)^2(2m-1) + 3m(n-1) + 1$.
As $n > 4, n_{min} = 5$, substituting $n = 5$ in the above inequality, we get $-40 < -15$. If the inequality holds for $n = 5$, it will hold for $n > 5$ also.
Hence, $s(u_{n-2}, v_m) < s(u_0, v_1)$.
$(n-1)^2(2m-1) + 3m(n-1) + 1 < 2(n-1)^2 m + (n-1)(\frac{n}{2} + 3m - 1) + 2$.

As $n > 4, n_{min} = 5$, substituting $n = 5$ in the above inequality, we get $-15 < 8$. If the inequality holds for $n = 5$, it will hold for $n > 5$ also.
Hence, $s(u_0, v_1) < s(u_0, v_2)$.

i.e., $2 < (n-1)^2 + (n-1)(3m-1) + 4$
i.e., $2 + (n-1)(3m-1) < (n-1)(3m-1) + (n-1)^2 + (n-1)(3m-1) + 4$
$2 + (n-1)(3m-1) + 2(n-1)^2 m + (n-1)\frac{n}{2} < 2(n-1)^2 m + (n-1)\frac{n}{2} + (n-1)(3m-1) + (n-1)^2 + (n-1)(3m-1) + 4$.
i.e., $2+2(n-1)^2 m+(n-1)(\frac{n}{2}+(3m-1)) < (n-1)^2(2m+1)+\frac{n(n-1)}{2}+(n-1)(6m-2)+4$.
Hence, $s(u_0, v_2) < s(u_0, v_3)$.
Since, $s(u_0, v_j) = (n-1)^2(2m + j - 2) + \frac{n(n-1)}{2} + (n-1)(6m-2) + 2j - 2$
for $j = 3, 4, \ldots, m-1$, it is obvious that $s(u_0, v_3) < s(u_0, v_4) < s(u_0, v_5) < \cdots < s(u_0, v_{m-1})$.
We have $-3(n-1)^2 + 2m - 4 < -2(n-1)^2 + 2m - 3$.
Thus $3m(n-1)^2 + \frac{n(n-1)}{2} + (n-1)(6m-2) - 3(n-1)^2 + 2m - 4 < 3m(n-1)^2 + \frac{n(n-1)}{2} + (n-1)(6m-2) - 2(n-1)^2 + 2m - 3$ and
$s(u_0, v_{m-1}) < s(u_0, v_m)$.
Hence the graph is antimagic.

Case 3. $n = 4$ and $m > 1$.
Consider the wheel W_4. Let the central vertex be u_0 and u_1, u_2, u_3, be the other vertices. For the path P_m, let v_1, v_2, v_3, $\ldots$, v_m be the vertices. Let the function $s : V(W_4 \Box P_m) \to \mathbb{N}$ be the vertex sum induced by the edge labeling $f : E(W_4 \Box P_m) \to \{1, 2, 3, \ldots, (10m-4)\}$. The Cartesian product gives rise to two parts, one contributed by W_4 and the other by P_m. The vertices of graph $W_4 \Box P_m$ are (u_i, v_j) for $0 \le i \le 3$ and $1 \le j \le m$.
We define the edge labeling for the graph $W_4 \Box P_m$ as follows.
$f((u_0, v_j), (u_i, v_j)) = (2i-1) + (j-1)10$ for $i = 1, 2, 3$ and $j = 1, 2, \ldots, m$,
$f((u_i, v_j), (u_{i+1}, v_j)) = 2i + (j-1)10$ for $i = 1, 2$ and $j = 1, 2, \ldots, m$,
$f((u_3, v_j), (u_1, v_j)) = 6 + (j-1)10$ for $j = 1, 2, \ldots, m$,
$f((u_i, v_j), (u_i, v_{j+2})) = 7 + i + (j-1)10.$ for $i = 0, 1, 2, 3$ and $j = 1, 2$,
$f((u_i, v_j), (u_i, v_{j+2})) = 14 + i + (j-1)10 + (j-3)10.$ for $i = 0, 1, 2, 3$ and $j = 3, 4, \ldots, m-1$ and
$f((u_i, v_{m-1}), (u_i, v_m)) = 7 + i + (m-2)10$ for $i = 0, 1, 2, 3$.
Using these labelings vertex sum is defined.
For all $i = 1, 2, 3$
$s(u_i, v_j) = 17 + (i-1)^i + 4(j-1)10$ for $j = 1, 2$.
$s(u_i, v_j) = 24 + (i-1)^i + i + (5j-7)10$ for $j = 3, 4, \ldots, m-1$,
$s(u_i, v_m) = 24 + (i-1)^i + i + (5m-8)10$,
$s(u_0, v_j) = 16 + 4(j-1)10$ for $j = 1, 2$,
$s(u_0, v_j) = 23 + (5j-7)10$ for $j = 3, 4, \ldots, m-1$ and
$s(u_0, v_m) = 23 + (5m-8)10$.
From the above vertex sum defined in this way, it is obvious that the inequality given below will follow.

