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Short Note

Remarks on the restrained Italian domination number in graphs

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Abstract: Let G be a graph with vertex set $V(G)$. An Italian dominating function (IDF) is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ having the property that that $f(N(u)) \geq 2$ for every vertex $u \in V(G)$ with $f(u) = 0$, where $N(u)$ is the neighborhood of u. If f is an IDF on G, then let $V_0 = \{v \in V(G) : f(v) = 0\}$. A restrained Italian dominating function (RIDF) is an Italian dominating function f having the property that the subgraph induced by V_0 does not have an isolated vertex. The weight of an RIDF f is the sum $\sum_{v\in V(G)} f(v)$, and the minimum weight of an RIDF on a graph G is the restrained Italian domination number. We present sharp bounds for the restrained Italian domination number, and we determine the restrained Italian domination number for some families of graphs.

Keywords: Italian domination, restrained Italian domination, restrained domination

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1. Introduction

For definitions and notations not given here we refer to [14]. We consider simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V|$. The *open neighborhood* of a vertex v is the set $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E\}$ and its *closed neighborhood* is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of vertex $v \in V$ is $d(v) = d_G(v) = |N(v)|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. The *complement* of a graph G is denoted by $\overline{G}$. For a subset D of vertices in a graph G, we denote by $G[D]$ the subgraph of G induced by D. A *leaf* is a vertex of degree one, and its neighbor is called a *support vertex*. A *strong support vertex* is a support vertex adjacent to more than one leaf. A set $S \subseteq V(G)$ is called a *dominating set* if every vertex is either an element of S or is adjacent to an element of S. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. A *restrained dominating set* is a set $S \subseteq V(G)$ where every vertex in $V(G) \setminus S$ is adjacent to a vertex in S as well as to another vertex in $V(G) \setminus S$. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G. Restrained domination was formally defined by Domke, Hattingh, S.T. Hedetniemi, Laskar and Markus in their 1999 paper [12]. For more information on this paramter we refer the reader to the survey paper [13]. We write P_n for the path of order n, C_n for the cycle of length n and K_n for the complete graph of order n. Also, let $K_{n_1,n_2,\ldots,n_p}$ denote the complete p-partite graph with vertex set $S_1 \cup S_2 \cup \ldots \cup S_p$ where $|S_i| = n_i$ for $1 \leq i \leq p$. For $n \geq 2$, the *star* $K_{1,n-1}$ has one vertex of degree $n-1$ and $n-1$ leaves. A *subdivision* of an edge uv is obtained by introducing a new vertex w and replacing the edge uv with edges uw and wv. A *wounded spider* is a tree obtained from $K_{1,r}$, $r \geq 1$, by subdividing at most $r-1$ of its edges. By $S_{p,q}$ we denote the *double star*, where one center vertex is adjacent to p leaves and the other one to q leaves.

Cockayne, Dreyer, S.M. Hedetniemi and S.T. Hedetniemi [11] introduced the concept of *Roman domination* in graphs, and since then a lot of related variations and generalizations have been studied (see [7–10]). A *Roman dominating function* (RDF) on a graph G is a function $f : V(G) \longrightarrow \{0,1,2\}$ satisfying the condition that each vertex u with $f(u) = 0$ has a neighbor v with $f(v) = 2$. The weight of an RDF f is the sum $w(f) = \sum_{v \in V(G)} f(v)$. The *Roman domination number* $\gamma_R(G)$ equals the minimum weight of a Roman dominating function on G. An RDF of G with weight $\gamma_R(G)$ is called a $\gamma_R(G)$-function. For an RDF f, one can denote $f = (V_0, V_1, V_2)$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0,1,2$.

In 2015, Roushini Leely Pushpam and Padmapriea [17] defined the *restrained Roman dominating function* (RRDF) as a Roman dominating function f with the property that $G[V_0]$ does not have an isolated vertex. The weight of an RRDF f is the sum $w(f) = \sum_{v \in V(G)} f(v)$. The *restrained Roman domination number* $\gamma_{rR}(G)$ equals the minimum weight of a restrained Roman dominating function on G. An RRDF of G with weight $\gamma_{rR}(G)$ is called a $\gamma_{rR}(G)$-function. The restrained Roman domination has been studied by several authors (see [3, 19]).

