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Research Article

More on the bounds for the skew Laplacian energy of weighted digraphs

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Abstract: Let $\mathscr{D}$ be a simple connected digraph with n vertices and m arcs and let $W(\mathscr{D}) = (\mathscr{D}, w)$ be the weighted digraph corresponding to $\mathscr{D}$, where the weights are taken from the set of non-zero real numbers. Let $\nu_1, \nu_2, \ldots, \nu_n$ be the eigenvalues of the skew Laplacian weighted matrix $\widetilde{SLW}(\mathscr{D})$ of the weighted digraph $W(\mathscr{D})$. In this paper, we discuss the skew Laplacian energy $\widetilde{SLEW}(\mathscr{D})$ of weighted digraphs and obtain the skew Laplacian energy of the weighted star $W(\mathscr{K}_{1,n})$ for some fixed orientation to the weighted arcs. We obtain lower and upper bounds for $\widetilde{SLEW}(\mathscr{D})$ and show the existence of weighted digraphs attaining these bounds.

Keywords: Weighted digraph, skew Laplacian matrix of weighted digraphs, skew Laplacian energy of weighted digraphs

AMS Subject classification: 05C30, 05C50

1. Introduction

A weighted digraph $W(\mathscr{D})$(or a weighted network) is defined to be an ordered pair $(\mathscr{D}^u, w)$, where $\mathscr{D}^u = (V, \mathscr{A})$ is the underlying digraph of $W(\mathscr{D})$ and $w : \mathscr{A} \to \mathbb{R} - \{0\}$ is the weight function. Weight of any arc $e = (u, v)$ is denoted by $w(e)$. Every digraph can be regarded as the weighted digraph with weight of each arc equal to one.

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Thus weighted digraphs are generalizations of digraphs. The weight $w(W(\mathscr{D}_1))$ of a weighted subdigraph $W(\mathscr{D}_1)$ of a weighted digraph $W(\mathscr{D})$ is defined as the product of weights of the arcs of $W(\mathscr{D}_1)$. Also, $W(\mathscr{D}_1)$ is said to be positive or negative according as $w(W(\mathscr{D}_1)) > 0$ or $w(W(\mathscr{D}_1)) < 0$, that is, it contains an even or odd number of negative weighted arcs respectively. The all-positive (resp. all- negative) weighted digraph $W(\mathscr{D}^+)$ (resp. $W(\mathscr{D}^-)$) of $W(\mathscr{D})$ is the weighted digraph obtained from $W(\mathscr{D})$ by replacing weight $w(e)$ of each arc e of $W(\mathscr{D})$ by $|w(e)|(resp. -|w(e)|)$. Our weighted digraphs will have simple underlying digraphs. For more about weighted digraphs, see [3, 9, 10, 19].

In theoretical chemistry, $\mathbb{E}(G)$ corresponds to the π-electron energy of a conjugated molecule, represented by the graph G (see [17] and the references therein). We note that in chemical graph theory, if the underlying molecule is a hydrocarbon, then G is a simple, unweighted graph. However, if the conjugated molecule contains atoms different from carbon and hydrogen (in chemistry referred to as "heteroatoms") then G must possess pertinently weighted edges with orientation. These weights are usually positive valued, but they may be negative also. Chemical theories based on weighted digraphs and their eigenvalues have been elaborated in detail (see [15] and the references therein). Hence the results on the energy of both unweighted and weighted digraphs are of some chemical significance.

The adjacency matrix of a weighted digraph $W(\mathscr{D})$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ matrix $A(W(\mathscr{D})) = (a_{ij})$, where

$$a_{ij} = \begin{cases} w(v_i, v_j), & \text{if there is an arc from } v_i \text{ to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

Conversely, given an $n \times n$ matrix $M = (m_{ij})$ of real numbers, the weighted digraph $W(\mathscr{D})$ of M consists of n vertices with vertex i joined to vertex j by a directed arc of weight w_{ij} if and only if $w_{ij} \neq 0$. The characteristic polynomial $|\lambda I - A(W(\mathscr{D}))|$ of the adjacency matrix $A(W(\mathscr{D}))$ of a weighted digraph $W(\mathscr{D})$ is called the characteristic polynomial of $W(\mathscr{D})$ and is denoted by $\phi_{W(\mathscr{D})}(\lambda)$. The eigenvalues of $A(W(\mathscr{D}))$ are called the eigenvalues of $W(\mathscr{D})$. The set of distinct eigenvalues of $W(\mathscr{D})$ together with their multiplicities is called the spectrum of $W(\mathscr{D})$. If $W(\mathscr{D})$ is a weighted digraph of order n with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and if their respective multiplicities are $m_1, m_2, \ldots, m_k$, we write the spectrum of $W(\mathscr{D})$ as spec(W($\mathscr{D}$)) $= (\lambda_1^{[m_1]}, \lambda_2^{[m_2]}, \ldots, \lambda_k^{[m_k]})$. The following result connects the coefficients of the characteristic polynomial of a weighted digraph $W(\mathscr{D})$ with the structure and weights of linear weighted subdigraphs of $W(\mathscr{D})$ and is also known as coefficient theorem for weighted digraphs [1]. Recently, Bhat [4] studied the sign alternating property of coefficients of characteristic polynomial in some classes of weighted digraphs.