$s(u_0, v_1) < s(u_1, v_1) < s(u_2, v_1) < s(u_3, v_1) <$
$s(u_0, v_2) < s(u_1, v_2) < s(u_2, v_2) < s(u_3, v_2) <$
$s(u_0, v_3) < s(u_1, v_3) < s(u_2, v_3) < s(u_3, v_3) <$
$\vdots$
$s(u_0, v_{m-1}) < s(u_1, v_{m-1}) < s(u_2, v_{m-1}) < s(u_3, v_{m-1}) <$
$s(u_0, v_m) < s(u_1, v_m) < s(u_2, v_m) < s(u_3, v_m)$.
Hence the graph $W_4 \square P_m$ is antimagic.
Subcase 3a. $m = 2$ and $n = 4$
The above edge labeling can be used but not the vertex sum. Vertex sum can be calculated specifically for it and observed to be distinct.
Therefore, from Case 1, Case 2, Subcase 2a, Case 3 and Subcase 3a, the theorem is proved. □

3. On $(K_1 + C_{n-1}) \square P_m$ and $K_1 + (C_{n-1} \square P_m)$

Wheel graphs are nothing but join of $K_1 + C_{n-1}$. From the above theorem, it can be seen that $(K_1 + C_{n-1})\square P_m$ is antimagic. Consider the graph $K_1 + (C_{n-1}\square P_m)$.
It is has been already proved that $(C_n \square P_m)$ is antimagic [4]. Now, join of K_1 and $(C_{n-1}\square P_m)$ gives rise to a graph with vertices $(u_1, v_1), (u_2, v_1), \ldots, (u_{n-1}, v_m)$ and u_0 of K_1. Comparing this with $W_n \square P_m$ for $n > 4$ and $m > 1$, the central vertices $(u_0, v_1), (u_0, v_2), \ldots, (u_0, v_m)$ are replaced by a single vertex say u_0 of K_1. The labels on spokes remain same as in Case 2 and are connected to u_0. It is already shown that the vertex sum of $(u_1, v_1), (u_2, v_1), (u_3, v_1), \ldots, (u_{n-1}, v_m)$ are distinct and vertex sum at u_0 will be the largest vertex sum. Thus, all vertices of the graph $K_1 + (C_{n-1}\square P_m)$ are distinct for $n > 4$ and hence antimagic.
As the labeling for C_{n-1} where $n = 4$ is different and also removing the central vertex from W_n for $n = 4$ affects the other labeling, we define a new labeling for $K_1 + (C_{n-1} \square P_m)$ for n = 4 and $m > 1$.
Let the function $s : V(K_1 + (C_{n-1}\square P_m)) \to N$ be the vertex sum induced by the edge labeling $f : E(K_1 + (C_{n-1}\square P_m)) \to \{1, 2, 3, \ldots, (9m - 3)\}$. The vertices of the graph $C_3 \square P_m$ are (u_i, v_j) for $1 \le i \le 3$ and $1 \le j \le m$ and u_0 is the vertex of K_1.
We define the edge labeling for the graph $K_1 + (C_{n-1} \square P_m)$for $n = 4$ and $m > 1$ as follows.
$f((u_i, v_j), (u_{i+1}, v_j)) = 6(j - 1) + 2i$ for $i = 1, 2$ and $j = 1, 2, \ldots, m$,
$f((u_3, v_j), (u_1, v_j)) = 6j$ for $j = 1, 2, \ldots, m$,
$f(u_0, (u_i, v_j)) = 6j + 2i - 7$ for $i = 1, 2, 3$ and $j = 1, 2, \ldots, m$,
$f((u_i, v_j), (u_i, v_{j+2}) = 6m + 3j + i - 3$ for $i = 1, 2, 3$ and $j = 1, 2, \ldots, m - 2$ and
$f((u_i, v_{m-1}), (u_i, v_m)) = 9m + i - 6$ for $i = 1, 2, 3$.
Using these labelings vertex sum is obtained as, $s(u_i, v_j) = 21j + 6m + i - 12$ for $i = 1, 2$ and $j = 1, 2$,
$s(u_3, v_j) = 21j + 6m - 3$ for $j = 1, 2$,
$s(u_i, v_j) = 24j + 12m + 2i - 21$ for $i = 1, 2$ and $j = 3, 4, \ldots, m - 1$,
$s(u_3, v_j) = 24j + 12m - 9$ for $j = 3, 4, \ldots, m - 1$,