In 2016, Chellali, Haynes, S.T. Hedetniemi and MacRae [6] defined a new variant of Roman dominating functions, the so called Italian dominating functions. An *Italian dominating function* (IDF) on a graph G is a function $f : V \longrightarrow \{0,1,2\}$ having the property that $f(N(u)) \geq 2$ for each vertex u with $f(u) = 0$. The weight of an IDF f is the sum $w(f) = \sum_{v \in V(G)} f(v)$, and the minimum weight of an IDF in a graph G is the *Italian domination number*, denoted by $\gamma_I(G)$. In [1, 2, 4, 5, 15, 20], the authors consider variants of Italian domination.

In 2021, Samadi, Alishahi, Masoumi and Mojdeh [18] defined the *restrained Italian dominating function* (RIDF) as an IDF f having the property that the subgraph induced by V_0 does not have an isolated vertex. The weight of an RIDF f is the sum $\sum_{v \in V(G)} f(v)$, and the minimum weight of an RIDF on a graph G is the *restrained Italian domination number*, denoted by $\gamma_{rI}(G)$. An RIDF of G with weight $\gamma_{rI}(G)$ is called a $\gamma_{rI}(G)$-function. For an RIDF f, one can denote $f = (V_0, V_1, V_2)$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0,1,2$. Clearly, $\gamma_{rI}(G) \leq \gamma_{rR}(G)$.

In this paper, we present further bounds and Nordhaus-Gaddum type results for the restrained Italian domination number. In addition, we determine the restrained Italian domination number for some families of graphs.

We make use of the following results.

Proposition 1. *[12] If $n \geq 2$ is an integer, then $\gamma_r(K_{1,n-1}) = n$.*

Proposition 2. *[12] If T is a tree of order $n \geq 3$, then $\gamma_r(T) \geq \Delta(T)$. Furthermore, $\gamma_r(T) = \Delta(T)$ if and only if T is a wounded spider which is not a star.*

Proposition 3. *[17] Let P_n be a path of order $n \geq 4$. Then $\gamma_{rR}(P_n) = \frac{2n+3+r}{3}$, where $n \equiv r \pmod 3$ for $r \in \{1, 2, 3\}$.*

Proposition 4. *[17] Let C_n be a cycle of order $n \geq 3$. Then $\gamma_{rR}(C_n) = \frac{2n+3+r}{3}$, when $n \equiv r \pmod 3$ for $r \in \{1, 2\}$ and $\gamma_{rR}(C_n) = \frac{2n}{3}$, when $n \equiv 0 \pmod 3$.*

Proposition 5. *If $n \geq 1$, then $\gamma_{rR}(P_n) = \gamma_{rI}(P_n)$ and $\gamma_{rR}(C_n) = \gamma_{rI}(C_n)$ for $n \geq 3$.*

Proof. Let $G \in \{P_n, C_n\}$. If f is an RIDF, then $\Delta(G) \leq 2$ implies that every vertex u with $f(u) = 0$ has a neighbor v with $f(v) = 0$ and a neighbor w with $f(w) = 2$. Therefore f is also an RRDF and thus $\gamma_{rR}(G) \leq \gamma_{rI}(G)$. Because of $\gamma_{rR}(G) \geq \gamma_{rI}(G)$, we obtain $\gamma_{rR}(G) = \gamma_{rI}(G)$. □

The following inequality chain is obviously.

Proposition 6. *If G is a graph, then $\gamma_r(G) \leq \gamma_{rI}(G) \leq \gamma_{rR}(G) \leq 2\gamma_r(G)$.*

Propositions 1 and 6 lead to the next observation immediately.

Proposition 7. *If $n \geq 2$, then $\gamma_{rI}(K_{1,n-1}) = n$.*

Proposition 8. *[18] If G is a connected graph of order $n \geq 2$, then $\gamma_{rI}(G) \leq n$ with equality if and only if G is star or $G \in \{C_4, C_5, P_4, P_5, P_6\}$.*

Let $C_{5,5}$ be the graph of order 9 consisting of two cycles of length five with one vertex in common. Let R_6 be the graph of order 6 consiting of a cycle $C_5 = v_1v_2v_3v_4v_5v_1$ with a further vertex y and two further edges yv_1 and yv_3. It is straightforward to verify that $\gamma_{rI}(C_{5,5}) = 8$ and $\gamma_{rI}(R_6) = 5$.