Theorem A. If $W(\mathscr{D})$ is a weighted digraph with characteristic polynomial

$$\phi_{W(\mathscr{D})}(\lambda) = \lambda^n + a_1(W(\mathscr{D}))\lambda^{n-1} + \cdots + a_{n-1}(W(\mathscr{D}))\lambda + a_n(W(\mathscr{D})),$$

then

$$a_k(W(\mathscr{D})) = \sum_{L \in \mathscr{L}_k} (-1)^{P(L)} |w(L)||s(L)|,$$

for all $k = 1, 2, \ldots, n$, where $L \in \mathscr{L}_k$ is the set of all linear weighted subdigraphs L of $W(\mathscr{D})$ of order k, $P(L)$ denotes number of components of L and $w(L)$ and $s(L)$ respectively denote the weight and sign of linear weighted subdigraph L.

The following result is a spectral characterization of balance in weighted digraphs [1].

Theorem B. A weighted digraph $W(\mathscr{D})$ is balanced if and only if it is cospectral with all its weighted digraph $W(\mathscr{D}^+)$.

The rest of the paper is organized as follows. In section 2, we obtain the skew Laplacian energy of a weighted star $W(\mathscr{K}_{1,n})$ for any orientation and with fixed weight a real number. We also obtain upper and lower bounds for the skew Laplacian energy $\widetilde{SLEW}(\mathscr{D})$ in terms of various parameters of the weighted digraph and also discuss the extremal cases. Finally, we mention some problems for the skew Laplacian energy of weighted digraphs for future research.

2. Skew Laplacian energy of weighted digraphs

Many results have been obtained on the skew spectra and skew spectral radii of simple digraphs [2, 6, 8, 13, 18, 20–22]. In [12], the authors obtained that every even positive integer is indeed the skew Laplacian energy of a digraph by taking weight of each edge equal to 1. Now, given a simple weighted digraph $W(\mathscr{D})$ with vertex set $V(W(\mathscr{D})) = \{v_1, v_2, \ldots, v_n\}$, let $w(d_i)^+$ and $w(d_i)^-$ denote respectively the out weighted degree and in weighted degree of a vertex v_i in $W(\mathscr{D})$. Denote $D^+W(\mathscr{D}) = \mathrm{diag}\big(w(d_1)^+, w(d_2)^+, \ldots, w(d_n)^+\big)$ and $D^-W(\mathscr{D}) = \mathrm{diag}\big(w(d_i)^-, w(d_2)^-, \ldots, w(d_n)^-\big)$ and define $\widetilde{D}W(\mathscr{D}) = D^+W(\mathscr{D}) - D^-W(\mathscr{D}) = \mathrm{diag}(w(d_1)^+ - w(d_1)^-, w(d_2)^+ - w(d_2)^-, \ldots, w(d_n)^+ - w(d_n)^-)$. Also, let $A^+W(\mathscr{D})$ be the $n \times n$ matrix, where $a_{ij} = w$ if (v_i, v_j) is a weighted arc of $W(\mathscr{D})$ and 0 otherwise and $A^-W(\mathscr{D})$ be the $n \times n$ matrix, where $a_{ij} = w$ if (v_j, v_i) is a weighted arc of $W(\mathscr{D})$ and 0 otherwise. Clearly, $A^-W(\mathscr{D}) = (A^+W(\mathscr{D}))^T$ (i,e in-adjacency matrix is equal to the transpose of the out adjacency matrix of a weighted digraph).
The skew adjacency matrix of a weighted digraph $W(\mathscr{D})$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ matrix $S(W(\mathscr{D})) = (a_{ij})$, where

$$a_{ij} = \begin{cases} w(v_i, v_j), & \text{if there is an arc from } v_i \text{ to } v_j, \\ -w(v_i, v_j), & \text{if there is an arc from } v_j \text{ to } v_i, \\ 0, & \text{otherwise.} \end{cases}$$

We define a new kind of skew Laplacian weighted matrix $\widetilde{SLW}(\mathscr{D})$ of $W(\mathscr{D})$ in a similar way as defined in [5] by

$$\begin{aligned}\widetilde{SLW}(\mathscr{D}) &= (D^+W(\mathscr{D}) - A^+W(\mathscr{D})) - (D^-W(\mathscr{D})) - A^-W(\mathscr{D}))\\ &= (D^+W(\mathscr{D}) - D^-W(\mathscr{D})) - (A^+W(\mathscr{D}) - A^-W(\mathscr{D}))\\ &= \widetilde{D}W(\mathscr{D}) - SW(\mathscr{D})\end{aligned}$$

Let $\nu_1, \nu_2, \ldots, \nu_n$ be the eigenvalues of the skew Laplacian weighted matrix $\widetilde{SLW}(\mathscr{D}) = \widetilde{D}W(\mathscr{D}) - SW(\mathscr{D})$. Since $\widetilde{SLW}(\mathscr{D})$ is not symmetric, the eigenvalues need not be always real. However, we have the following observations about the eigenvalues of $\widetilde{SLW}(\mathscr{D})$.