$s(u_i, v_m) = 36m + 2i - 24 \quad$ for $i = 1, 2$,
$s(u_3, v_m) = 36m - 12$ and
$s(u_0) = 9m^2$.
From the above defined vertex sum, it is obvious that the below inequality will follow except for $m = 2$ and $m = 3$.
$s(u_1, v_1) < s(u_2, v_1) < s(u_3, v_1) <$
$s(u_1, v_2) < s(u_2, v_2) < s(u_3, v_2) <$
$s(u_1, v_3) < s(u_2, v_3) < s(u_3, v_3) <$
$\vdots$
Thus $s(u_1, v_{m-1}) < s(u_2, v_{m-1}) < s(u_3, v_{m-1}) <$
$s(u_1, v_m) < s(u_2, v_m) < s(u_3, v_m) < s(u_0)$.

When $m = 2$, $s(u_1, v_1) < s(u_2, v_1) < s(u_3, v_1) < s(u_1, v_2) < s(u_2, v_2) < s(u_0) < s(u_3, v_2)$.
When $m = 3$, $s(u_1, v_1) < s(u_2, v_1) < s(u_3, v_1) < s(u_1, v_2) < s(u_2, v_2) < s(u_3, v_2) < s(u_0) < s(u_1, v_3) < s(u_2, v_3) < s(u_3, v_3)$.
Hence, the graph $K_1 + (C_{n-1} \square P_m)$ is antimagic.
Therefore, switching priority in operations for graph $(K_1 + C_{n-1}) \square P_m$ and $K_1 + (C_{n-1} \square P_m)$ does not affect the antimagicness.

References

[1] S. Arumugam, M. Miller, O. Phanalasy, and J. Ryan, *Antimagic labeling of generalized pyramid graphs*, Acta Math. Sin. (Engl. Ser.) **30** (2014), no. 2, 283–290.

[2] M. Bača, O. Phanalasy, J. Ryan, and A. Semaničová-Feňovčíková, *Antimagic labelings of join graphs*, Math. Comput. Sci. **9** (2015), no. 2, 139–143.

[3] D. Buset, M. Miller, O. Phanalasy, and J. Ryan, *Antimagicness for a family of generalized antiprism graphs*, Electron. J. Graph Theory Appl. **2** (2014), no. 1, 42–51.

[4] Y. Cheng, *Lattice grids and prisms are antimagic*, Theoret. Comput. Sci. **374** (2007), no. 1-3, 66–73.

[5] J.A. Gallian, *A dynamic survey of graph labeling*, Electro. J. Combin. **1** (2020), no. Dynamic Surveys.

[6] N. Hartsfield and G. Ringel, *Pearls in Graph Theory: A Comprehensive Introduction*, San Diego, CA: Academic Press, 1990.

[7] P.C.B. Li, *Antimagic labelings of power of cycles graphs*, Department of Computer Science, University of Manitoba, Canada, 2011.

[8] O. Phanalasy, M. Miller, C.S. Iliopoulos, S.P. Pissis, and E. Vaezpour, *Construction of antimagic labeling for the Cartesian product of regular graphs*, Math. Comput. Sci. **5** (2011), no. 1, 81–87.

[9] T. Wang, M.J. Liu, and D.M. Li, *A class of antimagic join graphs*, Acta Math. Sin. (Engl. Ser.) **29** (2013), no. 5, 1019–1026.

[10] T.-M. Wang, *Toroidal grids are anti-magic*, International Computing and Combinatorics Conference, Springer, 2005, pp. 671–679.

[11] Y. Zhang and X. Sun, *The antimagicness of the Cartesian product of graphs*, Theoret. Comput. Sci. **410** (2009), no. 8-10, 727–735.