Proposition 9. *[18] Let G be a connected graph of order n with $\delta(G) \geq 2$. If $G \notin \{C_3, C_4, C_5, C_7, C_8, K_{2,3}, R_6, C_{5,5}\}$, then $\gamma_{rI}(G) \leq n - 2$.*

Proposition 10. *[18] If G is a graph of order $n \geq 2$, then $\gamma_{rI}(G) \geq 2$, with equality if and only if $\Delta(G) = n-1$ and G contains a vertex w of maximum degree such that $\delta(G[N_G(w)]) \geq 1$ or G contains two vertices u and v such that the remaining $n-2$ vertices are adjacent to u and v and $G[V(G) \setminus \{u,v\}]$ has no isolated vertex.*

The proof of the next observation is easy and therefore omitted.

Proposition 11. *(i) $\gamma_{rI}(K_n) = 2$ for $n \geq 2$,*

(ii) $\gamma_{rI}(K_{m,n}) = 4$ for $m, n \geq 2$

(iii) Let $K_{n_1,n_2,\ldots,n_p}$ be the complete p-partite graph such that $p \geq 3$ and $n_1 \leq n_2 \leq \ldots \leq n_p$. Then $\gamma_{rI}(K_{1,n_2,\ldots,n_p}) = \gamma_{rI}(K_{2,n_2,\ldots,n_p}) = 2$ and $\gamma_{rI}(K_{n_1,n_2,\ldots,n_p}) = 3$ for $n_1 \geq 3$.

Let $p \geq 1$ and $0 \leq r \leq 2$ be integers, and let G_{3p+r} be the graph obtained from a cycle $C_{3p+r} = v_1v_2\ldots v_{3p+r}v_1$ by adding two leaves a_i and b_i to each vertex v_i for $1 \leq i \leq 3p+r$. Clearly, $\gamma_{rI}(G_{3p+r}) = 6p+2r$. Now let f be a $\gamma_{rR}(G_{3p+r})$-function. Then we observe that

$$f(v_i)+f(v_{i+1})+f(v_{i+2})+f(a_i)+f(a_{i+1})+f(a_{i+2}) \\ +f(b_i)+f(b_{i+1})+f(b_{i+2}) \geq 7.$$

This leads to $\gamma_{rR}(G_{3p+r}) \geq 7p+2r$. Hence we observe

Proposition 12. *There exist graphs G for which $\gamma_{rR}(G)-\gamma_{rI}(G)$ can be made arbitrarily large.*

2. Bounds

The *cliqe number* $c(G)$ of a graph G is the maximum order among the complete subgraphs of G.

Observation 1. Let G be a graph of order n.

(i) If $\delta(G) \geq 3$, then $\gamma_{rI}(G) \leq n+1-\delta(G)$.

(ii) If $c(G) \geq 3$, then $\gamma_{rI}(G) \leq n+2-c(G)$.

Proof. (i) Let $\delta = \delta(G) \geq 3$, and let z be a vertex of minimum degree with the neighbors $v_1, v_2, \ldots, v_\delta$. Define the function f by $f(z) = f(v_1) = f(v_2) = \ldots = f(v_{\delta-2}) = 0$ and $f(x) = 1$ otherwise. Then $G[V_0]$ is connected of order at least two, and every vertex of V_0 has at least two neighbors of weight one. Therefore f is an RIDF on G of weigh $n+1-\delta$ and thus $\gamma_{rI}(G) \leq n+1-\delta(G)$.

(ii) Let $c = c(G) \geq 3$, and let $u_1, u_2, \ldots, u_c$ be the vertices of a clique of G. Define the function f by $f(u_1) = f(u_2) = \ldots = f(u_{c-1}) = 0$, $f(u_c) = 2$ and $f(x) = 1$ otherwise. Then it is easy to see that f is an RIDF on G of weigh $n + 2 - c$ and thus $\gamma_{rI}(G) \leq n + 2 - c(G)$. □

The complete graphs demonstrate that Observation 1 (i) and (ii) are sharp. In addition, let H be the graph obtained from a complete graph K_{n-1} ($n \geq 4$) by adding a leaf w. Then it is straighforward to verify that $\gamma_{rI}(H) = n + 2 - c(H)$. This is a further example which shows that Observation 1 (ii) is sharp.

Theorem 2. *[18] If T is a tree of order $n \geq 3$ different from the double star $S_{2,2}$, then $\gamma_{rI}(T) \geq \frac{n+3}{2}$.*

If $\Delta(T) \geq \frac{n(T)+2}{2}$, then the next lower bound on $\gamma_{rI}(T)$ is better than this one in Theorem 2.