Lemma 1. *The sum of the eigenvalues of $\widetilde{SLW}(\mathscr{D})$ is zero.*

Proof. The proof of the lemma is simple as $\sum_{i=1}^{n} \nu_i = \text{trace}(\widetilde{SLW}(\mathscr{D})) = \sum_{i=1}^{n} (w(d_i)^+ - w(d_i)^-) = 0$. □

Lemma 2. *0 is an eigenvalue of $\widetilde{SLW}(\mathscr{D})$ with multiplicity at least η, the number of components of $W(\mathscr{D})$.*

Proof. Let $\Omega(\widetilde{SLW}(\mathscr{D}))$ denote the set of all eigenvalues of the skew Laplacian matrix $\widetilde{SLW}(\mathscr{D})$. Assume that $W(\mathscr{C}_1), W(\mathscr{C}_2), \ldots, W(\mathscr{C}_\eta)$ are the components of $W(\mathscr{D})$. Clearly, $\Omega(\widetilde{SLW}(\mathscr{D})) = \bigcup_{i=1}^{\eta} \Omega(\widetilde{SLW}(\mathscr{C}_i))$. To prove the result, it suffices to show that $0 \in \Omega(\widetilde{SLW}(\mathscr{C}_i))$, for $1 \leq i \leq \eta$. Clearly in the induced weighted subdigraph $W(\mathscr{C}_i)$, since sum of the weights in each row of $\widetilde{SLW}(\mathscr{C}_i)$ is zero, therefore zero is an eigenvalue of $\widetilde{SLW}(\mathscr{C}_i)$ with corresponding eigenvector $[1, 1, \cdots, 1]^T$. □

Gutman and Shao [15] and Gutman [14] defined energy of a weighted graph and energy of simple graphs respectively as the sum of absolute values of its eigenvalues. We extend the concept of skew Laplacian energy of digraphs to skew Laplacian energy of weighted digraphs. In [7, 11], the authors have obtained various bounds for the energy of weighted graphs and bounds for the skew Laplacian energy of weighted digraphs.

Definition 1. Let $W(\mathscr{D})$ be a simple weighted digraph on n vertices and m weighted arcs. The skew Laplacian energy of $W(\mathscr{D})$ is defined as

$$\widetilde{SLEW}(\mathscr{D}) = \sum_{i=1}^{n} |\nu_i|,$$

where $\nu_1, \nu_2, \ldots, \nu_n$ are the eigenvalues of the skew Laplacian weighted matrix $\widetilde{SLW}(\mathscr{D}) = \widetilde{D}W(\mathscr{D}) - SW(\mathscr{D})$ of $W(\mathscr{D})$.

Let $\mathscr{D}$ be a digraph and $W(\mathscr{D})$ be the weighted digraph with weight of each arc equal to one. Then $\widetilde{SLEW}(\mathscr{D}) = \widetilde{SLE}(\mathscr{D})$. This is one of the motivations that Definition 1 generalizes skew Laplacian energy of digraphs.

Theorem 1. *If $W(\mathscr{D})$ is an Eulerian weighted digraph, then $\widetilde{SLEW}(\mathscr{D}) = E_sW(\mathscr{D})$, where $E_sW(\mathscr{D})$ is the skew energy of weighted digraph $\mathscr{D}$.*

Proof. Since $W(\mathscr{D})$ is Eulerian, the weighted out-degree and the weighted in-degree are equal for each vertex in $W(\mathscr{D})$, and so $\widetilde{D}W = 0$, which implies in $\widetilde{SLW}(\mathscr{D}) = -SW(\mathscr{D})$ and hence $\widetilde{SLEW}(\mathscr{D}) = E_sW(\mathscr{D})$. □

We obtain the skew Laplacian energy of a weighted star for any orientation and as a consequence we show that every even positive integer is indeed the skew Laplacian energy of some weighted digraph.

Theorem 2. *For the weighted star $W(\mathscr{K}_{1,n})$ of order $n+1$, we have $\widetilde{SLE}(\mathscr{K}_{1,n}) = 2(n-1)w$, if all the arcs have same weight w and are oriented towards or away from the center, and $\widetilde{SLE}(\mathscr{K}_{1,n}) = (n-2)w + \sqrt{(nw-2kw)^2 - 4(n-1)}$, otherwise, where k, $1 \le k \le n-1$, is the number of weighted arcs oriented towards the center.*

Proof. Let $V(\mathscr{K}_{1,n}) = \{v_1, v_2, \dots, v_{n+1}\}$ be the vertex set of $\mathscr{K}_{1,n}$. If v_{n+1} is the center of $\mathscr{K}_{1,n}$, orient all the weighted edges toward v_{n+1}. Then

$$S(\mathscr{K}_{1,n}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & w \\ 0 & 0 & \cdots & 0 & w \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & w \\ -w & w & \cdots & -w & 0 \end{pmatrix} \text{ and } \widetilde{D}W(\mathscr{K}_{1,n}) = \begin{pmatrix} w & 0 & \cdots & 0 & 0 \\ 0 & w & \cdots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \cdots & w & 0 \\ 0 & 0 & \cdots & 0 & -nw \end{pmatrix}.$$

Therefore,

$$\widetilde{SLW}(\mathscr{K}_{1,n}) = \begin{pmatrix} w & 0 & \cdots & 0 & -w \\ 0 & w & \cdots & 0 & -w \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \cdots & w & -w \\ w & w & \cdots & w & -nw \end{pmatrix}.$$

It is easy to see that the eigenvalues of this weighted matrix are $\{-(n-1)w, 0, w^{[n-1]}\}$, and so $\widetilde{SLEW}(\mathscr{K}_{1,n}) = 2(n-1)w$. On the other hand, if we orient the weighted edges away from v_{n+1}, then it can be seen that

$$\widetilde{SLW}(\mathscr{K}_{1,n}) = \begin{pmatrix} -w & 0 & \cdots & 0 & w \\ 0 & -w & \cdots & 0 & w \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -w & w \\ -w & -w & \cdots & -w & nw \end{pmatrix},$$

having eigenvalues $\{(n-1)w, 0, -w^{[n-1]}\}$, so $\widetilde{SLEW}(\mathscr{K}_{1,n}) = 2(n-1)w$. Thus, for a directed weighted star $W(\mathscr{K}_{1,n})$, we have $\widetilde{SLEW}(\mathscr{K}_{1,n}) = 2(n-1)w$.
If all the weighted edges of the weighted star $W(\mathscr{K}_{1,n})$ are oriented away from the center v_{n+1} except $k,\ 1 \leq k \leq n-1$, weighted edges which are oriented towards the center v_{n+1}, then it can be seen that the skew Laplacian matrix of $W(\mathscr{K}_{1,n})$ is

$$\widetilde{SLW}(\mathscr{K}_{1,n}) = \begin{pmatrix} w & 0 & \cdots & 0 & 0 & \cdots & 0 & -w \\ 0 & w & \cdots & 0 & 0 & \cdots & 0 & -w \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & w & 0 & \cdots & 0 & -w \\ 0 & 0 & \cdots & 0 & -w & \cdots & 0 & w \\ \vdots & \vdots & \cdots & 0 & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -w & w \\ w & w & \cdots & w & -w & \cdots & -w & (n-2k)w \end{pmatrix}.$$

By direct calculation, it can be seen that the skew Laplacian weighted characteristic polynomial of this matrix is

$$x(x-w)^{k-1}(x+w)^{n-k-1} \times \Big(x^2 - (nw - 2kw)x + n - 1\Big)$$

and so its eigenvalues are
$\{0, w^{[k-1]}, -w^{[n-k-1]}, \frac{nw-2kw+\sqrt{(nw-2kw)^2-4(n-1)}}{2}, \frac{nw-2kw-\sqrt{(nw-2kw)^2-4(n-1)}}{2}\}$.
Therefore, $\widetilde{SLEW}(\mathscr{K}_{1,n}) = (n-2)w + \sqrt{(nw-2kw)^2 - 4(n-1)}$. Thus, we have $\widetilde{SLEW}(\mathscr{K}_{1,n}) = 2(n-1)w$, if all the weighted edges are oriented towards or away from the center, and $\widetilde{SLEW}(\mathscr{K}_{1,n}) = nw - 2w + \sqrt{(nw-2kw)^2 - 4(n-1)}$, otherwise, where $k,\ 1 \leq k \leq n-1$ is the number of edges oriented towards the center. This completes the proof. □

For a weighted digraph with n vertices, m weighted arcs with weights $\omega_1, \omega_2, \ldots, \omega_m$ having vertex out-degrees $w(d_i)^+$ and vertex in-degrees $w(d_i)^-,\ i = 1, 2, \ldots, n$, let $\mathscr{M} = -\sum\limits_{i=1}^{m} \omega_i^2 + \frac{1}{2}\sum\limits_{i=1}^{n}\big(w(d_i)^+ - w(d_i)^-\big)^2$ and $\mathscr{M}_1 = \mathscr{M} + 2\sum\limits_{i=1}^{m}\omega_i^2 = \sum\limits_{i=1}^{m}\omega_i^2 +$

$\frac{1}{2}\sum_{i=1}^{n}\big(w(d_i)^+ - w(d_i)^-\big)^2$. Clearly, $\mathscr{M}_1 \geq \sum_{i=1}^{m}\omega_i^2$, with equality if and only if $W(\mathscr{D})$ is Eulerian.