Theorem 3. *If T is a tree, then $\gamma_{rI}(T) \geq \Delta(T) + 1$.*

Proof. Let n be the order of T. If $1 \leq n \leq 3$, then $\gamma_{rI}(T) = n = \Delta(T) + 1$. Let now $n \geq 4$. If T is a star, then Proposition 7 implies $\gamma_{rI}(T) = n = \Delta(T) + 1$. If T is a wounded spider, which is not a star, then it is easy to verify that $\gamma_{rI}(T) \geq n - 1 \geq \Delta(T) + 1$. If T is not a wounded spider, then it follows from Propositions 2 and 6 that $\gamma_{rI}(T) \geq \gamma_r(T) \geq \Delta(T) + 1$. □

Let $S_{2,q}$ be the double star with $q \geq 1$. Then $\gamma_{rI}(S_{2,q}) = \Delta(S_{2,q}) + 1$. These double stars and the stars demonstsrate that Theorem 3 is sharp.

Theorem 4. *Let L be the set of leaves of a connected graph G. If $|L| \geq 1$, then $\gamma_{rI}(G) \geq |L|$ with equality if and only if G is not a star and each vertex $v \in V(G) \setminus L$ is a strong support vertex.*

Proof. Let f be an RIDF on G. Then $f(u) \geq 1$ for each $u \in L$ and so $\gamma_{rI}(G) \geq |L|$. Now let G be not a star, and let each vertex $v \in V(G) \setminus L$ be a strong support vertex. Define the function f by $f(x) = 1$ for $x \in L$ and $f(x) = 0$ for $x \in V(G) \setminus L$. Then $f(N(x)) \geq 2$ for each $x \in V(G) \setminus L$. Since G is connected and not a star, $G - L$ is connected and $|V(G) \setminus L| \geq 2$. Thus f is an RIDF on G and therefore $\gamma_{rI}(G) = |L|$.

Conversely, assume that $\gamma_{rI}(G) = |L|$. Then G is not a star. Let f be a $\gamma_{rI}(G)$-function. Since $f(u) \geq 1$ for each $u \in L$, we note that $f(u) = 1$ for each $u \in L$ and $f(x) = 0$ for each $x \in V(G) \setminus L$. Assume first that there exists a vertex $w \in V(G) \setminus L$ which is not a support vertex. Then $N[w] \subseteq V(G) \setminus L$ and $f(N[w]) \geq 1$. This leads to the contradiction $\gamma_{rI}(G) \geq |L| + 1$. Hence each vertex $x \in V(G) \setminus L$ is a support vertex. Assume that there exists a vertex $u \in V(G) \setminus L$ with exactly one leaf neighbor v. It follows that $N(u) \setminus \{v\} \subseteq V(G) \setminus L$. If $f(u) = 0$, then $f(v) = 2$ or $f(x) = 1$ for at least one vertex $y \in V(G) \setminus L$. In both cases we arrive at the contradiction

$\gamma_{rI}(G) \geq |L|+1$. If $f(u) \geq 1$, then we also obtain the contradiction $\gamma_{rI}(G) \geq |L|+1$. Consequently, each vertex $v \in V(G) \setminus L$ is a strong support vertex. □

3. Nordhaus-Gaddum type results

Results of Nordhaus-Gaddum type study the extreme values of the sum or product of a parameter on a graph and its complement. In their classical paper [16], Nordhaus and Gaddum discussed this problem for the chromatic number. We present such inequalities for the restrained Italian domination number

Theorem 5. *If G is a graph of order $n \geq 3$, then $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \geq 5$.*

Proof. Assume, without loss of generality, that $\gamma_{rI}(G) \leq \gamma_{rI}(\overline{G})$. According to Proposition 10 we only need to show that if $\gamma_{rI}(G) = 2$, then $\gamma_{rI}(\overline{G}) \geq 3$. Assume that $\gamma_{rI}(G) = 2$. It follows from Proposition 10 that $\Delta(G) = n-1$ and G contains a vertex w of maximum degree such that $\delta(G[N_G(w)]) \geq 1$ or G contains two vertices u and v such that the remaining $n-2$ vertices are adjacent to u and v and $G[V(G) \setminus \{u,v\}]$ has no isolated vertex. In the first case, $\overline{G} = H \cup \{w\}$, where w is an isolated vertex of $\overline{G}$. Since $n(H) \geq 2$, Proposition 10 leads to $\gamma_{rI}(\overline{G}) \geq \gamma_{rI}(H) + 1 \geq 3$. In the second case, we can assume, without loss of generality, that u and v are not adjacent in G. Thus $\overline{G} = H \cup \{uv\}$, where uv is an isolated edge of $\overline{G}$. Since $n(H) \geq 1$, we deduce that $\gamma_{rI}(\overline{G}) \geq \gamma_{rI}(H) + 2 \geq 3$. This completes the proof. □