We now obtain the lower and upper bounds for skew Laplacian energy of weighted digraphs $\widetilde{SLEW}(\mathscr{D})$ in terms of the number of vertices n, the number of components $\mathscr{P}$, $\mathscr{M}$ and $\mathscr{M}_1$ and also show that the left and right inequality holds for some family of graphs.

Theorem 3. *Let $W(\mathscr{D})$ be a simple weighted digraph with n vertices, m weighted arcs and $\mathscr{P}$ components. Assume that $w(d_i)^+$ and $w(d_i)^-$ respectively are the weighted out-degree and the weighted in-degree of the vertex v_i, $i = 1, 2, \ldots, n$ and $\nu_1, \nu_2, \ldots, \nu_n$ are the skew Laplacian weighted eigenvalues of $W(\mathscr{D})$. Then*

$$2\sqrt{|\mathscr{M}|} \leq \widetilde{SLEW}(\mathscr{D}) \leq \sqrt{2\mathscr{M}_1(n-\mathscr{P})}.$$

Moreover the inequality on the left holds if and only if for each pair of $\nu_{i_1}\nu_{j_1}$ and $\nu_{i_2}\nu_{j_2}$ $(i_1 \neq j_1, i_2 \neq j_2)$, there exists a non-negative real number z such that $\nu_{i_1}\nu_{j_1} = z\nu_{i_2}\nu_{j_2}$; and for each pair of $\nu_{i_1}^2$ and $\nu_{i_2}^2$, there exists a non-negative real number l such that $\nu_{i_1}^2 = l\nu_{i_2}^2$. Also the inequality on the right holds if and only if $\mathscr{D}$ is 0-regular or for each $v_i \in V(\mathscr{D}), w(d_i)^+ = w(d_i)^-$, and the eigenvalues of $\widetilde{SLW}(\mathscr{D})$ are $0^{[p]}, (\alpha i)^{[\frac{n-p}{2}]}, (-\alpha i)^{[\frac{n-p}{2}]}$ $(\alpha > 0)$.

Proof. Let $\nu_1, \nu_2, \ldots, \nu_n$ be the eigenvalues of the skew Laplacian weighted matrix $\widetilde{SLW}(\mathscr{D}) = \widetilde{DW}(\mathscr{D}) - SW(\mathscr{D})$, where $\widetilde{DW}(\mathscr{D}) = \mathrm{diag}(w(d_1)^+ - w(d_1)^-, w(d_2)^+ - w(d_2)^-, \ldots, w(d_n)^+ - w(d_n)^-)$ and $SW(\mathscr{D}) = [sw_{ij}]$ (where $[sw_{ij}]$ are the entries of the matrix $SW(\mathscr{D})$) is the skew adjacency weighted matrix of $\mathscr{D}$. Therefore, from Lemma 1, we have

$$\sum_{i=1}^{n}\nu_i = \sum_{i=1}^{n}[w(d_i)^+ - w(d_i)^-] = 0 \tag{1}$$

Also, note that

$$\begin{aligned}\sum_{i<j}\nu_i\nu_j &= \sum_{i<j}\det\begin{pmatrix} w(d_i)^+ - w(d_i)^- & sw_{ij} \\ -sw_{ji} & w(d_j)^+ - w(d_j)^- \end{pmatrix} \\ &= \sum_{i<j}\Big[[\widetilde{w(d_i)}\widetilde{w(d_j)}] + [sw_{ij}]^2\Big], \text{ where } \widetilde{w(d_i)} = w(d_i)^+ - w(d_i)^- \\ &= \sum_{i<j}[\widetilde{w(d_i)}\widetilde{w(d_j)}] + \sum_{i=1}^{m}\omega_i^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i\neq j}\nu_i\nu_j &= 2\sum_{i<j}\nu_i\nu_j = 2\Big[\sum_{i<j}[\widetilde{w(d_i)}\widetilde{w(d_j)}]\Big] + 2\sum_{i=1}^{m}\omega_i^2 \\ &= \sum_{i\neq j}[\widetilde{w(d_i)}\widetilde{w(d_j)}]\Big] + 2\sum_{i=1}^{m}\omega_i^2.\end{aligned} \tag{2}$$

Hence, from (1) and (2), we get

$$\begin{aligned}\sum_{i=1}^{n}\nu_i^2 &= \big(\sum_{i=1}^{n}\nu_i\big)^2-\sum_{i\neq j}\nu_i\nu_j\\ &= \big[\sum_{i=1}^{n}\widetilde{w(d_i)}\big]^2-\Big[\sum_{i\neq j}[\widetilde{w(d_i)}\widetilde{w(d_j)}]+2\sum_{i=1}^{m}\omega_i^2\Big]\\ &= \sum_{i=1}^{n}\big[\widetilde{w(d_i)}\big]^2-2\sum_{i=1}^{m}\omega_i^2=2\mathscr{M}. \end{aligned}\tag{3}$$

Let $\widetilde{SLW}(\mathscr{D})=(ws_{ij})$. By Schur's Unitary triangularization theorem [16], there exists a unitary matrix $\mathbb{U}$ such that $\mathbb{U}^*\widetilde{SLW}(\mathscr{D})\mathbb{U}=\mathbb{T}$, where $\mathbb{T}=(t_{ij})$ is an upper triangular matrix with diagonal entries $t_{ii}=\nu_i, i=1,2,\dots,n$.
Therefore,

$$\sum_{i,j=1}^{n}|sw_{ij}|^2=\sum_{i,j=1}^{n}|t_{ij}|^2\geq\sum_{i=1}^{n}|\nu_i|^2. \tag{4}$$

This implies that

$$\sum_{i=1}^{n}|\nu_i|^2\leq\sum_{i,j=1}^{n}|sw_{ij}|^2=\sum_{i=1}^{n}(\widetilde{w(d_i)})^2+2\sum_{i=1}^{m}\omega_i^2=2\mathscr{M}_1.$$

Without loss of generality, assume that $|\nu_1|\geq|\nu_2|\geq\dots|\nu_n|$. As $\nu_{n-i}=0$, for $i=1,2,\dots,\mathscr{P}-1$, by Cauchy-Schwarz Inequality, we get

$$\begin{aligned}\widetilde{SLEW}(\mathscr{D})=\sum_{i=1}^{n}|\nu_i|&\leq\sqrt{(n-\mathscr{P})\sum_{i=1}^{n-\mathscr{P}}|\mu_i|^2}=\sqrt{(n-\mathscr{P})\sum_{i=1}^{n}|\mu_i|^2}\\ &\leq\sqrt{2\mathscr{M}_1(n-\mathscr{P})},\end{aligned}\tag{5}$$

which proves the right inequality.
Now we prove the left-hand inequality. Since $\sum\limits_{i=1}^{n}\nu_i=0$, $\sum\limits_{i=1}^{n}\nu_i^2+2\sum\limits_{i<j}\nu_i\nu_j=0$, using (3), we have $2\sum\limits_{i<j}\nu_i\nu_j=-2\mathscr{M}$, and so

$$2|\mathscr{M}|=2|\sum_{i<j}\nu_i\nu_j|\leq2\sum_{i<j}|\nu_i||\nu_j|. \tag{6}$$

Using (3), we get

$$2|\mathscr{M}|=|\sum_{i=1}^{n}\nu_i^2|\leq\sum_{i=1}^{n}|\nu_i|^2. \tag{7}$$

Hence,

$$(\widetilde{SLEW}(\mathscr{D}))^2 = (\sum_{i=1}^{n} |\nu_i|)^2 = \sum_{i=1}^{n} |\nu_i|^2 + 2\sum_{i<j} |\nu_i||\nu_j| \geq 4|\mathscr{M}|,$$

so that

$$\widetilde{SLEW}(\mathscr{D}) \geq 2\sqrt{|\mathscr{M}|}.$$

To see that the left and the right inequalities of the above theorem are sharp, we proceed as follows.
Let $W(\mathscr{D}) = W(\mathscr{K}_{2n,2n})$ be a weighted digraph with weight of each edge ω. We assume that $\{A, B\}$ is the partite set of $W(\mathscr{D})$. We divide A into two disjoint sets A_1, A_2 such that $|A_1| = |A_2| = n$ and similarly B into B_1, B_2 such that $|B_1| = |B_2| = n$. The weighted arc set is $\{(x_1, y_1) : x_1 \in A_1, y_1 \in B_1, w(x_1, y_1) = \omega\} \cup \{(x_2, y_2) : x_2 \in A_2, y_2 \in B_2, w(x_2, y_2) = \omega\} \cup \{(y_1, x_2) : y_1 \in B_1, x_2 \in A_2, w(y_1, x_2) = \omega\} \cup \{(y_2, x_1) : x_1 \in A_1, y_2 \in B_2, w(y_2, x_1) = \omega\}$. It can be easily seen that $w(d_i)^+ = w(d_i)^-$ for each vertex v_i in $W(\mathscr{D})$. So we get $2\sqrt{|\mathscr{M}|} = 2\sqrt{m}\omega = 4n\omega$, and the skew Laplacian weighted matrix of $W(\mathscr{D}_1)$ is