Example 1. Let $Wd(2,p)$ be the *windmill graph* consiting of a center vertex z which is adjacent to the vertices of p copies of the complete graph K_2. If $p \geq 3$, then $\overline{Wd(2,p)}$ consists of an isolated vertex z and a complete p-partite graph $K_{n_1,n_2,\ldots,n_p}$ such that $n_1 = n_2 = \ldots = n_p = 2$. Hence it follows from Proposition 11 (iii) that $\gamma_{rI}(\overline{Wd(2,p)}) = 3$. Thus we obtain $\gamma_{rI}(Wd(2,p)) + \gamma_{rI}(\overline{Wd(2,p)}) = 5$.

Example 1 shows that Theorem 5 is sharp.

Theorem 6. *If G is a graph G of order $n \geq 6$, then $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 2n-2$.*

Proof. If G or $\overline{G}$ is neither a star nor the path P_6, then it follows from Proposition 8 that $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq (n-1) + (n-1) = 2n-2$. Assume next, without loss of generality, that G is a star. Then $\overline{G}$ is the union of an isolated vertex and a complete graph K_{n-1}. Thus Propositions 7 and 11 (i) imply $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq n+3 \leq 2n-2$. Finally, if, without loss of generality, $G = P_6$, then it is easy to see that $\gamma_{rI}(\overline{P_6}) = 3$ and so $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 6+3 = 9 \leq 2n-2$. □

If $G \in \{P_1, P_2, P_3, P_4, C_5\}$, then we observe that $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) = 2n$. Therefore the condition $n \geq 6$ in Theorem 6 is necessary.

Theorem 7. *If G is a graph of order $n \geq 7$, then $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 2n - 4$.*

Proof. Assume, without loss of generality, that $\delta(G) \leq \delta(\overline{G})$. We distinguish four case.

Case 1. Assume that $\delta(G) = 0$. Let u be a vertex such that $d_G(u) = 0$. Let $x_1, x_2, \ldots, x_{n-1}$ be the vertices of $G - u$ such that $d_{\overline{G}}(x_1), d_{\overline{G}}(x_2), \ldots, d_{\overline{G}}(x_k) \geq 2$ and $d_{\overline{G}}(x_{k+1}) = d_{\overline{G}}(x_{k+2}) = \ldots = d_{\overline{G}}(x_{n-1}) = 1$. If $k \geq 2$, then the function f with $f(u) = 2$, $f(x_1) = f(x_2) = \ldots = f(x_k) = 0$ and $f(x_{k+1}) = f(x_{k+2}) = \ldots = f(x_{n-1}) = 1$ is an RIDF on $\overline{G}$ of weight $n - k + 1$. If $k \geq 5$, then it follows that $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq n + (n - k + 1) \leq 2n - 4$. If $2 \leq k \leq 4$, then $\gamma_{rI}(G - u) = 2$ according to Proposition 10. This leads to $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 3 + n - k + 1 \leq 2n - 4$. If $d_{\overline{G}}(x_1) = d_{\overline{G}}(x_2) = \ldots = d_{\overline{G}}(x_{n-1}) = 1$, then $G - u$ is the complete graph, and we deduce from Proposition 11 (i) that $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 3 + n \leq 2n - 4$.

Case 2 Assume that $\delta(G) = 1$. Let u be a vertex such that $d_G(u) = 1$, and let v be adjacent to u in G. Let $H = G - v$, and let $x_1, x_2, \ldots, x_{n-2}$ be the vertices of $G - \{u, v\}$ such that $d_{\overline{H}}(x_1), d_{\overline{H}}(x_2), \ldots, d_{\overline{H}}(x_k) \geq 2$ and $d_{\overline{H}}(x_{k+1}) = d_{\overline{H}}(x_{k+2}) = \ldots = d_{\overline{H}}(x_{n-1}) = 1$. If $k \geq 2$, then the function f with $f(u) = 2$, $f(x_1) = f(x_2) = \ldots = f(x_k) = 0$ and $f(v) = f(x_{k+1}) = f(x_{k+2}) = \ldots = f(x_{n-1}) = 1$ is an RIDF on $\overline{G}$ of weight $n-k+1$. If $k \geq 5$, then it follows that $\gamma_{rI}(G)+\gamma_{rI}(\overline{G}) \leq n+(n-k+1) \leq 2n - 4$. If $2 \leq k \leq 4$ and $n \geq 8$, then $\gamma_{rI}(G) = 4$ according to Proposition 10. This leads to $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 4 + n - k + 1 \leq 2n - 4$.