$$\widetilde{SL}W(\mathscr{D}_1) = \begin{pmatrix} 0 & 0 & -J\omega & J\omega \\ 0 & 0 & J\omega & -J\omega \\ J\omega & -J\omega & 0 & 0 \\ -J\omega & J\omega & 0 & 0 \end{pmatrix},$$

where J is the $n \times n$ matrix in which each entry is 1. Therefore, the skew Laplacian weighted characteristic polynomial $W(P)_{\widetilde{SL}}(W(\mathscr{D}_1; \lambda)) = \det(\lambda I - \widetilde{SL}W(\mathscr{D}_1)) = \lambda^{4n-2}(\lambda^2 + 4n^2\omega^2)$ and the eigenvalues of $\widetilde{SL}W(\mathscr{D}_1)$ are $2n\omega i, -2n\omega i, [0]^{4n-2}$. Hence, the skew Laplacian weighted energy of $W(\mathscr{D}_1)$ is $4n\omega$, which shows that the lower bound is sharp.
From Schur's unitary triangularization theorem [16], we know that $T = (t_{ij})$ is a diagonal matrix if and only if $\widetilde{SL}W(\mathscr{D})$ is a normal matrix. That is, $\widetilde{SL}W(\mathscr{D})\widetilde{SL}^*W(\mathscr{D}) = \widetilde{SL}^*W(\mathscr{D})\widetilde{SL}W(\mathscr{D})$. Since $\widetilde{SL}W(\mathscr{D}) = \widetilde{D}W(\mathscr{D}) - SW(\mathscr{D})$ and $\widetilde{SL}^*W(\mathscr{D}) = \widetilde{D}W(\mathscr{D}) + SW(\mathscr{D})$,

$$\begin{aligned}(\widetilde{D}W(\mathscr{D}) - SW(\mathscr{D}))(\widetilde{D}W(\mathscr{D}) + SW(\mathscr{D}))& \\ = (\widetilde{D}W(\mathscr{D}) + SW(\mathscr{D}))(\widetilde{D}W(\mathscr{D}) - SW(\mathscr{D})).&\end{aligned}$$

This implies that $SW(\mathscr{D})\widetilde{D}W(\mathscr{D}) = \widetilde{D}W(\mathscr{D})SW(\mathscr{D})$.
Comparing the element on the i^{th} row and the j^{th} column of the matrices on both sides, we arrive at

$$sw_{ij}(w(d_j)^+ - w(d_j)^-) = (w(d_i)^+ - w(d_i)^-)sw_{ij}. \qquad (8)$$

If v_i and v_j are not adjacent, then $sw_{ij} = 0$ and so (8) always holds. Assume that v_i and v_j are adjacent, then $sw_{ij} \neq 0$ and so (8) gives

$$w(d_i)^+ - w(d_i)^- = w(d_j)^+ - w(d_j)^-.$$

Let $W(\mathscr{C}_1), W(\mathscr{C}_2), \ldots, W(\mathscr{C}_{\mathscr{P}})$ be the components of the weighted digraph $W(\mathscr{D})$. As $W(\mathscr{C}_k)$, $1 \leq k \leq \mathscr{P}$, is connected, there is a weighted path between any two vertices. Let $W(\mathscr{P}) : u = v_0, v_1, \ldots, v_t = y$ be a weighted path between u and w in $W(\mathscr{C}_k)$. Since for any two connected vertices in $W(\mathscr{D})$, the difference between the weighted out-degree and the weighted in-degree is same, it follows that $w(d)^+(u) - w(d)^-(u) = w(d)^+(y) - w(d)^-(y)$, for all $u, y \in W(\mathscr{C}_k)$. Therefore, using the fact that $\sum_{v \in W(\mathscr{C}_k)} (w(d)^+(v) - w(d)^-(v)) = 0$, it follows that $w(d)^+(v) - w(d)^-(v) = 0$, for all $v \in W(\mathscr{C}_k)$. That is, $wd_i^+ = wd_i^-$, for all $v_i \in W(\mathscr{D})$, giving that $\widetilde{D}W(\mathscr{D}) = 0$ and so $\widetilde{SLW}(\mathscr{D}) = -SW(\mathscr{D})$. From Lemma 2, we know that 0 is an eigenvalues of $\widetilde{SLW}(\mathscr{D})$ with multiplicity at least $\mathscr{P}$ and 0 is also an eigenvalue of $\widetilde{SLW}(\mathscr{C}_k)$, $k = 1, 2, \ldots, \mathscr{P}$, with multiplicity at least 1.

If $t = |\nu_1|$ and $|\nu_2| = \cdots = |\nu_{n-\mathscr{P}}| = 0$, then we must have $t = 0$. For if $t > 0$, using the fact that the eigenvalues of $\widetilde{SLW}(\mathscr{D})$ are either zero or purely imaginary, it follows that the spectrum of $\widetilde{SLW}(\mathscr{D})$ is $\{it\omega, -it\omega, 0^{[n-1]}\}$, which is not possible as order of $W\mathscr{D}$ is n. Therefore, we must have $t = 0$ and so the spectrum of $\widetilde{SLW}(\mathscr{D})$ contains 0 with multiplicity n. Since $w(d_i)^+ = w(d_i)^-$, for all v_i, it follows that $W(\mathscr{D})$ is a 0-regular weighted digraph.

If $t = |\nu_1|$ and $|\nu_2| = \cdots = |\nu_{n-\mathscr{P}}| = \alpha$, $\alpha > 0$, then we must have $t = \alpha$. For if $t > \alpha > 0$, using the fact that the eigenvalues of $\widetilde{SLW}(\mathscr{D})$ are either zero or purely imaginary, it follows that the spectrum of $\widetilde{SLW}(\mathscr{D})$ is $\{it\omega, -it\omega, (i\alpha\omega)^{[\frac{n-\mathscr{P}-1}{2}]}, (-i\alpha\omega)^{[\frac{n-\mathscr{P}-1}{2}]}, 0^{[\mathscr{P}]}\}$, which is not possible as order of $W(\mathscr{D})$ is n. Therefore, we must have $t = \alpha$ and so the eigenvalues of $\widetilde{SLW}(\mathscr{D})$ are $\{(i\alpha\omega)^{[\frac{n-\mathscr{P}}{2}]}, (-i\alpha\omega)^{[\frac{n-\mathscr{P}}{2}]}, 0^{[\mathscr{P}]}\}$. Now we show the existence of weighted digraphs in which the right inequality holds.

Let $W(\mathscr{D}_2) = W(\mathscr{K}_3) \cup W(\mathscr{K}_3) \cup W(\mathscr{K}_3) \cup W(\mathscr{K}_1) \cup W(\mathscr{K}_1)$, where $W(\mathscr{K}_3)$ is a weighted oriented graph with the weighted arc set $\{(1 \to 2), (2 \to 3), (3 \to 1)\}$ and weight of each arc equal to 5. The eigenvalues of the skew Laplacian weighted matrix $\widetilde{SLW}(\mathscr{D}_2)$ are $[8.6603i]^3, [-8.6603i]^2, [0]^5$. Hence $\widetilde{SLW}(\mathscr{D}_2) = 51.9618$ and $\sqrt{2\mathscr{M}_1(11-5)} = \sqrt{2 \times 9 \times 5^2(11-5)} = 51.9618$, which implies that the right inequality is also sharp for such types of weighted digraph. This completes the proof. $\square$

Corollary 1. *Let $W(\mathscr{D})$ be a simple weighted digraph with $\mathscr{P}$ components $W(\mathscr{C}_1), W(\mathscr{C}_2), \ldots, W(\mathscr{C}_{\mathscr{P}})$. If $\widetilde{SLEW}(\mathscr{D}) = \sqrt{2\mathscr{M}_1(n-\mathscr{P})}$, then each component $W(\mathscr{C}_i)$ is Eulerian (weighted oriented degree at each vertex is same) with odd number of vertices.*

Proof. If $W(\mathscr{D})$ is 0-regular, then each weighted component of the weighted digraph $W(\mathscr{D})$ is an isolated vertex with total weight equal to zero, which automatically satisfies the given condition.

Case 2. Now, if $w(d_i)^+ = w(d_i)^-$, for each vertex $v_i \in V(W(\mathscr{C}_k))$ and hence $W(\mathscr{C}_k)$ is Eulerian. Also, the eigenvalues of $\widetilde{SLW}(\mathscr{D})$ are $[0]^{\mathscr{P}}, [\alpha i]^{\frac{n-\mathscr{P}}{2}}, [-\alpha i]^{\frac{n-\mathscr{P}}{2}}$ $(\alpha > 0)$. It shows that for each component $W(\mathscr{C}_k)$, $[0]^1$ is an eigenvalue of $\widetilde{SLW}(\mathscr{C}_k)$ and all the other eigenvalues are αi and $-\alpha i$ which appear in pairs. It follows that the number of vertices in $W(\mathscr{C}_k)$ is odd. Hence the result. □

We conclude this paper with the following problems which will be of interest for the future research.

Problem 1. Interpret the coefficients of the weighted characteristic polynomial of $\widetilde{SLW}(\mathscr{D})$ in terms of structure of the weighted digraph $W(\mathscr{D})$.

Problem 2. Establish possible relations between the largest and smallest skew Laplacian eigenvalue of a weighted digraph $W(\mathscr{D})$ with the parameters associated with the weighted digraph.

Problem 3. For any orientation and giving any weight to the arcs of $W(\mathscr{D})$ give the complete description for the skew Laplacian weighted energy of the weighted cycle $W(\mathscr{C}_n)$.

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