Let next $k = 4$ and $n = 7$. If v is adjacent to x_5 in G, then the function f with $f(u) = 2$, $f(v) = f(x_5) = 0$ and $f(x_1) = f(x_2) = f(x_3) = f(x_4) = 1$ is an RIDF of G and therefore $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 6 + n - k + 1 = n + 3 = 10 = 2n - 4$. If v is adjacent to x_i in G for one $i \in \{1, 2, 3, 4\}$, say to x_4, then the function f with $f(v) = 2$, $f(x_4) = f(x_5) = 0$ and $f(x_1) = f(x_2) = f(x_3) = f(u) = 1$ is an RIDF of G and so $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 6 + n - k + 1 = n + 3 = 10 = 2n - 4$. It remains the case that v is adjacent to x_1, x_2, x_3, x_4 and x_5 in $\overline{G}$. Then the function f with $f(u) = f(v) = f(x_5) = 1$ and $f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ is an RIDF of $\overline{G}$ and so $\gamma_{rI}(G)+\gamma_{rI}(\overline{G}) \leq 7+3 = 10 = 2n-4$. Let next $2 \leq k \leq 3$ and $n = 7$. Then it easy to see that $\gamma_{rI}(G) \leq 4$ and hence $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 4+n-k+1 \leq 10 = 2n-4$. If $d_{\overline{H}}(x_1) = d_{\overline{H}}(x_2) = \ldots = d_{\overline{H}}(x_{n-2}) = 1$, then $G - \{u, v\}$ is the complete graph. If $n \geq 8$, then we have $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 4 + n \leq 2n - 4$ immediately. Let now $n = 7$. If v has at least two neighbors in $G - u$, say x_4 and x_5, then the fuction f with $f(x_5) = 2$, $f(u) = 1$ and $f(v) = f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ is an RIDF of G and thus $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 3 + 7 = 10 = 2n - 4$. It remains the case that v has at least 4 neighbors in $\overline{G} - u$, say x_1, x_2, x_3 and x_4 are neighbors of v in $\overline{G} - u$. Then the function f with $f(u) = f(x_1) = 2$, $f(x_5) = 1$ and $f(v) = f(x_2) = f(x_3) = f(x_4) = 0$ is an RIDF of $\overline{G}$, and we obtain $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 4 + 5 \leq 2n - 4$.

Case 3 Assume that $\delta(G) \geq 2$. Then the assumption $\delta(G) \leq \delta(\overline{G})$ leads to $\delta(\overline{G}) \geq 2$. Assume first that G is not connected. Let H_1 be a component of G and let $H_2 = G - H_1$. Since $\delta(G) \geq 2$, we note that $|H_1|, |H_2| \geq 3$. If $u \in V(H_1)$ and $v \in V(H_2)$, then the function f with $f(u) = f(v) = 2$ and $f(x) = 0$ otherwise is an RIDF of weight 4 on $\overline{G}$ and therefore $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq n + 4 \leq 2n - 4$ when $n \geq 8$. If $n = 7$, then let, without loss of generality, $|H_1| = 3$. Then $\gamma_{rI}(H_1) = 2$ and so $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 6 + 4 = 10 = 2n - 4$. The same holds when $\overline{G}$ is not connected. Assume next that G and $\overline{G}$ are connected. If $G, \overline{G} \notin \{C_7, C_8, C_{5,5}\}$, then Proposition 9 leads to $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 2n - 4$. If $G \in \{C_7, C_8, C_{5,5}\}$ or $\overline{G} \in \{C_7, C_8, C_{5,5}\}$, then it is straightforward to verify that $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 2n - 4$.

Case 4 Assume that $\delta(G) \geq 3$. Then $\delta(\overline{G}) \geq 3$. Now Observation 1 (i) leads to $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 2n - 4$. □

By Proposition 7, we have $\gamma_{rI}(K_{1,5}) = 6$. Hence it follows that $\gamma_{rI}(K_{1,5}) + \gamma_{rI}(\overline{K_{1,5}}) = 6 + 3 = 2n - 3$ for $n = 6$. In addition, we note that $\gamma_{rI}(R_6) + \gamma_{rI}(\overline{R_6}) = 6 + 3 = 2n - 3$ for $n = 6$. Consequently, the condition $n \geq 7$ in Theorem 7 is necessary.

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